Solution of systems of 
integral–differential equations by 
Adomian decomposition method

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Abstract

In this article Adomian decomposition method, as a well-known method for solving functional equations, has been employed to solve systems of integral–differential equations. Theoretical considerations are discussed, and convergence of the method for these systems is addressed. Some examples are presented to show the ability of the method for such systems.
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1. Introduction

A system of integral–differential equations can be considered, in general form, as:
were each equation represents the first derivative of one of the unknown functions as a mapping involving the independent variable \( t \), and \( n \) unknown functions \( f_1, \ldots, f_n \) which have appeared partly in the integral sign.

For economy of writing let us consider this system as the following;

\[
\frac{df_i(t)}{dt} = H_i\left(t, f_1(t), \ldots, f_n(t), \int_0^t K_i(t,s,f_1(s),\ldots,f_n(s))ds\right),
\]

\[
f_{i}(0) = a_{i}, \quad i = 1, \ldots, n.
\]

This system of integral–differential equation can be converted to a system of Volterra integral equations of the second kind. To derive this system let’s define;

\[
\frac{df_i(t)}{dt} = g_i(t), \quad \text{so} \quad f_i(t) = f_{i}(0) + \int_0^t g_i(t)dt, \quad i = 1, \ldots, n.
\]

By this definition the system of integral–differential equations (1) can be presented as the following system of Volterra integral equations consisting of 2\( n \) Volterra integral equations of the second kind.

\[
\begin{align*}
g_i(t) &= H_i\left(t, f_1(t), \ldots, f_n(t), \int_0^t K_i(t,s,f_1(s),\ldots,f_n(s))ds\right), \\
f_i(t) &= f_{i}(0) + \int_0^t g_i(t)dt, \quad i = 1, \ldots, n.
\end{align*}
\]

2. Using Adomian decomposition method to solve (4)

Adomian decomposition method is a powerful device for solving functional equations [1].

System of Volterra integral equations (4) can be rewritten in canonical form;

\[
\begin{align*}
g_i(t) &= u_i(t) + G_i(f_1(t), \ldots, f_n(t), g_1(t), \ldots, g_n(t)), \\
f_i(t) &= f_{i}(0) + v_i(t) + F_i(f_1(t), \ldots, f_n(t), g_1(t), \ldots, g_n(t)).
\end{align*}
\]

Adomian decomposition method considers the solutions of (5) as the summation of the following series;
\[ g_i(t) = \sum_{m=0}^{\infty} g_{i,m}, \quad (6a) \]
\[ f_i(t) = \sum_{m=0}^{\infty} f_{i,m}. \quad (6b) \]

And the functions \( G_i \) and \( F_i \) as the following of some polynomials;

\[ G_i(f_1(t), \ldots, f_n(t), g_1(t), \ldots, g_n(t)) = \sum_{m=0}^{\infty} A_{i,m}(f_{1,0}, \ldots, f_{1,m}, \ldots, f_{n,0}, \ldots, f_{n,m}, g_{1,0}, \ldots, g_{1,m}, \ldots, g_{n,0}, \ldots, g_{n,m}), \]

\[ F_i(f_1(t), \ldots, f_n(t), g_1(t), \ldots, g_n(t)) = \sum_{m=0}^{\infty} B_{i,m}(f_{1,0}, \ldots, f_{1,m}, \ldots, f_{n,0}, \ldots, f_{n,m}, g_{1,0}, \ldots, g_{1,m}, \ldots, g_{n,0}, \ldots, g_{n,m}), \]

where \( A_{i,m} \) and \( B_{i,m} \) are called Adomian polynomials [1], substituting these into (5) we derive

\[ \sum_{m=0}^{\infty} g_{i,m} = u_i(t) + \sum_{m=0}^{\infty} A_{i,m}(f_{1,0}, \ldots, f_{1,m}, \ldots, f_{n,0}, \ldots, f_{n,m}, g_{1,0}, \ldots, g_{1,m}, \ldots, g_{n,0}, \ldots, g_{n,m}) \quad (7a) \]

\[ \sum_{m=0}^{\infty} f_{i,m} = f_i(0) + v_i(t) + \sum_{m=0}^{\infty} B_{i,m}(f_{1,0}, \ldots, f_{1,m}, \ldots, f_{n,0}, \ldots, f_{n,m}, g_{1,0}, \ldots, g_{1,m}, \ldots, g_{n,0}, \ldots, g_{n,m}). \quad (7b) \]

From relations (7) Adomian procedure would be constructed as;

\[
\begin{cases}
  g_{i,0} = 0, \\
  f_{i,0} = f_i(0), \\
  g_{i,m} = A_{i,m}(f_{1,0}, \ldots, f_{1,m}, \ldots, f_{n,0}, \ldots, f_{n,m}, g_{1,0}, \ldots, g_{1,m}, \ldots, g_{n,0}, \ldots, g_{n,m}), \\
  f_{i,m} = B_{i,m}(f_{1,0}, \ldots, f_{1,m}, \ldots, f_{n,0}, \ldots, f_{n,m}, g_{1,0}, \ldots, g_{1,m}, \ldots, g_{n,0}, \ldots, g_{n,m}).
\end{cases} \quad (8)
\]

For computing Adomian polynomials we use the alternate algorithm for computing Adomian polynomials [3].
3. Convergence of the method

Since after the first step of the mentioned procedure we derive the system (4), which is a system of Volterra integral equations of the second kind, for the convergence of the method, we refer the reader to [2] in which the problem of convergence has been discussed briefly.

4. Numerical examples

In this part we present two examples. These examples are considered to illustrate the method for linear and nonlinear systems of ordinary integral–differential equations.

Example 1. In this example the following linear system of integral–differential equations is solved, the exact solutions are $y_1(x) = x + e^x$, $y_2(x) = x - e^x$.

$$
\begin{align*}
\begin{cases}
y_1' &= 1 + x + x^2 - y_2 - \int_0^x (y_1(t) + y_2(t))dt, \\
y_2' &= -1 - x + y_1 - \int_0^x (y_1(t) - y_2(t))dt.
\end{cases}
\end{align*}
$$

By applying the operator $\int_0^x (\cdot)dt$, inverse operator of differentiation, we derive

$$
\begin{align*}
\begin{cases}
y_1 &= y_1(0) + x + \frac{x^2}{2} + \frac{x^3}{6} - \int_0^x y_2(x)dx - \int_0^x \int_0^x (y_1(t) + y_2(t))dtdx, \\
y_2 &= y_2(0) - x - \frac{x^2}{2} + \int_0^x y_1(x)dx - \int_0^x \int_0^x (y_1(t) - y_2(t))dtdx.
\end{cases}
\end{align*}
$$

Using Adomian decomposition method and the alternate algorithm for computing Adomian polynomials [3], we would have the following procedure;

$$
\begin{align*}
\begin{cases}
y_{1,0} &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, \\
y_{2,0} &= -1 - x - \frac{x^2}{2}, \\
y_{1,n} &= -\int_0^x y_{2,n}(x)dx - \int_0^x \int_0^x (y_{1,n}(t) + y_{2,n}(t))dtdx, \\
y_{2,n} &= \int_0^x y_{1,n}(x)dx - \int_0^x \int_0^x (y_{1,n}(t) - y_{2,n}(t))dtdx.
\end{cases}
\end{align*}
$$

Seven terms approximations to the solutions are derived as;

$$
\begin{align*}
y_1 &= 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{2520}x^7 + \frac{1}{40320}x^8 \\
&\quad - \frac{1}{60480}x^9 + \frac{1}{1814400}x^{10} - \frac{1}{1995840}x^{11} - \frac{1}{119750400}x^{12} \\
&\quad - \frac{1}{259459200}x^{13} + \frac{1}{10897286400}x^{14} + \frac{1}{81729648000}x^{15},
\end{align*}
$$
The values of the four and seven terms approximations to the solutions at some points with the corresponding errors are presented in the following Tables 1 and 2.

As the results in tables show the more terms in approximations would cause the more accuracy in solutions.

Example 2. Consider the following system of two nonlinear integral–differential equations, with initial values \( n_1(0) = N_1 \) and \( n_2(0) = N_2 \).

\[
\begin{align*}
\frac{dn_1}{dt} &= n_1 \left[ K_1 - \gamma_1 n_2(t) - \int_{t_0}^{t} e^{-(t-s)} n_2(s) ds \right] & K_1 > 0, \\
\frac{dn_2}{dt} &= n_2 \left[ -K_2 + \gamma_2 n_1(t) + \int_{t_0}^{t} e^{-(t-s)} n_1(s) ds \right] & K_2 > 0.
\end{align*}
\]

(10)

To solve this system by Adomian method, let us consider;

\[
\begin{align*}
\frac{dn_1}{dt} &= m_1(t) \Rightarrow n_1(t) = n_1(0) + \int_0^t m_1(s) ds, \\
\frac{dn_2}{dt} &= m_2(t) \Rightarrow n_2(t) = n_2(0) + \int_0^t m_2(s) ds, 
\end{align*}
\]

(11)

So we have the following system of four integral equations

\[
\begin{align*}
n_1(t) &= n_1(0) + \int_0^t m_1(s) ds, \\
m_1(t) &= n_1(t) [K_1 - \gamma_1 n_2(t) - \int_{t_0}^{t} e^{-(t-s)} n_2(s) ds], \\
n_2(t) &= n_2(0) + \int_0^t m_2(s) ds, \\
m_2(t) &= n_2(t) [K_2 + \gamma_2 n_1(t) + \int_{t_0}^{t} e^{-(t-s)} n_1(s) ds].
\end{align*}
\]

(12)

Table 1
The values of four terms approximations with the related errors

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1(n=3) )</th>
<th>( e(y_1(n=3)) )</th>
<th>( y_2(n=3) )</th>
<th>( e(y_2(n=3)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>1.421331042</td>
<td>0.71716E-4</td>
<td>-1.021336356</td>
<td>0.66402E-4</td>
</tr>
<tr>
<td>0.4</td>
<td>1.890606688</td>
<td>0.1218010E-2</td>
<td>-1.090774759</td>
<td>0.1049939E-2</td>
</tr>
<tr>
<td>0.6</td>
<td>2.415655523</td>
<td>0.6463277E-2</td>
<td>-1.216907311</td>
<td>0.5211489E-2</td>
</tr>
<tr>
<td>0.8</td>
<td>3.004390424</td>
<td>0.21150504E-1</td>
<td>-1.409521835</td>
<td>0.16019093E-1</td>
</tr>
<tr>
<td>1.0</td>
<td>3.665465169</td>
<td>0.52816659E-1</td>
<td>-1.680566579</td>
<td>0.37715249E-1</td>
</tr>
</tbody>
</table>
Table 2
The values of seven terms approximations with the related errors

<table>
<thead>
<tr>
<th>x</th>
<th>$y_1(n = 6)$</th>
<th>$e(y_1(n = 6))$</th>
<th>$y_2(n = 6)$</th>
<th>$e(y_2(n = 6))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.3E–8</td>
<td>-1</td>
<td>0.2E–8</td>
</tr>
<tr>
<td>0.2</td>
<td>1.421402761</td>
<td>0.320E–6</td>
<td>-1.021402756</td>
<td>0.320E–6</td>
</tr>
<tr>
<td>0.6</td>
<td>2.422124164</td>
<td>0.364E–5</td>
<td>-1.222113441</td>
<td>0.5359E–5</td>
</tr>
<tr>
<td>0.8</td>
<td>3.025580019</td>
<td>0.39091E–4</td>
<td>-1.425501900</td>
<td>0.39028E–4</td>
</tr>
<tr>
<td>1.0</td>
<td>3.718461689</td>
<td>0.179861E–3</td>
<td>-1.718102549</td>
<td>0.179279E–3</td>
</tr>
</tbody>
</table>

Substituting from (11) into a part of (12) we obtain:

$$
\begin{cases}
\frac{d}{dt} n_1(t) = N_1 + \int_0^t m_1(s) ds, \\
\frac{d}{dt} m_1(t) = K_1 N_1 + K_1 \int_0^t m_1(s) ds - \gamma_1 n_1(t) n_2(t) - \int_{t-\tau_0}^t e^{-(t-s)} n_1(t) n_2(s) ds, \\
\frac{d}{dt} n_2(t) = N_2 + \int_0^t m_2(s) ds, \\
\frac{d}{dt} m_2(t) = K_2 N_2 + K_2 \int_0^t m_2(s) ds + \gamma_2 n_1(t) n_2(t) + \int_{t-\tau_0}^t e^{-(t-s)} n_2(t) n_1(s) ds.
\end{cases}
$$

(13)

Using the alternate algorithm for computing the Adomian polynomials [3], the Adomian procedure would be as the following,

$$
\begin{align*}
\frac{d}{dt} n_{10} &= N_1, \\
\frac{d}{dt} m_{10} &= K_1 N_1, \\
\frac{d}{dt} n_{20} &= N_2, \\
\frac{d}{dt} m_{20} &= K_2 N_2,
\end{align*}
$$

$$
\begin{align*}
\frac{d}{dt} n_{1, n+1} &= \int_0^t m_{1, n}(s) ds, \\
\frac{d}{dt} m_{1, n+1} &= K_1 \int_0^t m_{1, n}(s) ds - \gamma_1 \sum_{k=0}^{n} n_{1, k}(t) n_{2, n-k}(t) \\
&\quad - \int_{t-\tau_0}^t e^{-(t-s)} \left( \sum_{k=0}^{n} n_{1, k}(t) n_{2, n-k}(s) \right) ds, \\
\frac{d}{dt} n_{2, n+1} &= \int_0^t m_{2, n}(s) ds, \\
\frac{d}{dt} m_{2, n+1} &= K_2 \int_0^t m_{2, n}(s) ds + \gamma_2 \sum_{k=0}^{n} n_{1, k}(t) n_{2, n-k}(t) \\
&\quad + \int_{t-\tau_0}^t e^{-(t-s)} \left( \sum_{k=0}^{n} n_{1, k}(t) n_{1, n-k}(s) \right) ds,
\end{align*}
$$

A three terms approximation for $n_1$ and $n_2$ are as follows;

$$
\begin{align*}
\frac{d}{dt} n_1(t) &= N_1 + N_1 [K_1 - \gamma_1 N_2 - N_2 (1 - e^{-\tau_0})] t + 0.5 N_1 K_1^2 t^2, \\
\frac{d}{dt} n_2(t) &= N_2 + N_2 [K_2 + \gamma_2 N_1 + N_1 (1 - e^{-\tau_0})] t + 0.5 N_2 K_2^2 t^2.
\end{align*}
$$
5. Conclusion

Adomian decomposition method has been known as a powerful device for solving many functional equations as algebraic equations, ordinary and partial differential equations, Integral equations and so on. Here we used this method for solving systems of integral–differential equations. As it was shown this method has the ability of solving systems of both linear and nonlinear integral-differential equations. The package Maple 9 is used for computations. Adomian decomposition method has been used for solving systems of integral equations, systems of ordinary differential equations by the author. And extension of the method for solving systems of partial differential equations is in hand.

References