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Kermack-McKendrick epidemics vaccinated

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This paper proposes a deterministic model for the spread of an epidemic. We extend the classical Kermack–McKendrick model, so that a more general contact rate is chosen and a vaccination added. The model is governed by a differential equation (DE) for the time dynamics of the susceptibles, infectives and removals subpopulation.

We present some conditions on the existence and uniqueness of a solution to the non-linear DE. The existence of limits and uniqueness of maximum of infected individuals are also discussed.

In the final part, simulations, numerical results and comparisons of the different vaccination strategies are presented.

Keywords: SIR epidemic models, vaccination, differential equation

AMS Subject Classification: 92D25, 37N25

1. INTRODUCTION

Models that describe behaviour of highly infective diseases are frequently studied in the recent literature (see [2, 3, 5, 9]). There are not so many models that would enable to understand an optimal control of the infection by means of vaccination evaluate it (see [4, 9]). This paper extends the classical Kermack–McKendrick differential equation for epidemics (see [3, 5, 6, 7]) and suggests its control by a continuous vaccination.

The presented model describes the behaviour of highly infective diseases (e.g. flu) with short healing time (few days) and very short incubation time, which is omitted in the model.

We consider a population that consists of $N$ individuals and it is divided into three subpopulations. The first one is called Susceptibles (the individuals who are not infected, but who can be infected by the disease), its size at time $t$ is denoted by $x_t$. The second one is made by Infectives (the infected individuals, who are able to spread the disease), its size being denoted by $y_t$. The Removals subpopulation consists of individuals who were infected, but who are not able to spread the infection further or get themselves infected again, because they are either isolated or cured and became immune. It follows that its size $z_t$ equals to $N - x_t - y_t$ at arbitrary time $t$. 
The model is described by the following differential equations (see [8]):

\[
\begin{align*}
    dx_t &= -\beta(z_t) y_t \left[ x_t - \vartheta(z_t) \right]^+ dt, & x_0 > 0, \\
    dy_t &= \beta(z_t) y_t \left[ x_t - \vartheta(z_t) \right]^+ dt - \gamma y_t dt, & y_0 > 0, \\
    dz_t &= \gamma y_t dt, & z_0 = 0,
\end{align*}
\]

(R1)

where \( \beta(z_t) \) is a Susceptibles–Infectives contact rate that is time dependent through \( z_t \), where \( \gamma \) is a recovery rate of the infection and, finally, \( \vartheta(z_t) \) is the size of vaccinated susceptibles subpopulation controlled by \( z_t \) again.

We shall assume that \( \beta(z) : \mathbb{R} \to \mathbb{R}^+ \) is a nonincreasing continuous function, \( \gamma > 0 \) and that \( \vartheta(z) : \mathbb{R} \to \mathbb{R}^+ \) a nondecreasing continuous function. From the assumptions, we know that \( x_t + y_t + z_t = N = x_0 + y_0 \) for all \( t \geq 0 \).

It means that the size of individuals newly infected during the time interval \((t, t + \Delta)\) is approximately equal to the product \( y_t \left[ x_t - \vartheta(z_t) \right]^+ \beta(z_t) \Delta \), where \( y_t \left[ x_t - \vartheta(z_t) \right]^+ \) is the number of all possible contacts between infective and susceptible nonvaccinated people (i.e. the number of all possible pairs) in time \( t \), and \( \beta(z_t) \) is a probability that a randomly chosen susceptible nonvaccinated person is infected by a randomly chosen infective person during the time interval \((t, t + 1)\). Because the population consists of people with different rate of immunity (e.g. children are more inclined to diseases than adults) and people with weaker immunity fall ill more easily than strong immune people, the rate of immunity of susceptibles grows with increasing \( z_t \). Therefore, the Susceptibles–Infectives contact rate \( \beta \) is nonincreasing function of Removals.

After the consultation with practitioners in medicine, the function of vaccinated susceptibles \( \vartheta \) is also considered to be a function of the removals, because the number of the removals is usually known and also because it is an indicator of the extent of the epidemic used in practice. Moreover, if we choose \( \vartheta \) as an increasing linear function of vaccinated individuals, then we vaccinate more people in the case when we have more infected individuals, because the increment of Removals is proportional to the number of infectives.

2. THEORETICAL RESULTS

In this section, we will speak about a solution to DEs. By the solution to DE we mean the classical solution, i.e. we use the definition of solution as introduced in [1], p. 67.

The following lemma follows from more general results, e.g. Corollary 16.10, p. 219, in [1], but it could be unnoticed when using it for our case. Therefore we show more intuitive proof without using any special theorems.

**Lemma 2.1.** If \( l_t = (x_t, y_t, z_t) \) is a solution to (R1), then \( l_t \in [0, N]^3 \) for all \( t \geq 0 \). Moreover, \( x_t \) is a nonincreasing function and \( z_t \) is an increasing function.
Proof. From (R1), we can get
\[ y_t = y_0 \exp \left\{ \int_0^t \beta(z_s)[x_s - \vartheta(z_s)]^+ - \gamma ds, \right\} \tag{1} \]
therefore \( y_t > 0 \) for all \( t \).

Further, \( z_t = z_0 + \int_0^t \gamma y_s \, ds \), therefore we get \( z_t \) as a nonnegative increasing function as \( y_t > 0 \).

The size of susceptibles \( x_t \) is obviously a nonincreasing function.

If we denote by \( \tau_v := \inf\{ t \in \mathbb{R}^+ : x_t \leq \vartheta(z_t) \} \) the first time, when susceptibles are completely vaccinated, then
\[
\frac{dx_t}{dt} = -\beta(z_t)y_t[x_t - \vartheta(z_t)] \tag{2}
\]
for \( t \in [0, \tau_v] \). Moreover, as \( x_t \) is nonincreasing and \( z_t \) is increasing, we have for all \( t > \tau_v \) that \( x_t \leq \vartheta(z_t) \) and \( \frac{dx_t}{dt} = 0 \).

Solving equation (2) we get
\[
x_t = x_0 + \int_0^t \beta(z_s)\vartheta(z_s)y_s \exp \left\{ \int_0^s \beta(z_u)y_u \, du \right\} ds \exp \left\{ -\int_0^t \beta(z_s)y_s \, ds \right\} \geq 0.
\]
Since \( x_t = x_{\tau_v} \) for all \( t \in (\tau_v, \infty) \), \( x_t \) is nonnegative function.

We proved that any solution \( l_t \) to (R1) maps \([0, \infty)\) into the first octant. As \( x_t \geq 0, y_t \geq 0 \) and \( z_t \geq 0 \) for all \( t \geq 0 \) and \( x_t + y_t + z_t = N \) it follows that \( l_t \in [0, N]^3 \).

Hence \( \vartheta \) is nondecreasing and \( x(.) \) is nonincreasing, then using \( \tau_v \) from the previous proof we can rewrite (R1) to the form
\[
\begin{align*}
\frac{dx_t}{dt} &= -\beta(z_t)y_t[x_t - \vartheta(z_t)] \quad x_0 > 0, \\
\frac{dy_t}{dt} &= \beta(z_t)y_t[x_t - \vartheta(z_t)] - \gamma y_t \quad y_0 > 0, \tag{R2} \\
\frac{dz_t}{dt} &= \gamma y_t \quad z_0 = 0,
\end{align*}
\]
for \( t \in [0, \tau_v] \) and
\[
\begin{align*}
\frac{dx_t}{dt} &= 0, \\
\frac{dy_t}{dt} &= -\gamma y_t \quad \tag{R3} \\
\frac{dz_t}{dt} &= \gamma y_t \quad \tag{R3}
\end{align*}
\]
for \( t \in [\tau_v, \infty) \).

Lemma 2.2. Let \( \beta \) and \( \vartheta \) be Lipschitz bounded functions. Then the equation (R2) has a unique solution on the interval \([0, \tau_v]\).

Proof. Denote
\[
f(x, y, z) = (-\beta(z)[x - \vartheta(z)]y, \beta(z)[x - \vartheta(z)]y - \gamma y, \gamma y)
\]
and

\[ \hat{f}(l) = f(\hat{x}, \hat{y}, \hat{z}), \]

where \( \hat{x} = (x \vee -2N) \wedge 2N \). Then, using Lemma 2.1 and the fact that the unique solution to

\[ dl = \hat{f}(l), \quad l_0 = (x_0, y_0, z_0) \]

is the unique solution to (R2) on the interval \([0, \tau_v]\), Lemma 2.2 follows from more general theorem 7.6 in [1], p. 100. □

Define \( \tau_Y := \arg \max_{t \in \mathbb{R}_+} y_t \) the time of culmination of the epidemic (below we show that the time \( \tau_Y \) is unique).

The following theorem is our main result.

**Theorem 2.3.** Let \( \beta \) and \( \vartheta \) satisfy the conditions of Lemma 2.2. Then

(i) the equation (R1) has a unique solution on the time interval \([0, \infty)\),

(ii) there exist limits of \( x, y, z \) at infinity, \( y_\infty = 0 \). If \( \tau_v = \infty \) then \( z_\infty \) is a solution to the equation

\[ z = N - X(z), \]

where \( X(z) = x^0 + \int_0^z \frac{\beta(u)}{\gamma} \vartheta(u) \exp \left\{ \int_0^u \frac{\beta(s)}{\gamma} ds \right\} du \exp \left\{ -\int_0^z \frac{\beta(u)}{\gamma} du \right\}. \]

(iii) the size of infectives subpopulation \( y_t \) has a unique maximum \( y_{\tau_Y} \).

If \( \beta(z_0)[x_0 - \vartheta(z_0)] \) \( \gamma > 0 \), then \( \beta(z_{\tau_Y})[x_{\tau_Y} - \vartheta(z_{\tau_Y})] = \gamma. \)

If \( \beta(z_0)[x_0 - \vartheta(z_0)] \leq 0 \), then \( \tau_Y = 0. \)

**Proof.**

(i) The existence and uniqueness of a solution to (R2) on the time interval \([0, \tau_v]\) follows from Lemma 2.2. Therefore we need to prove its existence and uniqueness on the time interval \([\tau_v, \infty)\) in the case \( \tau_v < \infty \). Because the equation (R3) with the initial conditions \( x(\tau_v) = \hat{x}(\tau_v), y(\tau_v) = \hat{y}(\tau_v), z(\tau_v) = \hat{z}(\tau_v) \), where \( (\hat{x}, \hat{y}, \hat{z}) \) is a solution to (R2) on the time interval \([0, \tau_v]\), has a unique solution, it follows that

\[
\begin{align*}
x(t) &= x_{\tau_v}, \\
y(t) &= y_{\tau_v} - e^{-\gamma t} + e^{-\gamma \tau_v}, \\
z(t) &= N - x_{\tau_v} - y_{\tau_v} + e^{-\gamma t} - e^{-\gamma \tau_v}.
\end{align*}
\]

holds.

Joining these solutions, we get a unique solution to (R1) on the time interval \([0, \infty)\). Indeed, if we denote

\[
\hat{l}_t = (\hat{x}_t, \hat{y}_t, \hat{z}_t) = (\hat{x}_t, \hat{y}_t, \hat{z}_t) \quad t \in [0, \tau_v]
\]

\[
= (x_t, y_t, z_t) \quad t \in (\tau_v, \infty),
\]

Then, using Lemma 2.1 and the fact that the unique solution to
then
\[ \dot{x}_t = x_0 - \int_0^t \beta(\dot{z}_s) \dot{y}_s [\dot{x}_s - \vartheta(\dot{z}_s)]^+ \, ds \]
\[ \dot{y}_t = y_0 + \int_0^t \beta(\dot{z}_s) \dot{y}_s [\dot{x}_s - \vartheta(\dot{z}_s)]^+ - \gamma \dot{y}_s \, ds \]
\[ \dot{z}_t = \int_0^t \gamma \dot{y}_s \, ds. \]

Therefore \( \dot{t} \) is a solution to (R1).

(ii) Functions \( x \) and \( z \) are monotone and bounded, therefore they have their limits \( x_\infty, z_\infty \) at infinity. Because \( y_t = N - x_t - z_t \) for all \( t \in [0, \infty) \), the existence of the limits \( x_\infty \) and \( z_\infty \) implies the existence of the limit \( y_\infty \). Since \( z_\infty < \infty \), we get \( y_\infty = 0 \). Indeed, if \( y_\infty > 0 \), then there exists a time \( T \in [0, \infty) \) and a constant \( a > 0 \) such that \( y_t \geq a \) for all \( t > T \). Therefore

\[ z_\infty = \int_0^\infty \gamma y_s \, ds \geq \int_0^T \gamma y_s \, ds + \int_T^\infty \gamma a \, ds = \infty. \]

It means that \( y_\infty = 0 \).

As \( z_t \) is a continuous differentiable mapping of \([0, \infty)\) on \([0, z_\infty]\) with positive derivation, it has continuously differentiable inverse \( z_t^{-1} \), we can set \( X(z) = x(z_t^{-1}) \) and (R2) implies

\[ \frac{dX(z)}{dz} = \frac{dx_t/\,dt}{dz_t/\,dt} = -\frac{\beta(z_t)Y(z_t)[X(z_t) - \vartheta(z_t)]}{\gamma Y(z_t)}, \quad X(z_0) = x_0, \]

therefore

\[ X(z) = \left[ x^0 + \int_0^z \frac{\beta(u)}{\gamma} \vartheta(u) \exp \left\{ \int_0^u \frac{\beta(s)}{\gamma} \, ds \right\} \, du \right] \exp \left\{ -\int_0^z \frac{\beta(u)}{\gamma} \, du \right\}. \quad (3) \]

Finally, let \( t \to \infty \) in \( z(t) = N - x(t) - y(t) \) to get the equation \( z_\infty = N - x_\infty = N - X(z_\infty) \).

(iii) Because \( \beta(z_t) \) and \( x_t \) are nonincreasing functions of \( t \) and \( \vartheta \) a nondecreasing function of \( t \), it follows that \( \beta(z_t)[x_t - \vartheta(z_t)] \) is nonincreasing. Hence \( \beta(z_0)[x_0 - \vartheta(z_0)] - \gamma \leq 0 \) implies \( dy_t \leq 0 \), and \( y_t \) is nonincreasing. Thus, \( \tau_Y = 0 \). If \( \beta(z_0)[x_0 - \vartheta(z_0)] - \gamma > 0 \), then \( y_t \) is increasing in neighbourhood of zero and because moreover \( y_0 > y_\infty = 0 \), we have \( 0 < \tau_Y < \infty \). Hence continuity and the existence of derivative of \( y_t \) imply that \( y'_\tau = 0 \), therefore \( \beta(z_\tau)[x_\tau - \vartheta(z_\tau)] - \gamma = 0 \).

Let \( \beta(z_\tau)[x_\tau - \vartheta(z_\tau)] = \gamma \). Denote \( T := \inf \{ t \geq 0 : \beta(z_t)[x_t - \vartheta(z_t)] = \gamma \} \). From (R1) and \( y_\tau \geq y_0 > 0 \) it follows that \( x_t \) is decreasing in \( T \), and so there is no other time \( t \) satisfying \( \beta(z_t)[x_t - \vartheta(z_t)] = \gamma \). Therefore \( \tau_Y = T \) is unique. In the case \( \beta(z_0)[x_0 - \vartheta(z_0)] - \gamma < 0 \), the uniqueness of \( \tau_Y \) is obvious. \( \square \)
Example 2.4. We shall scrutinize the equation \( z_\infty = N - X(z_\infty) \) (see Theorem 2.3(ii)) and assume \( \beta \) to be a constant and \( \vartheta(z) \) a general function, later on a linear function.

We apply (3) to get
\[
z_\infty = N - \left[ x^0 + \int_0^{z_\infty} \frac{\beta}{\gamma} \vartheta(u) \exp \left\{ \frac{\int_0^u \beta \, ds}{\gamma} \right\} \right] \exp \left\{ -\int_0^{z_\infty} \frac{\beta}{\gamma} \, du \right\}. \tag{5}
\]

Denoting \( \rho = \beta / \gamma \) then (5) yields
\[
z_\infty = N - e^{-\rho z_\infty} \left[ x^0 + \rho \int_0^{z_\infty} \vartheta(u) e^{\rho u} \, du \right]. \tag{6}
\]

Choosing a linear vaccination, i.e. \( \vartheta(z) = \vartheta_0 + \vartheta_1 z \), where \( \vartheta_0 \geq 0 \) a \( \vartheta_1 \geq 0 \), we have
\[
C_1 z_\infty = C_2 - C_3 e^{-\rho z_\infty}, \tag{7}
\]
where
\[
C_1 = 1 + \vartheta_1, \\
C_2 = N - \vartheta_0 + \vartheta_1 / \rho, \\
C_3 = x^0 - \vartheta_0 + \vartheta_1 / \rho.
\]

The uniqueness of solution to equations (5) depends on the choice of functions \( \beta(z) \), \( \vartheta(z) \) and initial conditions. If we have more then one solution to the equation, we have to decide, which of them is \( z_\infty \).

To illustrate it, go back to the equation (7). What we know is that \( C_1 > 0 \) and \( C_2 > C_3 \) hold.

If \( C_3 \leq 0 \) then the number of vaccinated at \( t = 0 \) is larger than or equal to the number of susceptibles, hence \( \tau_v = 0 \) and the assumption of Theorem 2.3(ii) is not satisfied. In practice, this choice is not a very realistic one, mathematically it leads to \( z_\infty = y_0 \) by (R3) and, of course, to \( y_\infty = 0 \).

If \( C_3 > 0 \) then (7) possess two solutions, but only one positive. It follows that (7) has a unique solution \( z_\infty \in [0,N] \).

Example 2.5. Consider again constants \( \beta \), \( \gamma \) and a linear \( \vartheta \) in a way that \( \tau_v = \infty \) and \( \tau_Y \neq 0 \). Theorem 2.3(iii) yields
\[
[x_{\tau_Y} - \vartheta(z_{\tau_Y})] = \frac{\gamma}{\beta}. \tag{8}
\]

Computing
\[
X(z) = \left[ x^0 + \int_0^z \frac{\beta}{\gamma} \vartheta(u) \exp \left\{ \frac{\int_0^u \beta \, ds}{\gamma} \right\} \right] \exp \left\{ -\int_0^z \frac{\beta}{\gamma} \, du \right\} = \left[ x^0 + \frac{\beta}{\gamma} \int_0^z (\vartheta_0 + \vartheta_1 u) e^{\frac{\alpha u}{\tau}} \right] e^{-\frac{\alpha z}{\tau}} = \left( \vartheta_0 - \frac{\vartheta_1}{\rho} \right) + \vartheta_1 z + \left( x^0 + \frac{\vartheta_1}{\rho} - \vartheta_0 \right) e^{-\rho z} \tag{9}
\]
by (3) and substituting $X(z)$ into (8), we arrive at

$$ \left( \vartheta_0 - \frac{\vartheta_1}{\rho} \right) + \vartheta_1 z_{r_Y} + \left( x^0 + \frac{\vartheta_1}{\rho} - \vartheta_0 \right) e^{-\rho z_{r_Y}} - \vartheta_0 - \vartheta_1 z_{r_Y} = \frac{1}{\rho}. $$

This implies

$$ z_{r_Y} = \frac{1}{\rho} \left[ \log \left( x^0 + \frac{\vartheta_1}{\rho} - \vartheta_0 \right) - \log \left( \frac{1 + \vartheta_1}{\rho} \right) \right]. \quad (10) $$

Finally, having on mind that $N = x + y + z$, we get

$$ y_{\text{max}} = y_{r_Y} = N - X(z_{r_Y}) - z_{r_Y}, \quad (11) $$

where $z_{r_Y}$ and $X(z_{r_Y})$ are given by (10) and (9), respectively.

3. NUMERICAL RESULTS

This part deals with several problems that arise when one is trying to get some usable results concerning the time of culmination of the epidemics, the largest number of those infected, the influence of vaccination and the comparison of various vaccination strategies.

First, we consider a constant $\beta > 0$ and a linear vaccination, i.e. $\vartheta(z) = \vartheta_0 + \vartheta_1 z$. Having made this choice, we replace the differential equation (R1) by the equation

\[
\begin{align*}
x_{n+1} &= x_n - \beta(z_n)y_n \max\{[x_n - \vartheta(z_n)], 0\}\Delta, \quad x_0 = x^0 > 0, \\
y_{n+1} &= y_n + (\beta(z_n)y_n \max\{[x_n - \vartheta(z_n)], 0\} - \gamma y_n)\Delta, \quad y_0 = y^0 > 0, \quad (R4) \\
z_{n+1} &= z_n + \gamma y_n\Delta,
\end{align*}
\]

where $\Delta$ is a difference step.

We solved the equation (R4) with the number of steps 5000 and the difference step $\Delta = 0.016$, because we observed that a choice of smaller step does not change the results significantly. This corresponds to the time interval $(0, 80)$. We decided to use these values, because on this interval, the behavior of the epidemic can be well graphically shown (see Figure 1 and Figure 2). We chose the initial conditions $x^0 = 990, y^0 = 10$, what means that at the beginning, 1% of population suffers from the disease, and we observed the behavior of the epidemic with several choices of $\gamma, \beta, \vartheta_0$ and $\vartheta_1$. All computations and graphic results were made by software R.\footnote{Version R 2.3.1 was used.}

Figure 1 visualizes the differences in behavior of epidemic for different choices of $\beta$ with a permanent $\gamma$, when no vaccination is applied. Figure 2 shows the differences in behavior of epidemic for different vaccinations.

Although in the first case ($\vartheta_0 = 0$ and $\vartheta_1 = 1$), we have vaccinated 443 individuals by the time $t = 80$, while choosing $\vartheta_0 = 300$ and $\vartheta_1 = 0.2$ we have vaccinated only 363 individual in the same time interval, the evolution of epidemic is less favourable in the former case than in the latter one in the sense that the number of removals
Fig. 1. Behavior of epidemic with $\beta = 0.00025$, $\gamma = 0.25$ (left) and $\beta = 0.0005$, $\gamma = 0.25$ (right). The solid line describes the size of susceptibles, the dot-dashed line the size of infectives and the dashed line the size of removals.

Fig. 2. Behavior of epidemic with $\beta = 0.0005$, $\gamma = 0.25$ and the vaccination either $\vartheta_0 = 0$, $\vartheta_1 = 1$ (left) or $\vartheta_0 = 300$, $\vartheta_1 = 0.2$ (right). The solid line describes the size of susceptibles, the dot-dashed line the size of infectives and the dashed line the size of removals.

for the first choice is 443 in comparison with 317 for the second choice. Moreover, the maximal size of infected individuals (35) is also in favor of the latter vaccination compared with the former one (101).

Table 1 summarizes the values obtained by solving equation (R4) for several pairs of the coefficients $\vartheta_0$ and $\vartheta_1$. Here, we approximated $x_\infty$ and $z_\infty$ by $x_{12500}$ and $z_{12500}$, respectively. The approximation should be a satisfactory one as already the values $y_{12500}$ are observed to be close to zero. To get the results, we produced 12500 steps with difference step $\Delta = 0.016$ (i.e. we observed the time interval (0, 200)), choosing $\beta$ and $\gamma$ as before, i.e. $\beta = 0.0005$, $\gamma = 0.25$. The initial conditions were
again $x^0 = 990$ and $y^0 = 10$. The table lists the final values of $x, y$ and $z$, i.e. the numbers of the susceptibles, infectives and removals at time $t = 200$, the total number of vaccinated individuals by the time $t = 200$, the size of maxima of infected individuals and the time of maxima.

**Table 1.**

<table>
<thead>
<tr>
<th>$\vartheta_0$</th>
<th>$\vartheta_1$</th>
<th>$x_{12500}$</th>
<th>$z_{12500}$</th>
<th>$y_{12500}$</th>
<th>Vaccinated</th>
<th>max. $y$</th>
<th>$\tau_Y$</th>
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<td>0</td>
<td>0</td>
<td>199.5697</td>
<td>800.4303</td>
<td>5.3 e - 10</td>
<td>0</td>
<td>158.6046</td>
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<td>1.2 e - 05</td>
<td>363.7075</td>
<td>34.9557</td>
<td>22.896</td>
</tr>
<tr>
<td>300</td>
<td>0.5</td>
<td>736.9506</td>
<td>263.0494</td>
<td>2.8 e - 06</td>
<td>431.5247</td>
<td>30.6612</td>
<td>20.704</td>
</tr>
<tr>
<td>300</td>
<td>1</td>
<td>794.3346</td>
<td>205.6654</td>
<td>4.1 e - 07</td>
<td>505.6654</td>
<td>26.0612</td>
<td>17.952</td>
</tr>
<tr>
<td>300</td>
<td>2</td>
<td>854.0339</td>
<td>145.9661</td>
<td>2.0 e - 08</td>
<td>591.9322</td>
<td>21.1174</td>
<td>14.304</td>
</tr>
</tbody>
</table>

Numerical results have confirmed our expectations that having determined to provide a fixed number of vaccinations, an epidemic has a better evolution if choosing a more robust pre-vaccination (bigger $\vartheta_0$) because it decreases both the number and the global maximum of the infected individuals. Moreover, comparing 5th and 10th row in Table 1, we can see that for the same running of epidemic (in the mean of remained susceptibles), much less (almost one half) people need to be vaccinated in the case of pre-vaccination.

In Table 2, there are values of $z_\infty$, that we receive as a solution to equation (7) in Example 2.4. We choose again $\beta = 0.0005$, $\gamma = 0.25$, $x^0 = 990$ and $y^0 = 10$ and the vaccination which enters Table 1. We solved the equation by using the dividing interval method, we look for a solution in the interval $[0, 1000]$ and we require the error to be less than 0.001.

**Table 2.**

<table>
<thead>
<tr>
<th>$\vartheta_0$</th>
<th>$\vartheta_1 = 0$</th>
<th>$\vartheta_1 = 0.2$</th>
<th>$\vartheta_1 = 0.5$</th>
<th>$\vartheta_1 = 1$</th>
<th>$\vartheta_1 = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>800.2034</td>
<td>686.5820</td>
<td>567.7654</td>
<td>442.5726</td>
<td>309.4061</td>
</tr>
<tr>
<td>100</td>
<td>568.9032</td>
<td>469.4815</td>
<td>365.2218</td>
<td>255.0868</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>318.4655</td>
<td>262.9839</td>
<td>205.6079</td>
<td>145.9184</td>
<td></td>
</tr>
</tbody>
</table>
Comparing the values delivered by Table 1 with those delivered by Table 2, the differences are observed to be less than 0.3.

The values of maxima of infected individuals received by the formula (11) in Example 2.5 are presented by Table 3 choosing $\beta$, $\gamma$ and the initial conditions as above.

Table 3.

<table>
<thead>
<tr>
<th>$\vartheta_0$ = 0</th>
<th>$\vartheta_1 = 0$</th>
<th>$\vartheta_1 = 0.2$</th>
<th>$\vartheta_1 = 0.5$</th>
<th>$\vartheta_1 = 1$</th>
<th>$\vartheta_1 = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vartheta_0$ = 0</td>
<td>158.4516</td>
<td>141.7980</td>
<td>122.90494</td>
<td>101.2239</td>
<td>75.9957</td>
</tr>
<tr>
<td>$\vartheta_0$ = 100</td>
<td>99.5348</td>
<td>85.9673</td>
<td>70.6963</td>
<td>53.3324</td>
<td></td>
</tr>
<tr>
<td>$\vartheta_0$ = 300</td>
<td>34.9380</td>
<td>30.6450</td>
<td>26.0467</td>
<td>21.1049</td>
<td></td>
</tr>
</tbody>
</table>

Comparing Table 1 and Table 3, the differences are seen to be less than 0.2. Hence, we can conclude that (R4) provides approximations close enough to the theoretical values.

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