Super-twisting Adaptive Sliding Mode Control: a Lyapunov Design

Yuri B. Shtessel, Jaime A. Moreno, Franck Plestan, Leonid M. Fridman, Alexander S. Poznyak

Abstract—A novel super-twisting adaptive sliding mode controller is proposed. A drift uncertain term is assumed to be bounded with unknown boundary. The proposed Lyapunov-based approach consists in using dynamically adapted control gains that ensure the establishment, in a finite time, of a second order sliding mode. Finite convergence time is estimated. A numerical example confirms the efficacy of the proposed adaptive super-twisting control.

I. INTRODUCTION

Sliding mode control is one of the best choices for controlling perturbed systems with matched disturbances/uncertainties [1,2]. The price for achieving the robustness/insensitivity to these disturbances is control chattering [1,2,3]. The traditional ways for avoiding chattering are as follows:

(a) Replacing the discontinuous control function by “saturation” or “sigmoid ones” [4,5]. This approach yields continuous control and chattering elimination. However, it constrains the sliding system’s trajectories not to the sliding surface but to its vicinity loosing the robustness to the disturbances.

(b) Using the higher order sliding mode control techniques [6-8, 12, 13]. This approach allows driving to zero the sliding variable and its consecutive derivatives in the presence of the disturbances/uncertainties increasing the accuracy of the sliding variable stabilization.

One of the most powerful second order continuous sliding mode control algorithms is the super-twisting control law (STW) that handles a relative degree equal to one. It generates the continuous control function that drives the sliding variable and its derivative to zero in finite time in the presence of the smooth matched disturbances with bounded gradient, when this boundary is known. Since STW algorithm contains a discontinuous function under the integral, chattering is not eliminated but attenuated.

The main drawback of STW control algorithm is the requirements to know the boundaries of the disturbance gradient. In many practical cases this boundary cannot be easily estimated. The overestimating of the disturbance boundary yields to larger than necessary control gains, while designing the STW control law.

Contribution. In this work we propose the novel adaptive STW control law that continuously drives the sliding variable and its derivative to zero in the presence of the bounded disturbance with the unknown boundary. The finite convergence time is estimated. The proof is based on recently proposed Lyapunov function [9, 10] that is used for the derivation of the novel adaptive STW control algorithm.

II. PROBLEM FORMULATION

Consider a single-input uncertain nonlinear system

\[ \dot{x} = f(x,t) + h(x,t)u \]  

where \( x \in \mathbb{R}^n \) is a state vector, \( u \in \mathbb{R} \) is a control function, \( f(x,t) \in \mathbb{R}^n \) is a differentiable, partially known vector-field. Assume that

(A1) A sliding variable \( \sigma = \sigma(x,t) \in \mathbb{R} \) is designed so that the system’s (1) desirable compensated dynamics are achieved in the sliding mode \( \sigma(x,t) = 0 \).

(A2) The system’s (1) input-output \( (u \to \sigma) \) dynamics are of a relative degree one, and the internal dynamics are stable.

Therefore, the input-output dynamics can be presented

\[ \dot{\sigma} = \frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} f(x,t) + \frac{\partial \sigma}{\partial x} h(x,t)u \rightarrow \]

\[ \dot{\sigma} = \varphi(x,t) + b(x,t)u \rightarrow \]

\[ \dot{\sigma} = \varphi(x,t) + b(x,t)u \rightarrow u = b^{-1}(x,t)\omega \]

The solution of system (2) is understood in the sense of Filippov [11]. Assume that

(A3) the function \( b(x,t) \in \mathbb{R} \) is known and not equal to zero \( \forall x \) and \( t \in [0, \infty) \)

(A4) the function \( \varphi(x,t) \in \mathbb{R} \) is bounded

\[ |\varphi(x,t)| \leq |\sigma|^{1/2} \]

where the finite boundary \( \delta > 0 \) exists but is not known.

The problem is to drive the sliding variable \( \sigma \) and its derivative \( \dot{\sigma} \) to zero in finite time in the presence of the bounded perturbation with the unknown boundary by means of continuous control.

The classical SMC and the second order sliding mode (SOSM) controllers, including the continuous STW control algorithm, can robustly handle such problem if the boundary of the perturbation is known. The main disadvantage of the classical SMC is introducing control chattering, while
SOSM controllers are able to attenuate it. Classical sliding mode control with gain adaptation for a class of uncertain systems with unknown bounding function are proposed in works [14, 15]. In the work [14] in order to avoid chattering the discontinuous control is replaced by a continuous approximation. The price is a loss of robustness. The gain adaptation laws proposed in work [15] also guarantee convergence to a small domain but without overestimating the control gains.

In this work we are looking for an adaptive-gain STW algorithm that is able to address this problem via generating continuous control function (chattering attenuation) so that its gains are adapted to the unknown perturbation with the unknown boundary.

III. Control Structure

The following STW control is proposed

\[ \begin{align*}
\dot{\sigma} &= -\alpha [\sigma]^{1/2} \text{sign}(\sigma) + v \\
\dot{v} &= -\beta \text{sign}(\sigma)
\end{align*} \]  

(4)

where the adaptive gains

\[ \begin{align*}
\alpha &= \alpha(\sigma, \sigma, t) \\
\beta &= \beta(\sigma, \sigma, t)
\end{align*} \]

(5)

are to be defined.

Control system given by eqs. (2) and (4) is presented in a form

\[ \begin{align*}
\dot{\sigma} &= -\alpha [\sigma]^{1/2} \text{sign}(\sigma) + v + \varphi(x, t) \\
\dot{v} &= -\beta \text{sign}(\sigma)
\end{align*} \]  

(6)

The control design problem is reduced to designing adaptive STW control (4), (5) that drives \( \sigma, \dot{\sigma} \rightarrow 0 \) in finite time in the presence of the bounded perturbation with the unknown boundary.

IV. Main Results

The main result of the paper is formulated in the following theorem.

**Theorem:** Consider system (6). Suppose that the perturbation \( \varphi(x, t) \) satisfies Assumption (A4) for some unknown constant \( \delta > 0 \). Then for any initial conditions \( x(0), \sigma(0) \) the sliding surface \( \sigma = 0 \) will be reached in finite time via STW control (4) with the adaptive gains

\[ \begin{align*}
\dot{\alpha} &= \left\{ \begin{array}{ll}
\alpha_0 \sqrt{\frac{\gamma_1}{2}}, & \text{if } \sigma \neq 0 \\
0, & \text{if } \sigma = 0
\end{array} \right. \\
\dot{\beta} &= 2\alpha + \lambda + 4\varepsilon^2
\end{align*} \]  

(7)

where \( \varepsilon, \lambda, \gamma_1, \alpha_0 \) are arbitrary positive constants.

**Proof:** The proof is split into two steps. In the first step, we will present system (6) in a form convenient for the Lyapunov analysis. In order to do this a new state vector is introduced

\[ z = (z_1, z_2)^T = \left( [\sigma]^{1/2} \text{sign}(\sigma), v \right)^T \]  

(8)

and system (6) can be rewritten as

\[ \begin{align*}
\dot{z}_1 &= \frac{1}{|z_1|} \left(-\alpha \frac{1}{2} z_1 + \frac{1}{2} z_2 + \varphi(x, t)\right) \\
\dot{z}_2 &= -\beta \frac{1}{|z_1|} z_1
\end{align*} \]  

(9)

Equation (9) can be rewritten in a vector-matrix format

\[ \begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
-\alpha \frac{1}{2} z_1 & \frac{1}{2} z_2 \\
-\beta & 0
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + \frac{1}{|z_1|} \varphi(x, t) \Rightarrow
\]  

(10)

\[ \dot{z} = A(z_1) z + g(z_1) \varphi(x, t) \]

where

\[ A(z_1) = \frac{1}{|z_1|} \begin{bmatrix}
-\alpha \\
\frac{1}{2}
\end{bmatrix}, \quad g(z_1) = \frac{1}{|z_1|} \begin{bmatrix} 1 \\
0
\end{bmatrix} \]

(11)

It can be observed that

(a) If \( z_1, z_2 \rightarrow 0 \) in finite time then \( \sigma, \dot{\sigma} \rightarrow 0 \) in finite time;

(b) \( |z_1| = |\sigma|^{1/2} \) and \( \text{sign}(z_1) = \text{sign}(\sigma) \)

In the second step of the proof, the stability analysis of system given by eqs. (10) and (11) is performed. In order to do it, the following Lyapunov function candidate is introduced

\[ V(z_1, z_2, \alpha, \beta) = V_0 + \frac{1}{2\gamma_1} \left( \alpha - \alpha^* \right)^2 + \frac{1}{2\gamma_2} \left( \beta - \beta^* \right)^2 \]

(12)

where

\[ V_0(z) = \left( \lambda + 4\varepsilon^2 \right) z_1^2 + z_2^2 - 4\varepsilon z_1 z_2 = z^T P z \]

(13)

\[ P = \begin{bmatrix}
\lambda + 4\varepsilon^2 & -2\varepsilon \\
-2\varepsilon & 1
\end{bmatrix}, \quad \lambda > 0, \quad \varepsilon > 0. \]

(14)

and \( \alpha^* > 0, \beta^* > 0 \) are some constants. It is worth noting that the matrix \( P \) in eq. (14) is positive definite if \( \lambda > 0 \) and \( \varepsilon \) is any real number.

The derivative of the Lyapunov function candidate (13) is presented

\[ \dot{V}(z, \alpha, \beta) = \dot{z}^T P z + z^T P \dot{z} + \frac{1}{\gamma_1} \left( \alpha - \alpha^* \right) \dot{\alpha} + \frac{1}{\gamma_2} \left( \beta - \beta^* \right) \dot{\beta} \]

(15)

The first two terms of eq. (15) are computed taking into account eqs. (10) and (11)

\[ \dot{V}_0 = z^T P z + z^T P \dot{z} \leq -\frac{1}{|z_1|} z^T \hat{Q} z \]

(16)

The symmetric matrix \( \hat{Q} \) is computed taking into account inequality (3):
\[ \dot{Q} - 2\epsilon I = \begin{bmatrix} 2\lambda\alpha + 4\epsilon (2\lambda\alpha - \beta) - 2(\lambda + 4\epsilon^2)\delta & \ast \\ \beta - 2\epsilon\alpha - \lambda - 4\epsilon^2 + 2\epsilon\delta & 4\epsilon \end{bmatrix} \] (17)

In order to guarantee the positive definiteness of the matrix \( \dot{Q} \) we enforce
\[
\beta = 2\epsilon\alpha + \lambda + 4\epsilon^2
\] (18)

The matrix \( \dot{Q} \) will be positive definite with a minimal eigenvalue \( \lambda_{\text{min}}(\dot{Q}) \geq 2\epsilon \) if
\[
\alpha > \frac{\epsilon\delta^2 + (\lambda + 4\epsilon^2)(2\epsilon + \delta) + \epsilon}{\lambda}
\] (19)

In view of eq. (16) and assuming that eqs. (18), (19) hold, it is easy to show that
\[
\dot{V}_0 \leq -rV_0^{1/2}
\] (20)

where
\[
\epsilon = \frac{2\epsilon\lambda_{\text{min}}(P)}{\lambda_{\text{max}}(P)}
\] (21)

Indeed, since
\[
\dot{V}_0(z) \leq -\frac{1}{|z_1|}z_2^T\dot{Q}z \leq -\frac{\epsilon}{|z_1|}z_2^Tz = -\frac{\epsilon}{|z_1|}\|z\|^2
\] (22)

and
\[
\lambda_{\text{min}}(P)\|z\|^2 \leq z_2^TPz \leq \lambda_{\text{max}}(P)\|z\|^2
\] (23)

where \( \|z\|^2 = z_1^2 + z_2^2 = |\sigma| + |\zeta| \) and
\[
|z_1| = |\sigma|^{1/2} \leq \|z\|^{1/2} = \frac{V_0^{1/2}(z)}{\lambda_{\text{min}}(P)}
\] (24)

then
\[
\dot{V}_0(z) \leq -rV_0^{1/2}, \quad r = \frac{2\epsilon\lambda_{\text{min}}(P)}{\lambda_{\text{max}}(P)}
\]

Now, in view of eqs. (20), eq. (15) can be rewritten
\[
\dot{V}(z,\alpha,\beta) = z_2^TPz + z_2^TPz + \frac{1}{\gamma_1}(\alpha - \alpha^*)\dot{\alpha}
\]

+ \frac{1}{\gamma_2}(\beta - \beta^*)\dot{\beta} \leq -\frac{1}{|z_1|}z_2^TPz + \frac{1}{\gamma_1}(\alpha - \alpha^*)\dot{\alpha}

+ \frac{1}{\gamma_2}(\beta - \beta^*)\dot{\beta} \leq -rV_0^{1/2} + \frac{1}{\gamma_1}(\alpha - \alpha^*)\dot{\alpha}

+ \frac{1}{\gamma_2}(\beta - \beta^*)\dot{\beta} \leq -rV_0^{1/2} - \frac{\epsilon}{\gamma_1}\|\alpha - \alpha^*\| - \frac{\epsilon}{\gamma_2}\|\beta - \beta^*\| - \frac{\epsilon}{\gamma_2}\|\beta - \beta^*\| + \frac{1}{\gamma_1}(\alpha - \alpha^*)\dot{\alpha}

+ \frac{\epsilon}{\gamma_2}\|\beta - \beta^*\| + \frac{\epsilon}{\gamma_2}(\alpha - \alpha^*)\dot{\alpha}

+ \frac{\epsilon}{\gamma_2}(\beta - \beta^*)\dot{\beta} + \frac{\epsilon}{\gamma_2}(\alpha - \alpha^*)\dot{\alpha} + \frac{\epsilon}{\gamma_2}(\beta - \beta^*)\dot{\beta}
\] (25)

Taking into account the well-known inequality
\[
(x^2 + y^2 + z^2)^{1/2} \leq |x| + |y| + |z|
\] (26)

and in view of eq. (12), we can derive
\[
-\frac{1}{2}\gamma_2 \leq \frac{\alpha}{\sqrt{2\gamma_1}} - \frac{\alpha^*}{\sqrt{2\gamma_1}} - \frac{\beta}{\sqrt{2\gamma_2}} - \frac{\beta^*}{\sqrt{2\gamma_2}} \leq -\eta \sqrt{V(z,\alpha,\beta)}
\] (27)

where \( \eta = \min(r,\alpha_1,\alpha_2) \).

Taking into account eq. (27), we can rewrite eq. (25) as
\[
\dot{V}(z_1, z_2, \alpha, \beta) \leq -\eta [V(z_1, z_2, \alpha, \beta)]^{1/2} + \frac{1}{\gamma_1}(\alpha - \alpha^*)\dot{\alpha}

+ \frac{1}{\gamma_2}(\beta - \beta^*)\dot{\beta} + \frac{\alpha}{\sqrt{2\gamma_1}} - \frac{\alpha^*}{\sqrt{2\gamma_1}} + \frac{\beta}{\sqrt{2\gamma_2}} - \frac{\beta^*}{\sqrt{2\gamma_2}}
\] (28)

Now we assume that the adaptation law (7) makes the adaptive gains \( \alpha(t) \), \( \beta(t) \) bounded (this assumption will be proven later). Then there exist positive constants \( \alpha^*, \beta^* \) that \( \alpha(t) - \alpha^* < 0 \) and \( \beta(t) - \beta^* < 0 \) \( \forall t \geq 0 \).

In view of the above assumption, eq. (28) can be reduced to the following:
\[
\dot{V}(z_1, z_2, \alpha, \beta) \leq -\eta [V(z_1, z_2, \alpha, \beta)]^{1/2} - \frac{1}{\gamma_1}(\alpha - \alpha^*)\dot{\alpha}

- \frac{1}{\gamma_2}(\beta - \beta^*)\dot{\beta} + \frac{\alpha}{\sqrt{2\gamma_1}} - \frac{\alpha^*}{\sqrt{2\gamma_1}} + \frac{\beta}{\sqrt{2\gamma_2}} - \frac{\beta^*}{\sqrt{2\gamma_2}}
\] (29)

\[
\dot{V}(z_1, z_2, \alpha, \beta) \leq -\eta [V(z_1, z_2, \alpha, \beta)]^{1/2} + \xi
\] (30)

where
\[
\xi = -\alpha - \alpha^*\left[\frac{1}{\gamma_1}\dot{\alpha} - \frac{\alpha}{\sqrt{2\gamma_1}}\right] - \frac{\beta}{\sqrt{2\gamma_2}} - \frac{\beta^*}{\sqrt{2\gamma_2}}
\] (31)

For the finite time convergence we have to assure \( \xi = 0 \) that is supposed to be achieved via adaptation of the gains \( \alpha, \beta \).

\[
\dot{\alpha} = \alpha_1 \sqrt{\frac{\gamma_1}{2}}
\] (31)

\[
\dot{\beta} = \beta_2 \sqrt{\frac{\gamma_2}{2}}
\] (32)

After selecting \( \epsilon = \frac{\alpha_1}{2\alpha_1\sqrt{\frac{\gamma_1}{2}}} \) eq. (18) and (32) coincide, since
\[
\dot{\beta} = 2\epsilon\alpha + \lambda + 4\epsilon^2 \rightarrow \dot{\beta} = 2\epsilon\alpha \rightarrow \dot{\beta} = \epsilon\epsilon_1 \sqrt{\frac{\gamma_1}{2}}
\] (33)

It is worth noting that for the finite time convergence \( \alpha(t) \) must satisfy inequality (19). It means that \( \alpha(t) \) is supposed to increase in accordance with eq. (31) until eq. (19) is met.
that guarantees the positive definiteness of the matrix $\tilde{Q}$.
After that the finite convergence is guaranteed according to eq. (29). Also, as soon as the sliding variable $\sigma$ and its derivative converges to zero it does not make sense to increase $\alpha(t)$ and $\beta(t)$ by making $\dot{\alpha}=0$ as $\sigma=0$.
Therefore, we obtain the gain-adaptation law (7). Theorem is proven.

Now we can prove the assumption about boundedness of $\alpha(t)$ and $\beta(t)$.

**Proposition 1.** The adaptive gains $\alpha(t)$ and $\beta(t)$ are bounded.

**Proof.** A solution to eq. (7) can be constructed as

$$
\alpha(t) = \begin{cases} 
\alpha(0) + \omega_1 \sqrt{\frac{2}{\eta}} t_r, & 0 \leq t \leq t_r \\
\alpha(0) + \omega_1 \sqrt{\frac{2}{\eta}} t_r, & t > t_r 
\end{cases}
$$

(34)

Where $t_r$ is finite reaching time. Therefore $\alpha(t)$ is bounded. The adaptive gain $\beta(t)$ is also bounded, since $\beta(t) = 2\varepsilon\alpha(t) + \lambda + 4\varepsilon^2$. The proposition is proven.

Now we can easily estimate finite reaching time.

**Proposition 2.** As soon as inequality (19) is fulfilled in finite time the adaptive-gain STW control law (7) drives the sliding variable $\sigma$ and its derivative to zero in finite time that is estimated as:

$$
t_r \leq \frac{2V^{1/2}(0)}{\eta}
$$

(35)

where $\eta = \min(r, \omega_1, \omega_2)$.

**Proof.** Inequality (19) is fulfilled in finite time, since its right hand side is bounded and the adaptive gain $\alpha(t)$ is increasing linearly with respect to time in accordance with eq. (7). Inequality (35) is obtained by a direct integration of inequality (29) bearing in mind that $\xi=0$ due the adaptation law (7).

V. EXAMPLE

Consider the following uncertain nonlinear system

$$
\dot{\sigma} = \varphi(\sigma, t) + \omega
$$

(36)

The disturbance with the unknown gain-boundary is simulated

$$
\varphi(\sigma, t) = \begin{cases} 
10\sin t \sqrt{|\sigma|}, & \text{if } t \leq 5 \text{sec} \\
35\sin t \sqrt{|\sigma|}, & \text{if } t > 5 \text{sec} 
\end{cases}
$$

(37)

The initial value of the sliding variable is taken as $\sigma(0) = -10$.
The adaptive-gain STW control is designed

$$
\begin{cases} 
\omega = -\alpha(\sigma, t)|\sigma|^{1/2} \text{sign}(\sigma) + v, \\
\dot{v} = -\beta(\sigma, t) \text{sign}(\sigma) 
\end{cases}
$$

(38)

The adaptive gain $\alpha(\sigma, t)$ and $\beta(\sigma, t)$ dynamics follow eq. (7). The values of the parameters of the adaptive gain law (7) have been taken as $\omega_1 = 6, \gamma_1 = 1, \lambda = 1, \varepsilon = 1$. The parameter tuning has been made without “optimal” objectives but in order to get sufficiently accurate and fast convergence. Note, that, firstly, inequality (19) is to be fulfilled in finite time, while the gains are increasing, and only then the $\sigma(t)$ trajectory starts finite-time converging.

Finally, the adaptive-gain law (7) becomes

$$
\dot{\alpha} = \frac{\omega_1}{\sqrt{2}}, \quad \text{if } \sigma \neq 0, \\
0, \quad \text{if } \sigma = 0
$$

(39)

Figures 1 and 2 show the efficacy of the proposed adaptive STW control algorithm. Another parameter tuning has been made by increasing the parameter $\omega_1$ up to $\omega_1=15$. The simulations displayed by Figures 3 and 4 show that, as the gain is increasing faster, the perturbation is faster compensated: the convergence of $\sigma(t)$ to 0 is obtained in a shorter time.

![Figure 1 – Sliding variable $\sigma(t)$ versus time (sec)](image)

![Figure 2 – Gain $\alpha(t)$ versus time (sec)](image)
It is worth noting that the STW control gains adapt themselves while the unknown boundary of the disturbance changes during the system’s performance.

VI. CONCLUSIONS

A novel adaptive-gain super-twisting sliding mode controller is proposed. A drift uncertain term is assumed to be bounded with unknown boundary. The proposed Lyapunov-based approach consists in using dynamically adapted control gains that ensure the establishment, in a finite time, of a second order sliding mode. Finite convergence time is estimated. A numerical example confirms the efficacy of the proposed adaptive super-twisting control. The future work will be dedicated to the study of adaptive STW control with a broader classes of possible bounded disturbances with unknown bounds.

VII. REFERENCES