Vector $F$-implicit complementarity problems in topological vector spaces

A.P. Farajzadeh$^a$*, J. Zafarani$^b$

$^a$Department of Mathematics, Razi University, Kermanshah, 67149, Iran
$^b$Department of Mathematics, University of Isfahan, Isfahan 81745-163, Iran

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Abstract

Recently, Huang and Li [J. Li, N.J. Huang, Vector $F$-implicit complementarity problems in Banach spaces, Appl. Math. Lett. 19 (2006) 464–471] introduced and studied a new class of vector $F$-implicit complementarity problems and vector $F$-implicit variational inequality problems in Banach spaces. In this work, we study this class in topological vector spaces and drive some existence theorems for the vector $F$-implicit variational inequality and vector $F$-implicit complementarity problem. Also, their equivalence is presented under certain conditions.

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1. Introduction and preliminaries

Vector variational inequalities were first introduced and studied by Giannessi [4] in the setting of finite-dimensional Euclidean spaces. There are generalizations of scalar variational to the vector case. Vector variational inequalities have many applications in vector optimization, approximate vector optimization, and other areas (see [5]).

In 2001, Yin et al. [12] introduced a class of $F$-complementarity problems ($F$-CP), which consist in finding $x \in K$ such that

$$
\langle Tx, x \rangle + F(x) = 0, \quad \langle Tx, y \rangle + F(y) \geq 0, \quad \forall y \in K,
$$

where $X$ is a Banach space with topological dual $X^*$, and $\langle \cdot, \cdot \rangle$ duality pairing between them, $K$ a closed convex cone of $X$, and $T : K \to X^*$, $F : K \to \mathbb{R}$. They obtained an existence theorem for solving ($F$-CP) and also proved that if $F$ is positively homogeneous (i.e. $F(tx) = tF(x)$ for all $t > 0$ and $x \in K$) and convex, the problem ($F$-CP) is equivalent to the following generalized variational inequality problem (GVIP) which consists in finding $x \in K$ such that

$$
\langle Tx, y - x \rangle + F(y) - F(x) \geq 0, \quad \forall y \in K.
$$

* Corresponding author.

E-mail address: ali-ff@sci.ui.ac.ir (A.P. Farajzadeh).

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In 2003, Fang and Huang [3] introduced a new class of vector $F$-complementarity problems with demipseudomonotone mappings in Banach spaces. They presented the solvability of this class of vector $F$-complementary problems with demipseudomonotone mappings and finite-dimensional continuous mappings in reflexive Banach spaces. Later, Huang and Li [6] introduced and studied a new class of (scalar) $F$-implicit complementarity problems and $F$-implicit variational inequality problems in Banach spaces. They obtained some existence theorems for $F$-implicit complementarity and F-variational problems. Also, under special assumptions, they established the equivalence between $F$-implicit complementarity and F-variational problems. Recently, in [7], they extended those problems to a vector valued setting.

In this work our aim is to generalize some results of [7] to topological vector spaces under certain weaker conditions. We first consider the following vector $F$-implicit variational inequality (in short, VF-IVIP). Find $x \in K$ such that

\[(VF-IVIP) \quad (Tx, y - x) + F(y) - F(x) \in C(x), \quad \forall y \in K,\]

and the second problem which we study, is called vector $F$-implicit complementarity problem (in short, VF-ICP) which consists of finding $x \in K$ such that

\[(VF-ICP) \quad (Tx, x) = 0, \quad (Tx, y) + F(y) \in C(x), \quad \forall y \in K,\]

where $X, Y$ are topological vector spaces, $K$ is a nonempty convex subset of $X$, $C : K \to 2^Y$ a multi-valued map with convex cone values, $T : K \to L(X, Y)$, and $F : K \to Y$.

In the rest of this section, we recall some definitions and preliminary results which are used in next sections.

We shall denote by $2^A$ the family of all subsets of $A$ and by $\mathcal{F}(A)$ the family of all nonempty finite subsets of $A$. Let $X$ be a real Hausdorff topological vector space (in short, t.v.s.). A nonempty subset $P$ of $X$ is called convex cone if (i) $P + P = P$, (ii) $\lambda P \subseteq P$, for all $\lambda \geq 0$. Let $Y$ be a t.v.s. and $P \subseteq Y$ be a cone. The cone $P$ induces an order in $Y$ (in this case the pair $(Y, P)$ is called an ordered t.v.s.) which is defined as follows:

$$x \leq y \iff y - x \in P.$$ 

This ordering is anti-symmetrical if $P$ is pointed. Let $X$ and $Y$ be two t.v.s., $K$ a nonempty subset of $X$, and $C : K \to 2^Y$ a multi-valued map with nonempty convex cone values.

We say that $f : K \times K \to Y$ is vector $C$-upper semicontinuous ($C$-u.s.c.) in the first variable, if the set \(\{x \in K : f(x, y) \in C(x)\}\) is closed in $K$, for every $y \in K$. This definition reduces to vector 0-u.s.c., if $C(x) = P$ for every $x \in K$, where $P$ is a constant convex cone.

Let $X$ be a nonempty set, $Y$ a topological space, and $\Gamma : X \to 2^Y$ a multi-valued map. Then, $\Gamma$ is called transfer closed-valued if, for every $y \not\in \Gamma(x)$, there exists $x' \in X$ such that $y \not\in \text{cl} \Gamma(x')$, where cl denotes the closure of a set. It is clear that, $\Gamma : X \to 2^X$ is transfer closed-valued if and only if

$$\bigcap_{x \in X} \Gamma(x) = \bigcap_{x \in X} \text{cl} \Gamma(x).$$

If $B \subseteq Y$ and $A \subseteq X$, then $\Gamma : A \to 2^B$ is called transfer closed-valued if the multi-valued mapping $x \to \Gamma(x) \cap B$ is transfer closed-valued. In this case where $X = Y$ and $A = B$, $\Gamma$ is called transfer closed-valued on $A$.

Let $K$ be a nonempty convex subset of a t.v.s. $X$ and let $K_0$ be a subset of $K$. A multi-valued map $\hat{\Gamma} : K_0 \to 2^K$ is said to be a KKM map if

$$\text{co}A \subseteq \bigcup_{x \in A} \Gamma(x), \quad \forall A \in \mathcal{F}(K_0),$$

where co denotes the convex hull.

In the next section, we need the following theorem.

**Theorem 1.1** ([2]). Let $X$ be a t.v.s. and $K$ be a nonempty convex subset of $X$. Suppose that $\Gamma, \hat{\Gamma} : K \to 2^K$ are two multi-valued mappings such that:

(i) $\hat{\Gamma}(x) \subseteq \Gamma(x), \forall x \in K$;

(ii) $\hat{\Gamma}$ is a KKM map;
(iii) for each $A \in F(K)$, $\Gamma$ is transfer closed-valued on $\text{co}A$;
(iv) for each $A \in F(K)$, $\text{cl}_{K}(\bigcap_{x \in \text{co}A} \Gamma(x)) \subseteq \text{co}A = (\bigcap_{x \in \text{co}A} \Gamma(x)) \cap \text{co}A$;
(v) there is a nonempty compact convex set $B \subseteq K$ such that $\text{cl}_{K}(\bigcap_{x \in B} \Gamma(x))$ is compact.

Then, $\bigcap_{x \in K} \Gamma(x) \neq \emptyset$.

2. Main results

Throughout this section, let $X$ and $Y$ be real Hausdorff t.v.s. and $K$ be a nonempty convex subset of $X$. Denote by $L(X, Y)$ the space of all continuous linear mappings from $X$ into $Y$, and $\langle t, x \rangle$ be the value of the linear continuous mapping $t \in L(X, Y)$ at $x$. Suppose that $C : K \to 2^{Y}$ is a multivalued map with nonempty convex cone values, $f : K \to L(X, Y)$, $g : K \to K$ and $F : K \to Y$. We consider the following vector $F$-implicit complementarity problem (VF-ICP).

Find $x \in K$ such that

$$\langle f(x), g(x) \rangle + F(g(x)) = 0 \quad \text{and} \quad \langle f(x), y \rangle + F(y) \in C(x), \quad \forall y \in K.$$  

The above problem reduces to vector $F$-implicit complementarity problem considered in [7] for the case $C(x) = P$, where $(Y, P)$ is an ordered t.v.s. and $P$ is a convex cone subset of $K$.

Examples of (VF-ICP) in t.v.s.

(1) If $g$ is an identity mapping on $K$, then (VF-ICP) reduces to the vector $F$-complementary problem (in short VF-CP) which consists in finding $x \in K$ such that:

$$\langle f(x), x \rangle + F(x) = 0 \quad \text{and} \quad \langle f(x), y \rangle + F(y) \in C(x), \quad \forall y \in K.$$  

(2) If $F = 0$, then (VF-CP) reduces to the vector complementary problem (in short, VCP) which consists in finding $x \in K$ such that:

$$\langle f(x), x \rangle = 0 \quad \text{and} \quad \langle f(x), y \rangle \in C(x), \quad \forall y \in K,$n which has been studied by Chen and Yang [1], and Yang [11] in particular case $C(x) = P, \forall x \in K$.

(3) If $L(X, Y) = X^{*}$ and $F : K \to \mathbb{R}$, then (VF-ICP) reduces to the $F$-implicit complementarity problems (in short, F-ICP) which consists of finding $x \in K$ such that:

$$\langle f(x), g(x) \rangle + F(g(x)) = 0 \quad \text{and} \quad \langle f(x), y \rangle + F(y) \in C(x), \quad \forall y \in K$$

which were considered by Huang and Li [6] in the particular case, where $C(x) = P, \forall x \in K$.

(4) If $g$ is the identity mapping, then (F-ICP) reduces to the $F$-complementary problem (in short, F-CP) which consists in finding $x \in K$ such that:

$$\langle f(x), x \rangle + F(x) = 0 \quad \text{and} \quad \langle f(x), y \rangle + F(y) \in C(x), \quad \forall y \in K,$n which was studied by Yin et al. [12] in the particular case, where $C(x) = P, \forall x$.

(5) If $F = 0$, then (F-ICP) reduces to the implicit complementary problem (in short ICP) which consists in finding $x \in K$ such that:

$$\langle f(x), g(x) \rangle = 0 \quad \text{and} \quad \langle f(x), y \rangle \in C(x), \quad \forall y \in K,$$n which has been studied by Isac [9,10].

(6) If $g$ is the identity mapping and $F = 0$, then (F-ICP) reduces to the complementary problem (in short, CP) which consists in finding $x \in K$ such that:

$$\langle f(x), x \rangle = 0 \quad \text{and} \quad \langle f(x), y \rangle \in C(x), \quad \forall y \in K,$$n which has been studied by many authors, (for instance, see [10]). If $X = X^{*} = \mathbb{R}^{n}$, then (CP) becomes the classical complementary problem.
We also introduce the following vector $F$-implicit variational inequality problem (in short VF-IVIP) which consists in finding $x \in K$ such that

$$(f(x), y - g(x)) + F(y) - F(g(x)) \in C(x), \quad \forall y \in K.$$ 

This problem is a generalization of the problem (VF-IVIP) introduced in [7] in a Banach space setting.

**Remark 2.1.** Any solution of (VF-ICP) is a solution of (VF-IVIP). The following theorem says that the converse holds if $F$ is positively homogeneous; the proof is similar to Theorem 3.1 in [7] and thus will be omitted.

**Theorem 2.2.** If $F : K \to Y$ is positively homogeneous, then (VF-IVIP) and (VF-ICP) are equivalent.

The following example shows that if $F$ is not positively homogenous, the conclusion of Theorem 2.2 may be incorrect:

**Example 2.3.** Let $X = Y = \mathbb{R}$, $K = [0, +\infty)$, $g(x) = 0$, $F(x) = 1$, and $C(x) = [0, +\infty)$, for all $x \in K$. Define $f : K \to \mathbb{R}$ (note that $L(\mathbb{R}, \mathbb{R}) \equiv \mathbb{R}$) by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ -1 & \text{otherwise.} \end{cases}$$

Obviously, $x = 0$ is a solution of (VF-IVIP) but is not a solution of (VF-ICP).

In Theorem 2.2, if $g$ is the identity mapping, then we have the following corollary:

**Corollary 2.4.** Let $F : K \to Y$ be positively homogeneous. Then any solution of (VF-VIP) is a solution for (VF-CP).

The following theorem provides an existence result for the (VF-IVIP) in t.v.s. which improves Theorem 3.2. in [7].

**Theorem 2.5.** Assume that:

(a) the function $G : \text{co}A \times \text{co}A \to Y$ where,

$$G(x, y) = \langle f(x), y - g(x) \rangle + F(y) - F(g(x))$$

is C-u.s.c. in the first variable, $\forall A \in \mathcal{F}(K)$;

(b) let $A \in \mathcal{F}(K)$, $x, y \in \text{co}A$. If $(x_a)$ is any net on $K$ converging to $x$ then,

$$(f(x_a), tx + (1-t)y - g(x_a)) + F(tx + (1-t)y) - F(g(x_a)) \in C(x), \quad \forall t \in [0, 1]$$

implies

$$(f(x), y - g(x)) + F(y) - F(g(x)) \in C(x).$$

(c) There exists a mapping $h : K \times K \to Y$ such that:

(i) $h(x, x) \in C(x), \forall x \in K$;

(ii) $\langle f(x), y - g(x) \rangle + F(y) - F(g(x)) - h(x, y) \in C(x), \forall x \in K, \forall y \in K$;

(iii) the set $\{y \in K : h(x, y) \notin C(x)\}$ is convex, $\forall x \in K$;

(d) there exist a nonempty compact subset $B$ and a nonempty convex compact subset $D$ of $K$ such that, for each $x \in K \setminus B$, there exists $y \in D$ such that $(f(x), y - g(x)) + F(y) - F(g(x)) \notin C(x)$.

Then (VF-IVIP) has a solution. Moreover, the solution set of (VF-IVIP) is compact.

**Proof.** We define $\Gamma, \hat{\Gamma} : K \to 2^K$ as follows:

$$\Gamma(y) = \{x \in K : (f(x), y - g(x)) + F(y) - F(g(x)) \in C(x)\},$$

$$\hat{\Gamma}(y) = \{x \in K : h(x, y) \in C(x)\}.$$

We show that $\Gamma, \hat{\Gamma}$ satisfy conditions of Theorem 1.1. From assumption (ii) of (c), $\hat{\Gamma}(y) \subseteq \Gamma(y)$, for all $y \in K$.

If $A = \{x_1, x_2, \ldots, x_n\} \subseteq K, z \in \text{co}A$ and $z \in \bigcup_{i=1,2,\ldots,n} \hat{\Gamma}(x_i)$, then $h(z, x_i) \notin C(z)$ for $i = 1, 2, 3, \ldots, n$. It follows by (c)(iii) that, $h(z, z) \notin C(z)$ contradicting (c)(i). So $\hat{\Gamma}$ is a KKM map. Let $A \in \mathcal{F}(K), x \in \text{co}A$ and
\((x_a) \in \Gamma(x) \cap \text{co}A\) converges to \(z\). Then, \((f(x_a), x - g(x_a)) + F(y) - F(g(x_a)) \in C(x_a)\). By (a), we conclude that \(z \in \Gamma(x) \cap \text{co}A\). Since \(x\) is an arbitrary element of \(\text{co}A\), we obtain:

\[
\bigcap_{x \in \text{co}A} \Gamma(x) \cap \text{co}A = \bigcap_{x \in \text{co}A} \text{cl}(\Gamma(x) \cap \text{co}A).
\]

Similarly, using (b) we get:

\[
\bigcap_{x \in \text{co}A} \Gamma(x) \cap \text{co}A = \text{cl}_K \left( \bigcap_{x \in \text{co}A} \Gamma(x) \right) \cap \text{co}A, \quad A \in \mathcal{F}(K).
\]

From (d) we deduce that \(\text{cl}(\bigcap_{x \in K} \Gamma(x)) \subseteq B\). Hence, \(\Gamma, \hat{\Gamma}\) satisfy the conditions of Theorem 1.1. Then

\[
\bigcap_{x \in K} \Gamma(x) \neq \emptyset,
\]

which shows that the problem (VF-IVIP) has a solution. Now, let \((x_a)\) be a net of solutions of (VF-IVIP) which converges to \(x\). Then, for all \(y \in K\) and all \(t \in [0, 1]\), we have

\[
(f(x_a), tx + (1-t)y - g(x_a)) + F(tx + (1-t)y) - F(g(x_a)) \in C(x_a).
\]

Thus, from assumption (b) we obtain

\[
(f(x), y - g(x)) + F(y) - F(g(x)) \in C(x).
\]

Therefore, the solution set of (VF-IVIP) is closed and thanks to (d), it is a subset of \(B\) and consequently is compact. Thus the proof is completed. □

**Remark 2.6.** Let us endow \(L(X, Y)\) with the following topology. We say that a net \(F_a \in L(X, Y)\) converges to \(F \in L(X, Y)\) if, for each convergent net \(x_a \to x\) we have \(\langle F_a, x_a \rangle \to \langle F, x \rangle\). Now if, \(f, g, F\) are continuous and \(C\) is a map with the closed graph then, the assumptions (a) and (b) are satisfied. Also, if \(K\) is compact then, the condition (d) trivially holds.

**Corollary 2.7.** Assume that:

(a) the function \(G : \text{co}A \times \text{co}A \to Y\) where,
\[
G(x, y) = \langle f(x), y - x \rangle + F(y) - F(x)
\]
is \(C\)-u.s.c. in the first variable, \(\forall A \in \mathcal{F}(K)\);

(b) Let \(A \in \mathcal{F}(K), x, y \in \text{co}A\). If \((x_a)\) be any net on \(K\) converging to \(x\) then
\[
\langle f(x_a), tx + (1-t)y - g(x_a) \rangle + F(tx + (1-t)y) - F(g(x_a)) \in C(x_a), \quad \forall t \in [0, 1]
\]
implies
\[
\langle f(x), y - g(x) \rangle + F(y) - F(g(x)) \in C(x).
\]

(c) there exists a mapping \(h : K \times K \to Y\) such that
(i) \(h(x, x) \in C(x), \forall x \in K\);
(ii) \(\langle f(x), y - x \rangle + F(y) - F(x) - h(x, y) \in C(x), \forall x \in K, \forall y \in K\);
(iii) the set \(\{y \in K : h(x, y) \notin C(x)\}\) is convex, \(\forall x \in K\);

(d) there exist a nonempty compact subset \(B\) and a nonempty convex compact subset \(D\) of \(K\) such that, for each \(x \in K \setminus B\), there exists \(y \in D\) such that \(\langle f(x), y - x \rangle + F(y) - F(x) \notin C(x)\).

Then, (VF-VIP) has a solution. Moreover, the solution set of (VF-VIP) is compact.

By slight modifications of the proof of Corollary 2.4, we can obtain the following existence theorems.

**Theorem 2.8.** Assume that:

(a) the function \(G : \text{co}A \times \text{co}A \to Y\) where
\[
G(x, y) = \langle f(x), y - g(x) \rangle + F(y) - F(g(x))
\]
is \(C\)-u.s.c. in the first variable, \(\forall A \in \mathcal{F}(K)\);
(b) Let $A \in \mathcal{F}(K)$, $x, y \in \text{co}A$. If $(x_\alpha)$ be any net on $K$ converging to $x$ then, for all $t \in [0, 1]$ the following implication holds:

$$\langle f(x_\alpha), tx + (1-t)y - g(x_\alpha) \rangle + F(tx + (1-t)y) - F(g(x_\alpha)) \in C(x_\alpha)$$
then $$\langle f(x), y - g(x) \rangle + F(y) - F(g(x)) \in C(x).$$

(c) $\langle f(x), x - g(x) \rangle + F(x) - F(g(x)) \in C(x), \forall x \in K$;

(d) the set $\{y \in K : \langle f(x), y - g(x) \rangle + F(y) - F(g(x)) \notin C(x)\}$ is convex, $\forall x \in K$;

(e) there exist a nonempty compact set $B \subseteq K$ and a nonempty convex compact subset $D$ of $K$ such that, for each $x \in K \setminus B$, there exists $y \in D$ such that $\langle f(x), y - g(x) \rangle + F(y) - F(g(x)) \notin C(x)$.

Then, (VF-IVIP) has a solution. Moreover, the solution set of (VF-IVIP) is compact.

**Theorem 2.9.** Suppose that:

(a) the function $h$ is C-u.s.c. in the first variable on $\text{co}A$, $\forall A \in \mathcal{F}(K)$;

(b) for each $A \in \mathcal{F}(K)$, let $x, y \in \text{co}A$ and $(x_\alpha)$ be a net on $K$ converging to $x$, then, the following implication holds,

if $h(x_\alpha, tx + (1-t)y) \in C(x_\alpha), \forall t \in [0, 1]$, then $h(x, y) \in C(x)$;

(c) $h(x, x) \in C(x), \forall x \in K$;

(d) the set $\{y \in K : h(x, y) \notin C(x)\}$ is convex, $\forall x \in K$;

(e) there exist a nonempty compact subset $B$ and a nonempty convex compact subset $D$ of $K$ such that, for each $x \in K \setminus B$, there exists $y \in D$ such that $h(x, y) \notin C(x)$.

If, for every $y \in K$, the following implication holds:

$$\langle f(x), y - g(x) \rangle + F(y) - F(g(x)) - h(x, y) \in C(x), \forall x \in K.$$ 

Then, (VF-IVIP) has a solution. Moreover, the solution set of (VF-IVIP) is compact.

The following theorem improves Theorem 3.3. in [7].

**Theorem 2.10.** Suppose that all assumptions of one of the Theorems 2.5 and 2.8 or 2.9 are satisfied. If $F$ is positively homogeneous, then, (VF-ICP) has a solution. Moreover, the solution set of (VF-ICP) is compact.

**Proof.** The result follows by Theorems 2.2 and 2.5.

**Remark 2.11.** Consider the following vector $F$-implicit complementarity problems in t.v.s. which was studied in the special case $F(x) = 0$ and $g(x) = x$ in [8].

(Weak) vector $F$-implicit complementarity problem (W-VF-ICP): Find $x \in K$ such that:

$$\langle f(x), g(x) \rangle + F(g(x)) \notin \text{int}C(x), \forall y \in K.$$ 

(Positive) vector $F$-implicit complementarity problem (P-VF-ICP): Find $x \in K$ such that:

$$\langle f(x), g(x) \rangle + F(g(x)) \notin \text{int}C(x), \forall y \in K.$$ 

It is clear that the solution set of (VF-ICP), is a subset of the solution sets of (P-VF-ICP) and (W-VF-ICP). Thus, Theorems 2.5, 2.8 and 2.9 provide existence results for (W-VF-ICP) and (P-VF-ICP). If we take $F = 0$, which is obviously positively homogenous, then Theorem 2.8 gives a solution for the problems considered in [8].

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**References**


