Isomorphism classes of cycle permutation graphs

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Abstract


In this paper, we construct a cycle permutation graph as a covering graph over the dumbbell graph, and give a new characterization of when two given cycle permutation graphs are isomorphic by a positive or a negative natural isomorphism. Also, we count the isomorphism classes of cycle permutation graphs up to positive natural isomorphism, and find the number of distinct cycle permutation graphs isomorphic to a given cycle permutation graph by a positive/negative natural isomorphism. As a consequence, we obtain a formula for finding the number of double cosets of the dihedral group in the symmetric group.

1. Introduction

Permutation graphs were first introduced by Chartrand and Harary in [1] as a generalization of the Petersen graph. Let $C_n$ denote an $n$-cycle with consecutively labelled vertices $1, 2, \ldots, n$. For a permutation $\alpha$ in the symmetric group $S_n$ on $n$ elements, an $\alpha$-cycle permutation graph $P_{\alpha}(C_n)$ consists of two copies of $C_n$, say $C_x$ and $C_y$, with vertex sets $V(C_x) = \{x_1, x_2, \ldots, x_n\}$ and $V(C_y) = \{y_1, y_2, \ldots, y_n\}$, along with edges $x_i y_{\alpha(i)}$ for $1 \leq i \leq n$. When we wish to specify $n$, we will call $P_{\alpha}(C_n)$ $n$-cyclic: with neither $\alpha$ nor $n$ mentioned, it is simply a...
cyclic permutation graph. The copy of \( C_n \) labelled \( x_1, x_2, \ldots, x_n \) will be called the outer cycle, the copy of \( C_n \) labelled \( y_1, y_2, \ldots, y_n \) will be called the inner cycle, and the edges of the form \( x_i \) will be called permutation edges. Given two permutations \( \alpha \) and \( \beta \) in \( S_n \), \( P_\alpha(C_n) \) is said to be isomorphic to \( P_\beta(C_n) \) by a positive natural isomorphism \( \Theta \) if \( \Theta(y_i) = x_i \). The graph \( P_\alpha(C_n) \) is said to be isomorphic to \( P_\beta(C_n) \) by a negative natural isomorphism \( \Theta \) if \( \Theta(x_i) = y_i \). A natural isomorphism is either of these. Ringeisen counted the number of distinct cycle permutation graphs isomorphic to a \( k \)-twisted prism in [8]. Also, Stueckle [9], found the number of permutations which yield cycle permutation graphs isomorphic to a given cycle permutation graph by a natural isomorphism.

In this paper, we give a new characterization of when two given cycle permutation graphs are isomorphic by a positive or a negative natural isomorphism, by using a new construction of a cycle permutation graph as a covering graph over the ‘dumbbell graph’. Also, we give a complete numerical counting of the isomorphism classes of cycle permutation graphs up to positive natural isomorphism which gives in fact a formula for finding the number of double cosets of the dihedral group in the symmetric group, and find the number of distinct cycle permutation graphs isomorphic to a given cycle permutation graph by a positive/negative natural isomorphism. Stueckle [9] and Dörfler [3] gave values for some of these numbers in terms of double coset sizes, which they did not explicitly compute.

Let \( G \) be a connected graph with vertex set \( V(G) \) and edge set \( E(G) \) possibly with loops. Every edge of a graph \( G \) gives rise to a pair of oppositely directed edges. We denote the set of directed edges of \( G \) by \( D(G) \). By \( e^{-1} \) we mean the reverse edge to an edge \( e \). Each directed edge \( e \) has an initial vertex \( i_e \) and a terminal vertex \( t_e \). Following [4], a permutation voltage assignment \( \phi \) on a graph \( G \) is a map \( \phi : D(G) \rightarrow S_n \) with the property that \( \phi(e^{-1}) = \phi(e)^{-1} \) for each \( e \in D(G) \), where \( S_n \) is the symmetric group on \( n \) elements \( \{1, 2, \ldots, n\} \). Let \( D^+(G) \) denote the set of plus-directed edges in \( D(G) \). The permutation derived graph \( G^\phi \) has vertex set \( V(G^\phi) = V(G) \times \{1, 2, \ldots, n\} \) and edge set \( E(G^\phi) = D^+(G) \times \{1, 2, \ldots, n\} \); for each edge \( e \in D^+(G) \) and \( j \in \{1, 2, \ldots, n\} \) there is an edge \( (e, j) \) in \( E(G^\phi) \) with \( i_{(e,j)} = (i_e, j) \) and \( t_{(e,j)} = (t_e, \phi(e)j) \). The natural projection \( p^\phi: G^\phi \rightarrow G \) is actually a covering projection.

From now on, let \( G \) denote the ‘dumbbell’ graph with two vertices \( x, y \), an edge \( e = xy \) and two loops \( e_x = xx, e_y = yy \) pictured in Fig. 1.

![Fig. 1. The dumbbell graph.](image-url)
From the definition of the permutation derived graph $G^\phi$, we can see that the directions of loops $e_1$ and $e_2$ in $G$ do not affect the graph $G^\phi$, but the direction of the edge $e$ does. Let the edge $e = xy$ in $G$ be plus-directed, $e^{-1} = yx$ minus-directed and let two loops $e_1$ and $e_2$ be counterclockwise directed as their positive sense. Let $\rho$ denote the $n$-cycle $(1 \ 2 \ \cdots \ n)$ in $S_n$. Then the permutation derived graph $G^\phi$ with the voltage assignment $\phi$ defined by $\phi(e_1) = \phi(e_2) = \rho$ and $\phi(e) = \alpha, \ \alpha \in S_n$, is clearly the cycle permutation graph $P_\alpha(C_n)$. But the permutation derived graph $G^\phi$ is independent of the choice of the direction of the loop $e_1$, hence we can define $\phi(e_1) = \rho^{-1}$ instead of $\phi(e_1) = \rho$ in the construction of $G^\phi$, which is also the cycle permutation graph $P_\alpha(C_n)$. Any cycle permutation graph can be drawn as in Fig. 2, where the vertices of the outer cycle and inner cycle respectively are equally spaced around two concentric circles, and the edges of the outer cycle and the permutation edges are fixed. With a suitable relabelling of the vertices of the inner cycle $C_\gamma$ of $P_\alpha(C_n)$, we can assume that the permutation edges are $x_1y_1, i = 1, 2, \ldots, n$, as shown in Fig. 3 with $\alpha = (2 \ 4 \ 5 \ 3)$ and $n = 5$.

This relabelling of the vertices $\{y_1, y_2, \ldots, y_n\}$ of the inner cycle $C_\gamma$ of $P_\alpha(C_n)$ gives an $n$-cycle $\sigma$ in $S_n$ representing the inner cycle in the new labelling, for example $\sigma = (1 \ 3 \ 5 \ 2 \ 4)$ in Fig. 3. Such relabelling suggests the following theorem.
Theorem 1. A cycle permutation graph $P_n(C_n)$ is isomorphic to the permutation derived graph $G^\psi$ with voltage assignment $\psi$ defined by $\psi(e_x) = \rho$, $\psi(e) = \alpha^{-1}\rho\alpha$ (or $\psi(e_x) = \alpha^{-1}\rho^{-1}\alpha$), over the dumbbell graph $G$.

Proof. If we denote by $x_i$, $y_j$ the vertices $(x, i)$, $(y, j)$ of $G^\psi$ respectively, there exists a clear one to one correspondence ‘same second label’ between the vertex sets of $P_n(C_n)$ and $G^\psi$. To define a voltage assignment $\psi$ on the graph $G$ so that $P_n(C_n)$ is isomorphic to a derived graph $G^\psi$, we assume that the permutation edges in $P_n(C_n)$ are $x_i y_i$, $i = 1, 2, \ldots, n$, with a suitable relabelling of the vertices $\{y_i\}$ of the inner cycle $C$, of $P_n(C_n)$, as discussed above. Define $\psi(e_x) = \rho$ and $\psi(e)$ as the identity in $S_n$. With the given correspondence of vertices of $P_n(C_n)$ and $G^\psi$, there exists an $n$-cycle $\sigma$ in $S_n$ such that $P_n(C_n)$ is isomorphic to $G^\psi$ with $\psi(e_x) = \sigma$. And a path $x_i y_{\alpha(i)} y_{\rho\alpha(i)} x_{\alpha^{-1}\rho\alpha(i)}$ in $P_n(C_n)$ must correspond to a path $x_i y_{\alpha(i)} x_{\alpha(i)}$ in $G^\psi$ for all $i$. Hence, we get an isomorphism with $\sigma = \alpha^{-1}\rho\alpha$. But taking the minus-direction of $e_x$ in $G$, we can replace $\psi(e_x) = \alpha^{-1}\rho\alpha$ by $\psi(e_x) = \alpha^{-1}\rho^{-1}\alpha$. □

The permutation derived graph $G^\psi$ isomorphic to $P_n(C_n)$ defined in Theorem 1 will be denoted by $G^\sigma_n$, and $G^\sigma_n$ will be identified with $P_n(C_n)$ from now on.

Let $\Sigma_n$ denote the conjugacy class of $\rho = (12 \cdots n)$ in $S_n$, i.e., $\Sigma_n$ is the set of all $n$-cycles in $S_n$. From the identification above, it is enough to consider a permutation derived graph with a permutation voltage assignment which assigns the identity on the edge $e$, $\rho = (12 \cdots n)$ on the loop $e_x$ and $\sigma$ for $\sigma \in \Sigma_n$ on the loop $e_e$ of the dumbbell graph $G$ for a cycle permutation graph. Hence, the set $\Sigma_n$ can be identified as the set of all $n$-cyclic permutation graphs, which is crucial for the counting of their isomorphism classes.

Let two cycle permutation graphs $G^\sigma_n$ and $G^{\sigma'}_n$ be isomorphic by a natural isomorphism $\Theta$. Then it induces an automorphism $\theta$ on the dumbbell graph $G$ such that the following diagram commutes:

$\begin{array}{ccc}
G^{\sigma'}_n & \xrightarrow{\Theta} & G^\sigma_n \\
\downarrow{\rho_x} & & \downarrow{\rho_x} \\
G & \xrightarrow{\theta} & G
\end{array}$

Clearly, the automorphism group Aut($G$) of the dumbbell graph $G$ consists of two elements, 1, and $\iota$, where $\iota$ denotes the isomorphism of $G$ exchanging two vertices $x$ and $y$ (and then also the inner cycle and the outer cycle). Thus
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\[ \text{Aut}(G) = \mathbb{Z}_2. \]
A natural isomorphism \( \Theta \) is positive or negative according as it induces 1 or \( \tau \).

Let \( \mathcal{A}: \mathcal{S}_n \to \mathcal{S}_n \) be the map defined by \( \mathcal{A}(\sigma) = \sigma^{-1} \) for all \( \sigma \in \mathcal{S}_n \). Let \( D_n \) denote the dihedral group generated by two permutations \( \rho \) and \( \tau \), where \( \tau(i) = n + 1 - i \) and \( \rho(i) = i + 1 \); that is, the group of automorphisms of the \( n \)-cycle \( C_n \). Note that all arithmetic is done modulo \( n \), and the dihedral group \( D_n \) is the normalizer of \( \{ \rho, \rho^{-1} \} \) in \( \mathcal{S}_n \). Let us denote \( \Gamma = D_n \times \{ 1, \mathcal{A} \} \), and define an action \( \Gamma \times \mathcal{S}_n \to \mathcal{S}_n \) by \( (d, 1)(\sigma) = \sigma \alpha^{-1}d \) and \( (d, \mathcal{A})(\sigma) = \alpha \sigma^{-1}d^{-1} \).

**Theorem 2.** Let \( \alpha \) and \( \beta \) be two permutations in \( \mathcal{S}_n \).

1. \( P_{\alpha}(C_n) \) is isomorphic to \( P_{\beta}(C_n) \) by a positive natural isomorphism if and only if there exists \( \gamma \in \Gamma \) such that \( \beta^{-1}\rho \beta = \gamma(\alpha^{-1}\rho \alpha) \).
2. \( P_{\alpha}(C_n) \) is isomorphic to \( P_{\beta}(C_n) \) by a negative natural isomorphism if and only if there exists \( \gamma \in \Gamma \) such that \( \beta^{-1}\rho \beta = \gamma(\alpha \rho \alpha^{-1}) \).
3. \( P_{\alpha}(C_n) \) is isomorphic to \( P_{\beta}(C_n) \) by a natural isomorphism if and only if there exists \( \gamma \in \Gamma \) such that \( \beta^{-1}\rho \beta = \gamma(\alpha^{-1}\rho \alpha) \) or \( \beta^{-1}\rho \beta = \gamma(\alpha \rho \alpha^{-1}) \).

**Proof.** (1) Use the identifications \( P_{\alpha}(C_n) = G_{n}^{\alpha} \) and \( P_{\beta}(C_n) = G_{n}^{\beta} \). If \( G_{n}^{\alpha} \) and \( G_{n}^{\beta} \) are isomorphic by a positive natural isomorphism, say \( \Theta \), then \( \Theta \) maps the outer cycle of \( G_{n}^{\alpha} \) to the outer cycle of \( G_{n}^{\beta} \) isomorphically, which induces an element \( d \) in \( D_n \). The path \( x_i y_i x_{i-1}^{-1} \rho \sigma(i) x_{i-1}^{-1} \rho \sigma(i) \) (or \( x_i y_i x_{i-1}^{-1} \rho^{-1} \rho \sigma(i) x_{i-1}^{-1} \rho^{-1} \rho \sigma(i) \) depending on the orientation of \( e_i \)) in \( G_{n}^{\alpha} \) is mapped to the path \( x_{d(i)} y_{d(i)} x_{d(i)}^{-1} \rho \sigma(d(i)) x_{d(i)}^{-1} \rho \sigma(d(i)) \) (or \( x_{d(i)} y_{d(i)} x_{d(i)}^{-1} \rho^{-1} \rho \sigma(d(i)) x_{d(i)}^{-1} \rho^{-1} \rho \sigma(d(i)) \) depending on the orientation of \( e_i \)) in \( G_{n}^{\beta} \). In any case, we get \( \gamma \in \Gamma \) such that \( \beta^{-1}\rho \beta = \gamma(\alpha^{-1}\rho \alpha) \).

Conversely, if there exists an element \( \gamma \in \Gamma \) such that \( \beta^{-1}\rho \beta = \gamma(\alpha^{-1}\rho \alpha) \).
Then we have an element \( d \) in \( D_n \) which induces an automorphism in the \( n \)-cycle \( C_n \), and hence an isomorphism from the outer cycle of \( G_{n}^{\alpha} \) to the outer cycle of \( G_{n}^{\beta} \). It is easily extended to a positive natural isomorphism from \( G_{n}^{\alpha} \) to \( G_{n}^{\beta} \).

(2) The proof is the same as in (1), except that the negative natural isomorphism \( \Theta \) from \( G_{n}^{\alpha} \) to \( G_{n}^{\beta} \) induces the automorphism \( \iota \) of \( G \) and the automorphism \( \iota \) reverses the direction of the edge \( e \) in \( G \). Hence the voltage assignment value \( \alpha \) should be changed to \( \alpha^{-1} \) and vice versa.

(3) follows from (1) and (2). \( \square \)

This result corresponds to Theorems 3 and 4 and their corollaries in [9].

**Corollary 1.** Let \( \alpha \) and \( \beta \) be two permutations in \( \mathcal{S}_n \).

1. \( P_{\alpha}(C_n) \) is isomorphic to \( P_{\beta}(C_n) \) by a positive natural isomorphism if and only if \( \beta \in D_n\alpha D_n \).
2. \( P_{\alpha}(C_n) \) is isomorphic to \( P_{\beta}(C_n) \) by a negative natural isomorphism if and only if \( \beta \in D_n\alpha^{-1} D_n \).
3. \( P_{\alpha}(C_n) \) is isomorphic to \( P_{\beta}(C_n) \) by a natural isomorphism if and only if \( \beta \in D_n\alpha D_n \cup D_n\alpha^{-1} D_n \).
Proof. (1) Let \( P_n(C_n) \) be isomorphic to \( P_n(C_n) \) by a positive natural isomorphism. Then \( \beta^{-1} \rho \beta = \gamma(\alpha^{-1} \rho \alpha) \) for some \( \gamma \in \Gamma \), by Theorem 2 (1) and thus

\[
\rho = (\beta \alpha^{-1}) \rho (\alpha^{-1} \beta^{-1}) \rho (\alpha^{-1} \beta^{-1})^{-1},
\]

or

\[
\rho^{-1} = (\beta \alpha^{-1}) \rho (\beta \alpha^{-1})^{-1}
\]

for some \( \beta \in D_n \). Hence, \( \beta \alpha^{-1} \) is contained in the normalizer \( N(\rho, \rho^{-1}) \) of \( (\rho, \rho^{-1}) \) in \( S_n \) for some \( \beta \in D_n \). But \( N(\rho, \rho^{-1}) = D_n \). Therefore, \( \beta \in D_n \alpha D_n \).

Conversely, if \( \beta = d_1 \alpha d_2 \) for some \( d_1, d_2 \in D_n \) then \( \beta^{-1} \rho \beta = d_2^{-1} \alpha^{-1} \beta^{-1} \rho d_1 \alpha d_2 \), which is either \( d_2^{-1} \alpha^{-1} \rho \alpha \) or \( d_2^{-1} \alpha^{-1} \rho \alpha d_2 \), i.e., \( \beta^{-1} \rho \beta = \gamma(\alpha^{-1} \rho \alpha) \) for some \( \gamma \in \Gamma \).

A similar proof gives (2), and (3) follows from (1) and (2).

2. Counting formulas

A positive natural isomorphic or a natural isomorphic relation is clearly an equivalence relation on the set of all \( n \)-cyclic permutation graphs which is identified as the set \( S_n \), but a negative natural isomorphic relation is not. To count the corresponding equivalence classes of \( n \)-cyclic permutation graphs, we introduce the following symbols:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Collection counted</th>
<th>Up to what equivalence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Iso}_P(C_n) )</td>
<td>all</td>
<td>positive natural isomorphism</td>
</tr>
<tr>
<td>( \text{Iso}(C_n) )</td>
<td>all</td>
<td>natural isomorphism</td>
</tr>
<tr>
<td>( N_\rho(\alpha) )</td>
<td>graphs isomorphic to ( P_n(C_n) ) by a positive natural isomorphism</td>
<td>trivial</td>
</tr>
<tr>
<td>( N_\rho(\alpha) )</td>
<td>graphs isomorphic to ( P_n(C_n) ) by a negative natural isomorphism</td>
<td>trivial</td>
</tr>
<tr>
<td>( N(\alpha) )</td>
<td>graphs isomorphic to ( P_n(C_n) ) by a natural isomorphism</td>
<td>trivial</td>
</tr>
</tbody>
</table>

and, let \(|X|\) denote the cardinality of a set \( X \). Note that we obtain a formula for finding the number of double cosets of the dihedral group in the symmetric group \( S_n \), since \( \text{Iso}_P(C_n) \) is also the number of double cosets \( D_n \alpha D_n \) in \( S_n \).

Lemma 1. Let \( \sigma \) and \( \zeta \) be any two \( n \)-cycles in \( \Sigma_n \). Then:

1. \(|\{w \in S_n: w \sigma w^{-1} = \zeta\}| = n \). In particular,

\[
\{w \in S_n: w \sigma w^{-1} = \sigma\} = \{\sigma^i: i = 1, 2, \ldots, n\}.
\]

2. If \( w \sigma w^{-1} = \alpha^{-1} \) for some \( w \in S_n \), then \( w^2 \) is the identity in \( S_n \).
Proof. (1) For $\sigma = (a_1 \ a_2 \ \ldots \ a_n)$, $\zeta = (b_1 \ b_2 \ \ldots \ b_n)$ $n$-cycles in $\Sigma_n$, let $w\omega w^{-1} = \zeta$. Then $(b_1 \ b_2 \ \ldots \ b_n) = w(a_1 \ a_2 \ \ldots \ a_n)w^{-1} = (w(a_1)w(a_2) \ \ldots \ w(a_n))$ as $n$-cycles and $w(a_i)$ can be any $b_k$ in $\{b_1, b_2, \ldots, b_n\}$. Also, if $w(a_i) = b_k$, then $w(b_i) = b_{k+j-1}$ for all $j = 1, 2, \ldots, n$, where $k + j - 1$ is taken modulo $n$. Hence \(|\{w \in S_n : w\omega w^{-1} = \zeta\}| = n\) and $\sigma', i = 1, 2, \ldots, n$ are exactly $n$ such elements in \(\{w \in S_n : w\omega w^{-1} = \sigma\}\).

(2) If we let $w\omega w^{-1} = \sigma^{-1}$ for some $w \in S_n$, then $(w(a_1)w(a_2) \ \ldots \ w(a_n)) = (a_n a_{n-1} \ \ldots \ a_1)$. If $w(a_i) = a_k$, then $w(a_i) = a_{k-i+1}$ for all $i$ and $w^2(a_i) = w(a_{k-i+1}) = a_i$ for all $i$. Thus, $w^2$ is the identity in $S_n$. $\Box$

Lemma 1 shows that for any $\alpha$-cycle permutation graph $G^\alpha_n$, there are exactly $n$ permutations $\omega$ in $S_n$ such that $G^\omega_n = G^\alpha_n$. For any $n$-cycle $\sigma$ in $\Sigma_n$, let $I_\sigma$ denote the isotropy subgroup of $\sigma$:

\[I_\sigma = \{\gamma \in \Gamma : \gamma \sigma = \sigma\} ,\]

which is a subgroup of $\Gamma = D_n \times \{1, \phi\}$.

Lemma 2. For any $\alpha$ in $S_n$,

1. $\Gamma^{\alpha^{-1} \rho \alpha}$ is (group-) isomorphic to $\alpha D_n \alpha^{-1} \cap D_n$,
2. $|\Gamma^{\alpha^{-1} \rho \alpha}| = |\Gamma_{\alpha \rho \alpha^{-1}}|$,
3. $|\Gamma(\alpha^{-1} \rho \alpha)| = |\Gamma(\alpha \rho \alpha^{-1})|$.

Proof. (1) First, we observe that at most one of $(d, 1)$ and $(d, \phi)$ in $\Gamma = D_n \times \{1, \phi\}$ can be contained in the isotropy subgroup $\Gamma^{\alpha^{-1} \rho \alpha}$ of $\alpha^{-1} \rho \alpha$. Define a homomorphism $f$ from $\Gamma^{\alpha^{-1} \rho \alpha}$ to $\alpha D_n \alpha^{-1} \cap D_n$ by $f(d, \eta) = \alpha d \alpha^{-1}$, where $\eta \in \{1, \phi\}$. Then $f$ is clearly a monomorphism. To show the surjectivity of $f$, let $d$ be any element in $\alpha D_n \alpha^{-1} \cap D_n$, then $\alpha^{-1} d \alpha \in D_n$. Take $\gamma$ in $\Gamma$ as

\[\gamma = \begin{cases} \alpha^{-1} d \alpha, 1 & \text{if } d \in \{\rho^i : i = 1, 2, \ldots, n\}, \\ \alpha^{-1} d \alpha, \phi & \text{if } d \in \{\rho^i : i = 1, 2, \ldots, n\}. \end{cases}\]

Then, we can easily see that $\gamma \in \Gamma^{\alpha^{-1} \rho \alpha}$ and $f(\gamma) = d$.

(2) is clear from (1) and the fact that $|\alpha D_n \alpha^{-1} \cap D_n| = |\alpha^{-1} D_n \alpha \cap D_n|$ for any $\alpha$ in $S_n$.

(3) is clear because of $|\Gamma/\Gamma^{\alpha^{-1} \rho \alpha}| = |\Gamma(\alpha^{-1} \rho \alpha)|$. $\Box$

Lemma 3. For $\alpha \in S_n$, $\alpha^{-1} \rho \alpha$ and $\alpha \rho \alpha^{-1}$ lie in the same orbit of the $\Gamma$-action if and only if $\alpha D_n \alpha \cap D_n \neq \emptyset$.

Proof. Let $\alpha^{-1} \rho \alpha$ and $\alpha \rho \alpha^{-1}$ lie in the same orbit, then there exists a $\gamma \in \Gamma$ such that $\alpha^{-1} \rho \alpha = \gamma (\alpha \rho \alpha^{-1})$, that is, $\alpha^{-1} \rho \alpha = d \alpha \rho \alpha^{-1} d^{-1}$ or $\alpha^{-1} \rho \alpha = d (\alpha \rho \alpha^{-1})^{-1} d^{-1}$ for some $d \in D_n$, and then $\rho = (\alpha d) \rho (\alpha d)^{-1}$ or $\rho = (\alpha d) \rho^{-1} (\alpha d)^{-1}$ for some $d \in D_n$. Thus, $\alpha \rho \alpha = \rho'$ or $\rho \rho'$ for some $i$, and some $d \in D_n$, so, $\alpha D_n \alpha \cap D_n \neq \emptyset$. Conversely, if $\alpha D_n \alpha \cap D_n \neq \emptyset$ then $\alpha \rho \alpha = \rho'$ or $\rho \rho'$
for some $i$, and some $d \in D_n$. It is easy to see that $ho = (ada)\rho (ada)^{-1}$ or $ho = (ada)\rho^{-1}(ada)^{-1}$ for some $d \in D_n$. Hence, $\alpha^{-1}\rho \alpha = d\alpha\rho\alpha^{-1}d^{-1}$ or $\alpha^{-1}\rho \alpha = d(\alpha\rho\alpha^{-1})^{-1}d^{-1}$ for some $d \in D_n$, and $\alpha^{-1}\rho \alpha$ and $\alpha\rho\alpha^{-1}$ lie in the same orbit.

For the $\Gamma$-action on $\Sigma_n$, and any $\gamma \in \Gamma$, let $Fix_\gamma$ denote the set of fixed points of $\gamma$, i.e.,

$$Fix_\gamma = \{ \sigma \in \Sigma_n : \gamma\sigma = \sigma \}.$$

Part (3) of the following theorem was stated as Corollary 4 in [9].

**Theorem 3.** (1) $\text{Iso}_\rho(C_n) = \frac{|\Sigma_n|}{|\Gamma|} = \frac{1}{4n}\sum_{\gamma \in \Gamma} |\text{Fix}_\gamma|.$

(2) $\text{Iso}(C_n) = \frac{1}{2}\text{Iso}_\rho(C_n) + \frac{1}{2}|\Gamma(\alpha^{-1}\rho\alpha) : \alpha D_n \alpha \cap D_n \neq \emptyset, \alpha \in S_n|$, where $\Gamma(\alpha^{-1}\rho\alpha) = \{ \gamma(\alpha^{-1}\rho\alpha) : \gamma \in \Gamma \}$ denotes the orbit of $\alpha^{-1}\rho\alpha$ under the $\Gamma$-action, and the right term counts such orbits.

(3) For any $\alpha \in S_n$,

$$N_\rho(\alpha) = N_\alpha(\alpha) = \frac{4n^2}{|\Gamma_{\alpha^{-1}\rho\alpha}|} = \frac{4n^2}{|\alpha D_n\alpha^{-1} \cap D_n|}.$$

(4) For any $\alpha \in S_n$,

$$N(\alpha) = \begin{cases} N_\rho(\alpha) & \text{if } \alpha D_n \alpha \cap D_n \neq \emptyset, \\ 2N_\rho(\alpha) & \text{otherwise.} \end{cases}$$

**Proof.** (1) is clear by Burnside’s Lemma for the $\Gamma$-action on $\Sigma_n$.

Since the set $\Gamma(\alpha^{-1}\rho\alpha) \cup \Gamma(\alpha\rho\alpha^{-1})$ is the set of all voltage assignments representing natural isomorphism classes of $G_n^\alpha$ by Theorem 2, (2) comes from Lemma 3.

For any $\sigma \in \Sigma_n$, there are exactly $n$ permutations $\beta$ in $S_n$ such that $\beta \rho \beta^{-1} = \sigma$ by Lemma 1, and if $\sigma_1$ and $\sigma_2$ are distinct in $\Sigma_n$, then the corresponding sets of $n$ such elements are clearly disjoint. Hence,

$$N_\rho(\alpha) = n |\Gamma(\alpha^{-1}\rho\alpha)| = n |\Gamma| / |\Gamma_{\alpha^{-1}\rho\alpha}| = 4n^2 / |\Gamma_{\alpha^{-1}\rho\alpha}|,$$

by Theorem 2. Also, $N_\alpha(\alpha)$ is also equal to the same number by Lemma 2 and Theorem 2, giving (3).

By Theorem 2, $N(\alpha) = n |\Gamma(\alpha^{-1}\rho\alpha) \cup \Gamma(\alpha\rho\alpha^{-1})|$, and two orbit sets $\Gamma(\alpha^{-1}\rho\alpha)$ and $\Gamma(\alpha\rho\alpha^{-1})$ are either identical or disjoint. Hence, (4) follows from (3) and Lemma 3.

**Corollary 2.** For any $\alpha$ in $S_n$, $N_\rho(\alpha) \geq 2n$. Moreover, if $n$ is prime then $N_\rho(\alpha)$ is one of $2n$, $4n$, $2n^2$, or $4n^2$.

**Proof.** For any $\alpha \in S_n$ and any $d \in D_n$, at most one of $(d, 1)$ and $(d, \emptyset)$ can be contained in $\Gamma_{\alpha^{-1}\rho\alpha}$. Thus, $|\Gamma_{\alpha^{-1}\rho\alpha}| \leq 2n$, and $|\Gamma_{\alpha^{-1}\rho\alpha}|$ is a divisor of $|\Gamma| = 4n$. Hence, $N_\rho(\alpha)$ is one of $2n$, $4n$, $2n^2$, or $4n^2$ if $n$ is prime, by Theorem 3.
3. Countings

Now, we compute $|\text{Fix}_r|$ for the $r$-action on $\Gamma_n$, $n > 3$. Let $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$, and for $k \in \mathbb{Z}_n$ we let $o(k)$ denote the order of $k$ in the cyclic group $\mathbb{Z}_n$, $\iota(k)$ the index of the subgroup generated by $k$, and $\phi(k)$ the Euler phi-function, giving the number of integers relatively prime to $k$ between 1 and $k$.

**Lemma 4.** (1) $|\text{Fix}_{(\rho^*, 1)}| = \phi(o(k))(\iota(k) - 1)!o(k)^{\iota(k) - 1}$.

(2) $|\text{Fix}_{(\rho^*, 1)}| = \begin{cases} \left(\frac{n}{2} - 1\right)!2^{n/2 - 1} & \text{if } n \text{ is even and } k \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$

(3) $|\text{Fix}_{(\rho^*, s)}| = \begin{cases} \left(\frac{n}{2} - 1\right)!2^{n/2 - 1} & \text{if } n \text{ is even and } k = n/2, \\ 0 & \text{otherwise}. \end{cases}$

(4) $|\text{Fix}_{(\rho^*, s)}| = \begin{cases} \left(\frac{n}{2} - 1\right)!2^{n/2 - 1} & \text{if } n \text{ is odd}, \\ \left(\frac{n - 1}{2}\right)!2^{(n - 1)/2} & \text{if } n \text{ is odd and } k \text{ is even}, \\ \left(\frac{n}{2} - 1\right)!2^{n/2 - 1} & \text{if } n \text{ is even and } k \text{ is odd}. \end{cases}$

**Proof.** (1) Let $\sigma = (a_1 a_2 \cdots a_n)$ be an element of $\text{Fix}_{(\rho^*, 1)}$. Then $\rho^k \sigma \rho^{-k} = \sigma$ and $\sigma' = \rho^k$ for some $l \in \{1, 2, \ldots, n\}$, by Lemma 1. Clearly, the number of candidates for such $l$ is $\phi(o(k))$, and $\sigma' = \rho^k$ is a product of mutually disjoint $\iota(k)$ cycles of length $o(k)$. For a given such $l$ and such a product for $\sigma' = \rho^k$, we can have $\iota(k)!o(k)^{\iota(k)}$ different expressions for $\sigma' = \rho^k$ by permuting $\iota(k)$ cycles and rotating the members of each cycle. But each of these expressions corresponds to a $\sigma$ with $\sigma' = \rho^k$, uniquely up to rotation of members of $\sigma$. (For example, if $n = 6$, $k = 3$, $\rho = (123456)$, $\rho^3 = (14)(25)(36)$, the two different expressions (25)(14)(36) and (41)(25)(36) of $\rho^3$ correspond to two different $\sigma_1 = (123546)$ and $\sigma_2 = (423156)$, respectively.) Hence, for a fixed $l$, there are $\iota(k)!o(k)^{\iota(k)/n} = (\iota(k) - 1)!o(k)^{\iota(k) - 1}$ candidates of $\sigma$ such that $\sigma' = \rho^k$. For any two different $l$ with $\sigma' = \rho^k$, the corresponding sets for the candidates of $\sigma$ are disjoint. Hence, we have $|\text{Fix}_{(\rho^*, 1)}| = \phi(o(k))(\iota(k) - 1)!o(k)^{\iota(k) - 1}$.

(2) Let $n$ be odd. Then, $\rho^k \tau$ is a reflection of $C_n$ about the axis through a vertex and the middle point of its opposite edge for any $k$, hence, $\rho^k \tau$ has a unique fixed point. If $n$ is even and $k$ is odd, then $\rho^k \tau$ is a reflection of $C_n$ about
the axes through two opposite vertices, hence \( \rho^k \tau \) has two fixed points. But, if \( \sigma \in {\text{Fix}}(\rho^k \tau, 1) \), then \((\rho^k \tau)\sigma(\rho^k \tau)^{-1} = \sigma \) and \( \sigma' = \rho^k \tau \) for some \( 1 < n \). But for any \( 1 < n \), \( \sigma' \) has no fixed point. Hence, in either case \( {\text{Fix}}(\rho^k \tau, 1) = \emptyset \).

Finally, let both \( n \) and \( k \) be even. Then, \( \rho^k \tau \) is a reflection of the \( n \)-cycle \( C_n \) about the axis through the middle points of two opposite edges, hence \( |{\text{Fix}}(\tau, 1)| = |{\text{Fix}}(\rho^k \tau, 1)| \) for any even \( k \). Let \( \sigma \in {\text{Fix}}(\tau, 1) \), then \( \tau \sigma \tau^{-1} = \sigma \) and \( \sigma = \rho^k \tau \) by Lemma 1. Since \( \sigma^{-1} = \tau \) is a product of \( n/2 \) mutually disjoint cycles of length 2, a similar method to (1) gives

\[
|{\text{Fix}}(\rho^k \tau, 1)| = (n/2 - 1)!2^{n/2 - 1}.
\]

(3) Let \( \sigma = (a_1 a_2 \cdots a_n) \in \Sigma_n \) be an element of \( {\text{Fix}}(\rho^k \tau, 1) \), that is, \( \sigma = \rho^k \sigma^{-1} \rho^{-k} \). Then \((a_n a_{n-1} \cdots a_1) = \sigma^{-1} = \rho^k \sigma \rho^{-k} \). By Lemma 1, \( \rho^k \) is the identity in \( S_n \). Hence, \( k \) must be equal to \( n/2 \), \( n \) even, and

\[
(a_n a_{n-1} \cdots a_1) = \rho^k \sigma \rho^{-k} = ((a_1 + n/2)(a_2 + n/2) \cdots (a_n + n/2)).
\]

If \( a_1 + n/2 = a_l \) for some \( l \), then \( l \) must be even, because if \( l \) is odd, then

\[
a_{(l+1)/2} + n/2 = a_{(l-1)/2} + 1 = a_{(l+1)/2},
\]

which is impossible. Thus for a given \( a_1 \), there are exactly \( n/2 \) candidates for \( a_1 + n/2 \) in \( \{a_1, a_2, \ldots, a_n\} \). Set \( a_1 + n/2 = a_m \), \( m \) even. There are exactly \( n/2 \) candidates for \( a_2 \), and \( a_{m-1} \) is given uniquely as \( a_{2} + n/2 \), and there are exactly \( n - 4 \) candidates for \( a_3 \), and so on. Hence, the number \( |{\text{Fix}}(\rho^k \tau, 1)| \) of all candidates for \( \sigma = (a_1 a_2 \cdots a_n) \) in \( {\text{Fix}}(\rho^k \tau, 1) \), or equivalently

\[
(a_n a_{n-1} \cdots a_1) = ((a_1 + n/2)(a_2 + n/2) \cdots (a_n + n/2)),
\]

is equal to \((n/2)(n-2)(n-4) \cdots 2 = (n/2)!2^{n/2 - 1}\).

(4) Let \( n \) be odd. Then \( \rho^k \tau \) is a reflection of \( C_n \) about the axis through a vertex and the middle point of its opposite edge for any \( k \). Hence \( |{\text{Fix}}(\tau, 1)| = |{\text{Fix}}(\rho^k \tau, 1)| \) for any \( k \). Let \( \sigma = (a_1 a_2 \cdots a_n) \). Then \( \sigma \in {\text{Fix}}(\tau, 1) \) if and only if \( \tau \sigma \tau^{-1} = \sigma^{-1} \), i.e.,

\[
(\tau(a_1) \tau(a_2) \cdots \tau(a_n)) = (a_n a_{n-1} \cdots a_1). \]

Since \( \tau \) has a unique fixed point \((n + 1)/2\), we can assume that \( \tau(a_1) = a_1 = (n + 1)/2 \). Then there are \( n - 1 \) candidates for \( a_2 \) and if \( a_2 \) is given then \( \tau(a_2) = a_n \) is fixed, and next there are \( n - 3 \) candidates for \( a_3 \) and if \( a_3 \) is given then \( \tau(a_3) = a_{n-1} \) is fixed, and so on. Hence, the number \( |{\text{Fix}}(\tau, 1)| \), the number of all candidates of \( \sigma \) with \( \tau \sigma \tau^{-1} = \sigma^{-1} \), is

\[
(n-1)(n-3) \cdots 2 = 2^{(n-1)/2}((n-1)/2)!. \]

Next, let \( n \) be even. Then \( \rho^k \tau \) is a reflection of \( C_n \) without a fixed point, if \( k \) is even. Hence a similar method used in (3) gives \( |{\text{Fix}}(\rho^k \tau, 1)| = 2^{n/2 - 1}(n/2)! \) for any even \( k \). For any odd \( k \), \( \rho^k \tau \) is a reflection of \( C_n \) having two fixed points. Hence

\[
|{\text{Fix}}(\rho^k \tau, 1)| = |{\text{Fix}}(\rho^k \tau, 1)|
\]

for any odd \( k \). In particular, \( \rho \tau \) fixes two points \( 1 \) and \( n/2 + 1 \). We also can see that

\[
|{\text{Fix}}(\rho^k \tau, 1)| = (n/2 - 1)!2^{n/2 - 1}
\]

for any odd \( k \). □
By Theorem 3 and Lemma 4, we have the following.

**Theorem 4.** (1) \( \text{Iso}_P(C_2) = \text{Iso}(C_2) = 1. \)

(2) If \( n \) is even greater than 2,

\[
4n \text{ Iso}_P(C_n) = (n - 1)! + \sum_{k=1}^{n-1} \phi(o(k))(i(k) - 1)!o(k)^{i(k)-1}
\]

\[
+ \frac{n}{2} \left( 3 + \frac{n}{2} \right) 2^{\frac{n^2}{2} - 1}.
\]

(3) If \( n \) is odd,

\[
4n \text{ Iso}_P(C_n) = (n - 1)! + \sum_{k=1}^{n-1} \phi(o(k))(i(k) - 1)!o(k)^{i(k)-1}
\]

\[
+ n2^{\left( \frac{n^2}{2} - 1 \right)} \left( \frac{n-1}{2} \right)!
\]

**Corollary 3.** (1) For a prime \( q \),

\[
\text{Iso}_P(C_q) = \begin{cases} 
1 & \text{if } q = 2, \\
\frac{1}{4q} \left[ (q - 1)! + (q - 1)^2 + q2^{(q-1)/2} \left( \frac{q-1}{2} \right)! \right] & \text{otherwise.}
\end{cases}
\]

(2) \( \frac{1}{2} \text{ Iso}_P(C_n) < \text{Iso}(C_n) \leq \text{Iso}_P(C_n). \)

A short calculation gives the following table for \( \text{Iso}_P(C_n) \):

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<th>( \text{Iso}_P(C_n) )</th>
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<td>( \cdots )</td>
</tr>
</tbody>
</table>

**Example.** Let \( n = 5 \), and let \( \text{id} \) denote the identity in \( S_5 \). By Corollary 3, we have \( \text{Iso}_P(C_5) = 4 \), and these four non-isomorphic 5-cyclic permutation graphs are given in Fig. 2 with their representative permutations \( \alpha \). Also, we can find the number of distinct 5-cyclic permutation graphs isomorphic to each of them, by Theorem 3, as follows:

\( \text{id} \) \( D_5 \) \( \text{id} \cap D_3 \neq \emptyset \), \( |\Gamma_{\text{id}^{-1}\text{id}}| = 10 \), and

\( N_P(\text{id}) = N_P(\text{id}) = N(\text{id}) = 10; \)

\( (2354)D_5(2354) \cap D_3 \neq \emptyset \), \( |\Gamma_{(2354)^{-1}2(2354)}| = 10 \), and

\( N_P((2354)) = N_P((2354)) = N((2354)) = 10; \)

\( (35)D_5(35) \cap D_5 \neq \emptyset \), \( |\Gamma_{35}\rho(35)| = 2 \), and

\( N_P((35)) = N_P((35)) = N((35)) = 50; \)

\( (23)(45)D_5(23)(45) \cap D_5 \neq \emptyset \), \( |\Gamma_{(23)(45)^{-1}2(23)(45)}| = 2 \), and

\( N_P((23)(45)) = N_P((23)(45)) = N((23)(45)) = 50. \)
Moreover,

\[ \text{Iso}_p(C_5) = \text{Iso}(C_5) = 4. \]

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**References**