ON THE DECAY OF THE ENERGY
FOR SYSTEMS WITH MEMORY
AND INDEFINITE DISSIPATION

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ABSTRACT. In this paper we study the asymptotic behavior of the viscoelastic system with non dissipative kernels. We show that the uniform decay of the energy depends on the decay of the kernel, the positivity of the kernel in \( t = 0 \) and some smallness condition. That is, if the kernel \( g \in C^2(\mathbb{R}^+) \) with \( g(0) > 0 \), decays exponentially to zero then the solution decays exponentially to zero. On the other hand, if the kernel decays polynomially as \( t^{-p} \) then the corresponding solutions also decays polynomially to zero with the same rate of decay.

1. Introduction

Let us denote by \( H \) a Hilbert space with inner product \( \langle \cdot, \cdot \rangle_H \), norm \( \| \cdot \|_H \) and let \( A \) be a self adjoint positive definite operator over \( H \), such that

\[
\langle A^{1/2} \varphi, A^{1/2} \varphi \rangle_H \geq \lambda_1 \| \varphi \|_H^2, \quad \forall \varphi \in \mathcal{D}(A^{1/2}),
\]

for some \( 0 < \lambda_1 \in \mathbb{R} \). We introduce a class of second-order abstract models

\[
u_{tt}(t) + Au(t) - (g * Au)(t) = 0 \quad \text{in } L^2(\mathbb{R}^+: H),
\]

\[
u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in } H,
\]

where \( u_0 \in \mathcal{D}(A) \) and \( u_1 \in \mathcal{D}(A^{1/2}) \). Here the subscript \( \cdot_t \) denotes time derivative and \( * \) is the convolution product in the variable \( t \), namely

\[
(g * u)(t) = \int_0^t g(t - \tau) u(\tau) d\tau,
\]

where \( g \in L^1(\mathbb{R}^+) \). The system above rules a linear viscoelastic isotropic model when for instance \( Au = -\mu \Delta u - (\mu + \lambda) \nabla \text{div } u \) and \( H = L^2 \).

Our main interest concerns the asymptotic behavior of the solution of the system above. That is, what type of rate of decay may we expect? What are the sharp conditions to produce uniform stabilization?

The dynamics of a viscoelastic body has been studied extensively by several authors. For more details we address the reader to the papers [6, 8, 12, 17] and to the textbooks [15, 24, 33] and references therein. It is well know that, in the case \( g \equiv 0 \), the energy of the system is constant. When \( g \neq 0 \), we mention some known results related to the asymptotic behavior of the solution of viscoelastic systems. We start our long list with

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the pioneer work of C.M. Dafermos [7] who showed that the solution to the viscoelastic system tends to zero as time goes to infinity, but without giving explicit rate of decay. Other works in this sense are [11, 13, 14, 26]. J.M. Greenberg [20] and W.J. Hrusa [21] obtain an exponential rate of decay for the nonlinear viscoelastic equation when the relaxation function \( g \) is of the form \( g(t) = e^{-\mu t} \). A similar result is found by G. Dassios and F. Zafiropoulos [9, 10] for homogeneous and isotropic viscoelastic materials which occupy the whole three-dimensional space. They prove that the longitudinal and transverse waves decay to zero uniformly like \( t^{-m-3/2} \), where \( m \) increases depending on the symmetry of the initial data, provided the relaxation is an exponential function like \( f(t) = \mu_0 e^{-\gamma t} \).

Subsequently, in [4, 5, 22] it is proved that the rate of decay of the solution depends on the rate of decay of the relaxation function, that is if the relaxation function decays exponentially then the solution decays exponentially, while if the relaxation function decays polynomially then the solution decays also polynomially with the same rate.

For localized damping in viscoelasticity, we refer to [25, 28], where the authors proved the exponential stability, provided the kernel \( g \) satisfies \( g(t) > 0, \ g'(t) < 0, \ t \in \mathbb{R}^+ \), and the relaxation kernel also decays exponentially to zero.

In [16], M. Fabrizio and S. Polidoro consider the viscoelastic equation with an additional frictional damping. The authors showed that the solution decays exponentially only if the relaxation kernel decays exponentially. Analogous results are proved for polynomial decay.

On the other hand, in [27] is considered the initial and boundary value system related to the following equation

\[
u_{tt}(t) + Au(t) - (g * A^\alpha u)(t) = 0,
\]

where \( A \) is a strictly positive, self-adjoint operator with domain \( D(A) \subset H \) a Hilbert space. The authors showed that the dissipation given by the memory effect is not strong enough to produce exponential stability, when \( 0 \leq \alpha < 1 \). That is, the corresponding semigroup, associated to this problem does not decay exponentially, when \( 0 \leq \alpha < 1 \), but such dissipation is capable to produce polynomial decay in appropriate norms.

Finally, we recall some others related results. The Timoshenko system with memory was studied in [1, 18, 19]. Viscoelastic plates with memory was considered in [3, 23, 29, 30]. In [34], N.-E. Tatar obtained the exponential decay of the solution to the viscoelastic equation with inertial term

\[
u_{tt}(t) - \Delta u_{tt}(t) - \Delta u(t) - (g * \Delta u)(t) = 0.
\]
One important point to remark is that in all the above works, it is considered the kernel function $g$ with dissipative properties. That is, $g(t) > 0$ and $g'(t) < 0$, for any $t \geq 0$ and the exponential decay of $g$ as a consequence of the property $g'(t) \leq -\gamma g(t)$, $\gamma > 0$. In a recent work [2] the exponential decay of solutions for a linear viscoleastic equation is found provided the kernel $g(t) = Ae^{-\delta t} (\delta \cos \omega t + \omega \sin \omega t)$, with $A, \delta, \omega \in \mathbb{R}^+$. 

The main result of our paper is to remove all these hypotheses and instead to consider

$$g \in W^{2,1}(\mathbb{R}^+) \cap C^2(\mathbb{R}^+),$$

$$g_0 = g(0) > 0,$$

$$|g_0' + g_0''| < \lambda_1,$$

$$|g(t)| \leq C_g e^{-\gamma t},$$

$$|g'(t)| \leq C_g' e^{-\gamma t},$$

$$|g''(t)| \leq C_g'' e^{-\gamma t}, \quad \forall t \geq 0,$$  \hspace{1cm} (1.6)

where $g_0' = g'(0)$ and $\lambda_1$ is given in (1.1), $\gamma$ and $C_g$ are positive constants. In particular this implies that the system is not of dissipative type. That is to say, we will show the exponential decay of the solution of system (1.2) when $g$, $g'$ and $g''$ may change of sign.

As an example of a kernel satisfying the above hypotheses we have, $g(t) = ce^{-\gamma t} \cos(\beta t)$, $\gamma \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, for a convenient choice of $c$.

To show the polynomial decay of the solution we substitute (1.6) by the following one: we assume that there exist positive constants $C_i$, $i = 1, 2, 3$, such that for any $t \geq 0$ and $p > 0$

$$|g(t)| \leq C_1 (1 + t)^{-p},$$

$$|g'(t)| \leq C_2 (1 + t)^{-p},$$

$$|g''(t)| \leq C_3 (1 + t)^{-p}. \quad (1.7)$$

We briefly sketch the plan of the paper. Section 2 contains functional setting and notation used in the paper. In Section 3 we find, by using the spectral method, a rate of decay for the abstract wave equation with frictional damping, in terms of the coefficient. In Sections 4 and 5, we show that the uniform decay of the energy depends on the decay of the kernel and the positivity of the kernel in $t = 0$. The method we use combines the result of Section 3, some energy estimates and a fixed point theorem. Finally, in Section 6 we suggest some applications to our result.

## 2. Notations and mathematic tools

For $r \in \mathbb{R}$, we introduce the scale of Hilbert spaces $H_r = D(A^{r/2})$, $H_0 = H$, endowed with the usual inner products

$$\langle v_1, v_2 \rangle_{H_r} = \langle A^{r/2}v_1, A^{r/2}v_2 \rangle.$$

In particular, for (1.1) there holds

$$\lambda_1 \|\phi\|^2 \leq \|\phi\|^2_{H_1} \quad \forall \phi \in H_1. \quad (2.1)$$
The inner product and the norm on $H_0$ are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, without subscript.

To give a precise formulation of the evolution problem we introduce therefore the Hilbert space

\[ Z = H_1 \times H_0, \]

endowed with the following inner product

\[ \langle \varphi, \psi \rangle_Z = \langle \varphi_1, \psi_1 \rangle_{H_1} + r_0' \langle \varphi_1, \psi_1 \rangle + \langle \varphi_2, \psi_2 \rangle, \]

(2.2)

where $\varphi = [\varphi_1, \varphi_2]^T$, $\psi = [\psi_1, \psi_2]^T$, and $r_0' = g_0^2 + g_0'$. Let us denote by

\[ K_{C,\varepsilon} = \left\{ \phi \in L^\infty(\mathbb{R}^+; H_0) : \sup_{t \in \mathbb{R}} e^{\varepsilon t} \| \phi(t) \| \leq C \right\}, \]

\[ J_{C,q} = \left\{ \phi \in L^\infty(\mathbb{R}^+; H_0) : \| \phi(t) \| \leq C (1 + t)^{-q}, \quad t \geq 0 \right\}, \]

where $C, \varepsilon, q, k$ are positive constants to be fixed later.

### 3. The resolvent kernel and the homogeneous problem

Let $r$ be the resolvent kernel of $g$, that is, the solution of the following Volterra equation

\[ r(t) - (g \ast r)(t) = g(t). \]

(3.1)

Then we have

\[ r'(t) = g'(t) + r_0 g(t) + (r' \ast g)(t). \]

By assumptions (1.4)–(1.5)

\[ r_0 = r(0) = g_0 > 0, \quad r_0' = r'(0) = g_0^2 + g_0' \quad \text{and} \quad |r_0'| < \lambda_1. \]

(3.2)

Under the above conditions we have

**Lemma 3.1.** Let $h$ and $g$ be functions satisfying the following condition

\[ |h(t)| \leq C_h e^{-\gamma t} \quad \text{and} \quad |g(t)| \leq C_g e^{-\gamma t}, \quad \forall t > 0 \]

and $\gamma, C_h, C_g > 0$ with $C_g < \gamma$, then the solution $r$ of the Volterra equation

\[ r(t) = h(t) + (g \ast r)(t), \]

satisfies

\[ |r(t)| \leq \frac{C_h(\gamma - \gamma_r)}{\gamma - \gamma_r - C_g} e^{-\gamma_r t}, \quad \forall t > 0 \]

for $\gamma_r > 0$ such that $C_g < \gamma - \gamma_r$.

**Proof.** We use a fixed point theorem. Let us define $P : L^\infty(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ be the function such that

\[ P(f) = h + g \ast f. \]
We will show that $P(K_{C;\gamma_r}) \subseteq K_{C;\gamma_r}$ for some large $C$. In fact, using the hypotheses

$$|P(f)(t)| \leq C_h e^{-\gamma t} + C g \int_0^t e^{-\gamma(t-s)} e^{-\gamma_r s} ds$$

$$\leq C_h e^{-\gamma t} + \frac{C C_g}{\gamma - \gamma_r} e^{-\gamma_r t}$$

$$\leq \left(C_h + \frac{C C_g}{\gamma - \gamma_r}\right) e^{-\gamma_r t}.$$ 

Since $C_g < \gamma - \gamma_r$ we can take

$$C \geq \frac{C_h(\gamma - \gamma_r)}{\gamma - \gamma_r - C_g},$$

to find that $P(f)(t) \in K_{C;\gamma_r}$. Finally, we prove that $P$ is a contraction over $L^\infty(\mathbb{R}^+)$. Let us denote by $\phi = f_1 - f_2$, where $f_i \in K_{C;\gamma_r}$, $i = 1, 2$, are such that $P(f_i) = h + g * f_i$. Thus we have

$$|P(\phi)(t)| \leq C_g \int_0^t e^{-\gamma(t-s)} |\phi(s)| ds \leq \frac{C_g}{\gamma} \|\phi\|_{L^\infty(\mathbb{R}^+)},$$

and consequently

$$\|P(\phi)\|_{L^\infty(\mathbb{R}^+)} \leq \frac{C_g}{\gamma} \|\phi\|_{L^\infty(\mathbb{R}^+)}. $$

Then we find that $P$ is a contraction. Since $K_{C;\gamma_r}$ is a closed subset of $L^\infty(\mathbb{R}^+)$, using the fixed point theorem to contractions our conclusion follows.

Let us denote by $C_p$ the term

$$C_p = \sup_{0 \leq t < \infty} \int_0^t (1 + t)^p (1 + t - \tau)^{-p} (1 + \tau)^{-p} d\tau.$$ 

For the polynomial case we recall the following Lemma [31, Lemma 3.1].

**Lemma 3.2.** Let us suppose that $g$ satisfies

$$|g(t)| \leq C_g (1 + t)^{-p} \text{ for any } t \geq 0,$$

for some $p > 1$ and $C_g > 0$, assuming that

$$\frac{1}{C_g} \geq C_p$$

holds, then the resolvent kernel $r$ of $g$ satisfies

$$|r(t)| \leq \frac{C_g}{1 - C_g C_p} (1 + t)^{-p} \text{ for any } t \geq 0.$$ 

**Remark 3.3.** By previous Lemmas and assuming hypothesis (1.6) or (1.7), we conclude that $r''$ decays exponentially or polynomially, respectively. In particular, given $g$ defined as in the previous Lemma, we conclude that that

$$r''(t) = h(t) + (r'' * g)(t).$$
where $h(t) = g''(t) + r_0 g'(t) + r'_0 g(t)$. Hence, we get
\[ |r''(t)| \leq \frac{C_h(\gamma - \gamma_r)}{\gamma - \gamma_r - C_g} e^{-\gamma t}, \quad \forall t > 0. \]
where $\gamma_r > 0$ is such that $C_g < \gamma - \gamma_r$.

Let us introduce, $v$ given by
\[ v(t) = u(t) - (g * u)(t), \quad (3.3) \]
Using the resolvent kernel identity we can rewrite (3.3) as
\[ u(t) = v(t) + (r * v)(t). \quad (3.4) \]
Differentiation with respect to $t$ yields
\[ u_{tt}(t) = v_{tt}(t) + r_0 v_t(t) + r'_0 v(t) + (r'' * v)(t). \quad (3.5) \]
Since
\[ u_{tt}(t) + Av(t) = 0, \]
and considering (3.5), the previous equation becomes
\[ v_{tt}(t) + Av(t) + r_0 v_t(t) + r'_0 v(t) + (r'' * v)(t) = 0 \quad \text{in} \quad L^2(\mathbb{R}^+; H_0), \quad (3.6) \]
\[
\begin{align*}
v(0) &= u_0, \quad v_t(0) = u_1 - r_0 u_0, & \text{in} \quad H_0.
\end{align*}
\]
The above system is equivalent to (1.2). Therefore to show the exponential decay of equation (1.2) it is enough to show exponential decay to (3.6). Our starting point is to study the homogeneous equation
\[ v_{tt}(t) + Av(t) + r_0 v_t(t) + r'_0 v(t) = 0 \quad \text{in} \quad L^2(\mathbb{R}^+; H_0), \quad (3.7) \]
\[
\begin{align*}
v(0) &= u_0, \quad v_t(0) = u_1 - r_0 u_0, & \text{in} \quad H_0.
\end{align*}
\]
Using the semigroup theory and a result by Prüss [32] we will show the exponential decay. Then using a fixed point argument we prove that this decays also holds for system (3.6) which in particular implies the exponential decay to system (1.2).

To introduce the semigroup framework let us denote by $w = v_t$ and
\[ z(t) = [v(t), w(t)]^T, \quad z_0 = [u_0, u_1 - r_0 u_0]^T \in \mathcal{Z}. \]
Problem (3.7) can be rewritten as the abstract linear evolution equation in the Hilbert space $\mathcal{Z}$
\[ \begin{cases} \frac{d}{dt} z(t) = L z(t), & t > 0, \\ z(0) = z_0, \end{cases} \quad (3.8) \]
where the linear operator $L$ is defined as
\[ L \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} w \\ -Av - r'_0 v - r_0 w \end{bmatrix}, \]
with domain
\[ \mathcal{D}(L) = \{ z \in \mathcal{Z} : v \in H_2, w \in H_1 \} \],
and the embedding $D(L) \hookrightarrow Z$ is compact. It is very well known that $L$ is the infinitesimal generator of a $C_0$-semigroup $T(t) = e^{tL}$ on $Z$. So we have,

**Lemma 3.4.** Assume conditions (1.4)--(1.5). The semigroup $T(t)$ on $Z$ is exponentially stable.

**Proof.** Relying on a result by Prüss [32], we will show that the necessary and sufficient condition in order to have exponential stability, namely

$$
\{ \beta \in \mathbb{C} : \mathrm{Re} \beta \geq \mu \}
$$

is involved into the resolvent set $\rho(L)$ and there is $M \geq 1$ such that

$$
\| (\beta I - L)^{-1} \| \leq M, \quad \text{for all } \mathrm{Re} \beta \geq \mu, \quad (3.9)
$$

is verified for a negative value of $\mu$. The infimum value of $\mu$ verifying (3.9) will be the optimal rate of decay of the semigroup. For $\lambda \in \mathbb{C}$ and $h = [h_1, h_2]^T \in Z$, we introduce the (complex) equation for the unknown variable $z = [v, w]^T$

$$(\lambda I - L)z = h,$$

which, written in components, reads

$$
\begin{cases}
\lambda v - w = h_1, \\
\lambda w + Av + r_0'v + r_0w = h_2.
\end{cases} \quad (3.10)
$$

Multiplying the first equation and the second equation of (3.10) by $Av + r'_0v$ and $w$, respectively, summing the real part of the resulting equations we find

$$
\left( \| A^{1/2} v \|^2 + r'_0 \| v \|^2 + \| w \|^2 \right) \mathrm{Re} \lambda + r_0 \| w \|^2 = \mathrm{Re} \langle h_2, w \rangle + \mathrm{Re} \langle h_1, Av + r'_0v \rangle, \quad (3.11)
$$

and then we give

$$
\| z \|^2 \leq \| h \| \| z \|. \quad (3.12)
$$

Multiplying the second equation of (3.10) by $v$, substituting $v = w/\lambda + h_1/\lambda$ into the right-hand side, and considering the above inequality, there exists a positive constant $C$ such that

$$
\| A^{1/2} v \|^2 + r'_0 \| v \|^2 = - \mathrm{Re} \lambda \langle w, v \rangle - r_0 \langle w, v \rangle + \mathrm{Re} \langle h_2, v \rangle
$$

$$
\leq - \frac{\lambda}{\lambda} \| w \|^2 - r_0 \| v \|^2 + C \| h \| \| z \|. \quad (3.13)
$$

Then we have

$$
\| A^{1/2} v \|^2 + r'_0 \| v \|^2 + r_0 \| v \|^2 \leq \| w \|^2 + C \| h \| \| z \|. \quad (3.13)
$$
Multiplying (3.13) by $r_0/2$ and summing to (3.12), we find
\[
\|z\|^2_2 \Re \lambda + \frac{r_0}{2} \left[ \frac{\|A^{1/2}v\|^2 + r'_0\|v\|^2 + \|w\|^2}{\|z\|^2_2} \right] + \frac{r_0^2}{2} \Re \lambda \|v\|^2 \leq C \|h\|_2 \|z\|_2.
\]

Let $\varrho$ a positive constant to be fixed later, from the above inequality we get
\[
\|z\|_2 \left( \Re \lambda + \varrho \right) + \frac{r_0^2}{2} \left( \Re \lambda + \varrho \right) \|v\|^2 + \mathcal{R} \leq C \|h\|_2 \|z\|_2,
\]
where
\[
\mathcal{R} = (\frac{r_0^2}{2} - \varrho) \|z\|_2 - \frac{r_0^2}{2} \varrho \|v\|^2.
\]

We will choose $\varrho$ such that $\mathcal{R} \geq 0$. Recalling the definition of $\| \cdot \|_Z$, and using inequality (1.1) we have
\[
\mathcal{R} \geq \left( \frac{r_0}{2} - \varrho \right) (\lambda_1 + r'_0) - \frac{r_0}{2} \varrho \|v\|^2.
\]

Then our best of $\varrho$ such that $\mathcal{R} \geq 0$ is
\[
\varrho = \frac{r_0}{2} \left( \frac{\lambda_1 + r'_0}{\lambda_1 + r'_0 + \frac{r_0^2}{2}} \right),
\]
and this implies that
\[
\|(\lambda I - L)^{-1}\| \leq C, \quad \text{for all } \Re \lambda > -\varrho.
\]

Note that the rate of exponential decay to the solution is given by $\varrho$. \hfill \Box

4. EXPONENTIAL DECAY

We consider now the following system
\[
\begin{align*}
v_{tt}(t) + Av(t) + r_0 v_t(t) + r'_0 v(t) &= -f(t) - (r''*f)(t) & \text{in } L^2(\mathbb{R}^+; H_0), \\
v(0) &= u_0, \quad v_t(0) = u_1 - r_0 u_0 & \text{in } H_0.
\end{align*}
\]
where $f$ is a function belonging to a suitable space which will be chosen later. Setting $w(t) = v_t(t)$, $z(t) = [v(t), w(t)]^T$, $f_r(t) = -(r''*f)(t)$ and $B(t) = [0, f_r(t)]^T$, system (4.1) can be rewritten as
\[
\begin{align*}
z_t(t) &= L z(t) + B(t), \\
z(0) &= z_0.
\end{align*}
\]
The operator $L$ is defined as in the previous Section. The mild solution of system (4.1) can be written as
\[
z(t) = T(t) z(0) + \int_0^t T(t-s) B(s) \, ds.
\]
We introduce now a function $T$ defined as
\[
\forall f \in \mathcal{K}_{C,e}, \quad T(f) = v,
\]
where \( v \) is the solution of system (4.1). We want to prove that \( T(K_{C,\varepsilon}) \subseteq K_{C,\varepsilon} \) and \( T \) is a contraction. Moreover, \( T(C(0, T; H_2)) \subseteq C(0, T; H_2) \).

Recalling the constants \( \gamma, C_h, C_g, \gamma_r \) as used in Lemma 3.1, the following Lemma holds.

**Lemma 4.1.** Assume hypotheses (1.3)–(1.6), and function \( f \in K_{C,\varepsilon} \), with \( 0 < \varepsilon < \min\{\varrho, \gamma_r\} \) and

\[
\frac{C_h(\gamma - \gamma_r)}{(\gamma - \gamma_r - C_g)(\gamma_r - \varepsilon)\sqrt{\lambda_1 + r_0}} < 1.
\]

Then \( T(K_{C,\varepsilon}) \subseteq K_{C,\varepsilon} \) is verified for \( C \) large enough.

**Proof.** From (4.2) we have

\[
\|z(t)\|_Z \leq \|T(t)z(0)\|_Z + \int_0^t \|T(t-s)B(s)\|_Z \, ds
\]

\[
\leq \|z(0)\|_Z e^{-\varrho t} + \int_0^t e^{-\varrho(t-s)}\|f_r(s)\|_Z \, ds.
\]

Recalling \( f_r(s) = -(r'' \ast f)(s) \), we assume condition (1.6). By Lemma 3.1, for any \( f \in K_{C,\varepsilon} \) we have

\[
|\langle r'' \ast f \rangle(s)| \leq \frac{CC_h(\gamma - \gamma_r)}{\gamma - \gamma_r - C_g}(\gamma_r - \varepsilon) \int_0^s e^{-\gamma_r(s-r)} e^{-\varepsilon \tau} \, d\tau
\]

\[
\leq \frac{CC_h(\gamma - \gamma_r)}{(\gamma - \gamma_r - C_g)(\gamma_r - \varepsilon)} e^{-\varepsilon s},
\]

provided that \( 0 < \varepsilon < \gamma_r \) and \( C_g < \gamma - \gamma_r \). Then we have

\[
\|z(t)\|_Z \leq \|z(0)\|_Z e^{-\varrho t} + \frac{CC_h(\gamma - \gamma_r)}{(\gamma - \gamma_r - C_g)(\gamma_r - \varepsilon)(\varrho - \varepsilon)} e^{-\varepsilon t}
\]

\[
\leq \left[ \|z(0)\|_Z + \frac{CC_h(\gamma - \gamma_r)}{(\gamma - \gamma_r - C_g)(\gamma_r - \varepsilon)(\varrho - \varepsilon)} \right] e^{-\varepsilon t},
\]

provided that \( 0 < \varepsilon < \varrho \). Since

\[
\|z(t)\|_Z^2 = \|v(t)\|_H^2 + r'_0 \|v(t)\|_H^2,
\]

we find that

\[
\|v(t)\| \leq \frac{1}{\sqrt{\lambda_1 + r_0}} \left[ \|z(0)\|_Z + \frac{CC_h(\gamma - \gamma_r)}{(\gamma - \gamma_r - C_g)(\gamma_r - \varepsilon)(\varrho - \varepsilon)} \right]^{1/2} e^{-\varepsilon t}.
\]

By the hypotheses, there exists \( 0 < \varepsilon < \min\{\varrho, \gamma_r\} \) such that

\[
\frac{C_h(\gamma - \gamma_r)}{(\gamma - \gamma_r - C_g)(\gamma_r - \varepsilon)(\varrho - \varepsilon)\sqrt{\lambda_1 + r_0}} < 1,
\]

this implies that \( T(f) = v \in K_{C,\varepsilon} \) for some \( C \) large enough.

**Lemma 4.2.** Assume the same hypotheses as in Lemma 4.1, then \( T \) is a contraction.
Proof. Consider the following systems
\begin{align*}
v_1''(t) + Av_1(t) + g_0 v_1'(t) + r_0 v_1(t) &= -(r'' * f^1)(t) \quad \text{in } L^2(\mathbb{R}^+; H_0), \\
v_1(0) &= u_0, \quad v_1'(0) = u_1 - r_0 u_0 \quad \text{in } H_0,
\end{align*}
and
\begin{align*}
v_2''(t) + Av_2(t) + g_0 v_2'(t) + r_0 v_2(t) &= -(r'' * f^2)(t) \quad \text{in } L^2(\mathbb{R}^+; H_0), \\
v_2(0) &= u_0, \quad v_2'(0) = u_1 - r_0 u_0 \quad \text{in } H_0.
\end{align*}
Setting \( v = v_1 - v_2 \), \( \phi = f^1 - f^2 \), we find
\begin{align*}
v(t) + Av(t) + g_0 v'(t) + r_0 v(t) &= -(r'' * \phi)(t) \quad \text{in } L^2(\mathbb{R}^+; H_0), \\
v(0) = 0, \quad v'(0) = 0 \quad \text{in } H_0.
\end{align*}
Denoting by \( \omega = [v, \nu]^\top \) and \( \phi_* = [0, -r'' * \phi]^\top \), we can write \( \omega \) in terms of the semigroup in the following way
\[ \omega(t) = \int_0^t T(t-s)\phi_*(s) \, ds. \]
Since \( T \) is exponentially stable we have that
\[ \|\omega(t)\|_Z \leq \int_0^t e^{-\theta(t-s)}\|\phi_*(s)\|_Z \, ds \]
\[ \leq \int_0^t e^{-\theta(t-s)}\| (r'' * \phi)(s) \| \, ds. \]
Introducing the space \( X(H_0) \), we can estimate
\[ \| (r'' * \phi)(s) \| \leq \int_0^s |r''(s-\tau)| \| \phi(\tau) \| \, d\tau \]
\[ \leq \| \phi \|_{L^\infty(0,T; H_0)} \int_0^s |r''(\tau)| \, d\tau. \]
Hence we find
\[ \|\omega(t)\|_Z \leq \int_0^t e^{-\theta(t-s)}\| (r'' * \phi)(s) \| \, ds \]
\[ \leq \int_0^t e^{-\theta(t-s)} \, ds \| \phi \|_{L^\infty(0,T; H_0)} \int_0^t |r''(\tau)| \, d\tau \]
\[ \leq \frac{1}{\theta} \| \phi \|_{L^\infty(0,T; H_0)} \int_0^t |r''(\tau)| \, d\tau. \]
On the other hand, there holds
\[ \|\omega(t)\|_Z \geq \sqrt{\lambda_1 + r_0^t \| \nu(t) \|.} \]
From the above two inequalities we get that
\[ \sqrt{\lambda_1 + r_0^t \| \nu(t) \|} \leq \frac{1}{\theta} \| \phi \|_{L^\infty(0,T; H_0)} \int_0^\infty |r''(\tau)| \, d\tau, \]
and then from Remark 3.3 we find
\[ \|v\|_{L^\infty(0,T;H_0)} \leq \frac{1}{\theta \sqrt{\lambda_1 + r_0}} \int_0^\infty |v''(\tau)| \, d\tau \|\|L^\infty(0,T;H_0) \]
\[ \leq \frac{C_h(\gamma - \gamma_r)}{\theta \sqrt{\lambda_1 + r_0 (\gamma - \gamma_r - C_g) \gamma_r}} \|\|L^\infty(0,T;H_0), \]
which implies that \( T \) is a contraction. \( \square \)

**Theorem 4.3.** Assume the same hypotheses as in Lemma 4.1. Then there exists only one solution of system (1.2) which decays exponentially to zero.

**Proof.** Since the operator \( T \) is a contraction over \( L^1(R_+;H_0) \) and it is invariant over \( K_{C,\varepsilon} \), there exists a fixed point, say \( v \), in \( K_{C,\varepsilon} \).

Therefore equation (4.1) can be written as
\[ v_{tt}(t) + A v(t) + g_0 v(t) + r_0 v(t) = -r'' \ast v(t) \quad \text{in} \quad L^2(R_+;H_0), \]
\[ v(0) = u_0, \quad v_t(0) = u_1 - g_0 u_0 \quad \text{in} \quad H_0. \tag{4.3} \]

Note that \( v \in K_{C,\varepsilon} \) implies that \( v \) decays exponentially. Since the above problem is equivalent to system (1.2), our conclusion follows. \( \square \)

**Remark 4.4.** For instance, our result can be applied to study the asymptotic behavior of the solution related to the problem (1.2) when \( A = -(\cdot)_{xx} \) with Dirichlet boundary conditions, the Hilbert space \( H = L^2(0,1) \), it is easy to see that \( \lambda_1 = \pi^2 \). Let us consider the relaxation function \( g \) as follows
\[ g(t) = c \cos(\beta t) \, e^{-\gamma t}, \]
with \( c, \gamma \in \mathbb{R}_+, \beta \in \mathbb{R} \). In this case,
\[ C_g = c, \quad g_0 = c, \quad \theta = \frac{c(\pi^2 - c^2)}{2\pi^2 - c^2}, \quad \beta = 3c. \]

Since
\[ g'(t) = -c [\beta \sin(\beta t) + \gamma \cos(\beta t)] e^{-\gamma t}, \quad g_0' = -c \gamma, \quad r_0' = (c - \gamma)c, \]
\[ g''(t) = c [-(\gamma^2 - \beta^2) \cos(\beta t) + 2\beta \gamma \sin(\beta t)] \, e^{-\gamma t}, \]
which can be written as
\[ g'(t) = -c \sqrt{\gamma^2 + \beta^2} \sin(\beta t + \theta_1) e^{-\gamma t}, \quad \theta_1 = \arccos \frac{\beta}{\sqrt{\beta^2 + \gamma^2}} \]
\[ g''(t) = -c (\gamma^2 + \beta^2) \sin(\beta t + \theta_2) e^{-\gamma t}, \quad \theta_1 = \arccos \frac{2\beta \gamma}{\beta^2 + \gamma^2} \]
and recalling Remark 3.3, function \( h \) can be written as
\[ h(t) = -c (\gamma^2 + \beta^2) \sin(\beta t + \theta_2) e^{-\gamma t} - c^2 \sqrt{\gamma^2 + \beta^2} \sin(\beta t + \theta_1) e^{-\gamma t} + (c - \gamma)c^2 \cos(\beta t) e^{-\gamma t} \]
\[ + (c - \gamma) c^2 \cos(\beta t) e^{-\gamma t} \]

Then, \( C_h = c(\gamma^2 + \beta^2) + c^2 \sqrt{\gamma^2 + \beta^2} + |c - \gamma| c^2 \). Therefore, setting \( \gamma = 2c \), we have
\[ C_h \leq (13 + \sqrt{13} + 1)c^3. \]
Our task is to find $c$ and $\gamma$ such that
\[
\frac{C_h(\gamma - \gamma_r)}{(\gamma - \gamma_r - C_g)\gamma r_0 \sqrt{\lambda_1 + r_0^2}} < 1.
\]
Let us take $\gamma_r = c/2$ and $\lambda_1 = \pi^2$. We look for $c$ such that
\[
G(c) := \frac{6(13 + \sqrt{13} + 1)(2\pi^2 - c^2)c}{\sqrt{(\pi^2 - c^2)^3}} < 1.
\]
Choosing $c = 1/4$, we have that $G(c) < 1$. Then the relaxation function can be chosen as
\[
g(t) = \frac{1}{4} \cos \frac{3t}{2} e^{-\frac{t}{2}}.
\]
Note that
\[
\int_0^\infty g(t) \sin(wt) \, dt = -0.02139751278, \quad \text{for } w > 0.
\]

5. POLYNOMIAL DECAY

We introduce a function $T$ defined as
\[
\forall f \in \mathcal{J}_{C,q}, \quad T(f) = v,
\]
where $v$ is the solution of system (4.1). We want to prove that $T(\mathcal{J}_{C,q}) \subseteq \mathcal{J}_{C,q}$ and $T$ is a contraction.

**Lemma 5.1.** Assume hypotheses (1.3)–(1.5), (1.7), and function $f \in \mathcal{J}_{C,p}$, then $T(\mathcal{J}_{C,p}) \subseteq \mathcal{J}_{C,p}$ provided
\[
\frac{C_g C_p^2}{1 - C_g C_p} < 1,
\]
where $\gamma$ is such that $x^p e^{p\gamma x} < \gamma$ for any $x \geq 0$.

**Proof.** From (4.2) we have
\[
\|z(t)\| \leq \|T(t) z(0)\| + \int_0^t \|T(t-s) B(s)\| \, ds \\
\leq \|z(0)\| e^{p_0 t} + \int_0^t e^{p_0 (t-s)} \|f_r(s)\| \, ds.
\]
Recalling that $f_r(s) = -(r'' * f)(s)$, and assuming condition (1.7) and $f \in \mathcal{J}_{C,q}$, by Lemma 3.2, we have
\[
|(r'' * f)(s)| \leq \int_0^s \frac{C_g}{1 - C_g C_p} (1 + s - \tau)^{-p} C (1 + \tau)^{-p} d\tau \\
\leq \frac{C_g C}{1 - C_g C_p} (1 + t)^{-p} \int_0^s (1 + s - \tau)^{-p} (1 + t)^p (1 + \tau)^{-p} d\tau \\
\leq \frac{C_g C_p C}{1 - C_g C_p} (1 + t)^{-p}.
\]
Therefore we find
\[ \|z(t)\| \leq \|z(0)\| e^{\rho_0 t} + \frac{C_g C_p C}{1 - C_g C_p} \int_0^t e^{\rho_0 (t-s)} (1+s)^{-p} \, ds \]
\[ \leq \|z(0)\| e^{\rho_0 t} + \frac{C_g C_p C}{1 - C_g C_p} \gamma \int_0^t (1+t-s)^{-p}(1+t)^{-p} \, ds \]
\[ \leq \|z(0)\| e^{\rho_0 t} + \frac{C_g C_p^2 C}{1 - C_g C_p} \gamma (1+t)^{-p}. \]

Using similar arguments as in Lemma 4.1, and taking \( C \) large enough, our conclusion follows.

**Theorem 5.2.** Assume that conditions (1.3)–(1.5), (1.7) hold. Then there exists only one solution of system (1.2) which decays to zero polynomially.

**Proof.** The proof follows the same procedure as in Theorem 4.3. Since \( T \) is a contraction and invariant over \( J_{C, p} \), then the solution of system (4.1) belong to \( J_{C, p} \), which implies that in this case the solution of (1.2) decays polynomially as \( 1/t^p \).

\[ \square \]

### 6. Applications

Here we introduce some examples of our result.

#### 6.1. Elasticity

Let us denote by \( \Omega \) a bounded domain of \( \mathbb{R}^n \). Then we have that \( \lambda_1 \) given in (1.1) is the first eigenvalue of \( A \). The model is
\[ \begin{align*}
\varrho u_{tt}(t) + Au(t) - g \ast Au(t) & = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\
u(t) & = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+, \\
u(0) = u_0, \quad u_t(0) = u_1 \quad & \text{in } \Omega,
\end{align*} \]
where \( \varrho > 0, \, Au = -\mu \Delta u - (\mu + \lambda) \nabla \text{div } u, \) and \( \mu, \lambda \) are the Lamé coefficients satisfying \( \mu > 0 \) and \( \lambda + \mu > 0 \).

#### 6.2. Plates

Let \( \Omega \) and \( \lambda_1 \) be introduced as in the elastic case above. The equations are
\[ \begin{align*}
\varrho u_{tt}(t) - \delta \Delta u_{tt} + \gamma \Delta^2 u(t) - g \ast \Delta^2 u(t) & = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\
u(t) & = 0, \quad \Delta u(t) = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+, \\
u(0) = u_0, \quad u_t(0) = u_1 \quad & \text{in } \Omega,
\end{align*} \]
where \( \varrho > 0 \). This system holds by taking \( Au = (\varrho I - \delta \Delta)^{-1} \Delta^2 u \) in (1.2). Note that \( \delta \geq 0 \).
6.3. **Elasticity with inertial term.** Let $\Omega$ and $\lambda_1$ be introduced as in the elastic case above. The model is
\[
\rho u_{tt}(t) - \delta \Delta u_{tt} - \alpha_0 \Delta u(t) + g \ast \Delta u(t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,
\]
\[
u(t) = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+,
\]
\[
u(0) = u_0, \quad \nu_t(0) = u_1 \quad \text{in } \Omega,
\]
where $\rho > 0$, $\delta \geq 0$ and $\alpha_0 > 0$. This system holds by taking $Au = (\rho I - \delta \Delta)^{-1}\Delta u$ in (1.2).

6.4. **Unbounded domains.** Let us denote by $\Omega$ an unbounded domain of $\mathbb{R}^n$. The model is
\[
\rho u_{tt}(t) + Au(t) - g \ast Au(t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,
\]
\[
u(t) = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+,
\]
\[
u(0) = u_0, \quad \nu_t(0) = u_1 \quad \text{in } \Omega,
\]
where $Au = -\mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \delta u$. For $\Omega = \mathbb{R}^n$ we remove the boundary condition.

**References**


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