Quantum Separability of the vacuum for Scalar Fields with a Boundary

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Using the Green's function approach we investigate separability of the vacuum state of a massless scalar field with a single Dirichlet boundary. Separability is demonstrated using the positive partial transpose criterion for effective two-mode Gaussian states of collective operators. In contrast to the vacuum energy, entanglement of the vacuum is not modified by the presence of the boundary.

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Recently theories and experiments in quantum information science[1, 2] have been progressed significantly. This has led to interest in studying entanglement in many-particle systems such as Bose Einstein condensations[3], Fermion systems[4, 5],lattice systems[6], thermal boson systems[7, 8] and even in the vacuum[9, 10, 11]. Entanglement is now treated as a physical quantity, as well as a resource for quantum information processing. On the other hand, scalar particles such as the Higgs particles in the standard model are essential ingredients of modern physics. Although it is well known that the vacuums in some quantum field models have quantum nonlocality[12], the existence of entanglement of the general vacuum is still controversial. From the particle physics viewpoint, everything in the universe is made of quantum fields and massless non-interacting quantum scalar field is the simplest and, hence, the most basic quantum field. Furthermore, scalar fields can be treated as an infinite-mode extension of harmonic oscillators[13]. Therefore, studying entanglement of the vacuum state of scalar fields is very important. For decades the nature of the vacuum, concerning the Casimir energy[14, 15], in a bounded space has been extensively studied using the Green's function method[16]. In this paper, we use this Green's function approach to investigate the quantum entanglement of the vacuum for non-interacting scalar fields with an infinite Dirichlet boundary (i.e., fields are constrained to vanish at the boundary). Since the scalar fields are continuous variables we need a separability criterion for continuous variables. Recently there has been a renewal of interest in entanglement of Gaussian states in the context of continuous variable quantum information, and entanglement measures such as the purity[17] or the negativity has been suggested[18] for two-mode Gaussian states. In this work we shall average the fields over two tiny boxes to make the vacuum state an effective two-mode Gaussian state.

We start by introducing the massless (real) Klein-Gordon fields, which are described by a Lagrangian density,

$$\mathcal{L}(\vec{x},t)=\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi,\quad(\alpha=t,x,y,z),\tag{1}$$

which leads to the equation of motion $\partial_\alpha\partial^\alpha\phi=0$ in unbounded four-dimensional Minkowski spacetime. In this free-space (without a boundary) the quantum field operator of the scalar field can be expanded as

$$\phi(\vec{x},t)=\int\frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}}\left(a(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega_k t}+H.C.\right),\tag{2}$$

where $\omega_k=|\vec{k}|$. The equal time commutation relations of the scalar field are

$$\begin{aligned}[\phi(\vec{x},t),\phi(\vec{x}',t)]&=0,[\pi(\vec{x},t),\pi(\vec{x}',t)]=0,\\ [\phi(\vec{x},t),\pi(\vec{x}',t)]&=i\delta(\vec{x}-\vec{x}'),\end{aligned}\tag{3}$$

where $\pi(\vec{x},t)\equiv\partial_t\phi(\vec{x},t)$ is the momentum operator for $\phi(\vec{x},t)$. Consider a vector of the field and its momentum operator at two points $\vec{x}$ and $\vec{x}'$ at a time t, i. e., $\xi=(\phi(\vec{x},t),\pi(\vec{x},t),\phi(\vec{x}',t),\pi(\vec{x}',t))$. The vacuum for the scalar field ($|0\rangle$) has a variance matrix having the following form:

$$V_{\alpha\beta}\equiv\frac{1}{2}\langle 0|\{\triangle\xi_\alpha,\triangle\xi_\beta\}|0\rangle,\tag{4}$$

where $\{A,B\}=AB+BA$ and $\triangle\xi_\alpha\equiv\xi_\alpha-\langle 0|\xi_\alpha|0\rangle$ with $\langle 0|\xi_\alpha|0\rangle=0$ in this paper. This matrix represents zero-temperature quantum fluctuation of the vacuum. Using the equal time commutation relations one can easily note that $\langle 0|\xi_1\xi_3|0\rangle=\langle 0|\phi(\vec{x},t)\phi(\vec{x}',t)|0\rangle=\langle 0|\xi_3\xi_1|0\rangle$, hence $V_{13}=V_{31}$. For systems having time-translational symmetry, like ours, $\partial_t\langle 0|\phi(\vec{x},t)\phi(\vec{x},t)|0\rangle=0$. Therefore, $\langle 0|\xi_1\xi_2|0\rangle=\langle 0|\phi(\vec{x},t)\pi(\vec{x},t)|0\rangle=-\langle 0|\xi_2\xi_1|0\rangle$, and $V_{12}=V_{21}=0$. Hence the variance matrix has a following form

$$V=\begin{bmatrix}a&0&c&0\\0&b&0&d\\c&0&a'&0\\0&d&0&b'\end{bmatrix}\tag{5}$$

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where $a = \langle 0|\{\phi(\vec{x},t),\phi(\vec{x},t)\}|0\rangle/2$, $b = \langle 0|\{\pi(\vec{x},t),\pi(\vec{x},t)\}|0\rangle/2$, $a' = \langle 0|\{\phi(\vec{x'},t),\phi(\vec{x'},t)\}|0\rangle/2$, and so on. Note that for the free-space field operator (Eq. (2)) a and a' term diverge. The divergence, however, can be made disappear if we consider a scalar field with a boundary. The effect of the boundary can be calculated by subtracting a free-space Green's function from the Green's function with a boundary (See Eq.(8) and below). Divergent terms which are in the both Green's functions cancel each other. The components of V can be calculated from the Green's function called the Hadamard's elementary function [19]

$$G(\vec{x},t;\vec{x}',t') = \langle 0|\{\phi(\vec{x},t),\phi(\vec{x}',t')\}|0\rangle. \tag{6}$$

Once we know the Green's function we can calculate the components of the variance matrix V from it, i.e.,

$$\begin{aligned} c(\vec{x},\vec{x}') &= \frac{1}{2}\langle 0|\{\phi(\vec{x},t),\phi(\vec{x}',t)\}|0\rangle \\ &= \lim_{t\to t'}\frac{1}{2}G(\vec{x},t;\vec{x}',t'), \\ d(\vec{x},\vec{x}') &= \frac{1}{2}\langle 0|\{\partial_t\phi(\vec{x},t),\partial_t\phi(\vec{x}',t)\}|0\rangle \\ &= \lim_{t\to t'}\partial_t\partial_{t'}\frac{1}{2}G(\vec{x},t;\vec{x}',t'). \end{aligned} \tag{7}$$

Then, $a = \lim_{\vec{x}'\to\vec{x}} c(\vec{x},\vec{x}')$, $b = \lim_{\vec{x}'\to\vec{x}} d(\vec{x},\vec{x}')$ and so on.

With the boundary the mode expansion in Eq. (2) is no longer valid. Instead one can directly work with a modified Green's function to obtain V as follows. First let us choose two points $\vec{x} = (x,y,z)$ and $\vec{x'} = (x',y',z')$. Using the method of images[20] one can obtain a scalar field Green's function G_B with a Dirichlet boundary positioned at $z=0$ [19];

$$G_B = G_0 - \frac{1}{2\,\pi^2\left(-(t-t')^2 + r^2 + (z+z')^2\right)}, \tag{8}$$

where the free-space Green's function is $G_0 = [2\,\pi^2\left(-(t-t')^2 + r^2 + (z-z')^2\right)]^{-1}$ and $r^2 \equiv (x-x')^2 + (y-y')^2$. By subtracting the free-space Green's function from G_B we obtain a regularized Green's function; $G = G_B - G_0$ which is not divergent. From G one can obtain the components of V using Eq. (7);

$$\begin{aligned}(a,b,a',b',c,d) = \frac{1}{2\pi^2}(&\frac{-1}{8\,z^2},\frac{1}{16\,z^4},\frac{-1}{8\,z'^2},\frac{1}{16\,z'^4}, \\ &\frac{-1}{2\left(r^2+(z+z')^2\right)},\frac{1}{\left(r^2+(z+z')^2\right)^2}).\end{aligned} \tag{9}$$

Now we discuss separability of the vacuum. Since 'if and only if ' separability test for infinite-mode states are unknown, we need to reduce the infinite-mode states to effective two-mode Gaussian states. We follow the approach in ref. [13], i.e., we spatially average the field operator over two tiny boxes centered at $\vec{x}$ and $\vec{x}'$, respectively (See Fig. 1). Defining these collective operators is reasonable, because in a physical situation probes always have finite spatial resolution. We, however, do not average but integrate the momentum operators within the box, since momentum is additive. Thus the collective operators are

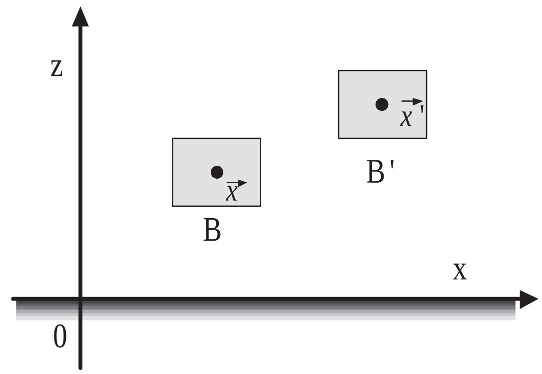

FIG. 1: We average fields over two tiny boxes B and B' centered at $\vec{x}$ and $\vec{x}'$, respectively. The Dirichlet boundary is at $z=0$.

$$\begin{aligned}\tilde{\xi} &\equiv (\Phi(\vec{x},t),\Pi(\vec{x},t),\Phi(\vec{x}',t),\Pi(\vec{x}',t)) \\ &= (\frac{1}{L^3}\int_B d^3\vec{y}\phi(\vec{x}+\vec{y},t),\int_B d^3\vec{y}\pi(\vec{x}+\vec{y},t), \\ &\quad \frac{1}{L^3}\int_{B'} d^3\vec{y'}\phi(\vec{x}'+\vec{y'},t),\int_{B'} d^3\vec{y'}\pi(\vec{x}'+\vec{y'},t)).\end{aligned} \tag{10}$$

Here $\int_B d^3\vec{y} f(\vec{x}+\vec{y})$ denotes an integration of $f(\vec{x})$ over a box B centered at $\vec{x}$ with volume L^3. Then, the commutation relations in Eq. (3) reduce to the canonical commutation relations for the collective operators [13]

$$[\tilde{\xi}_\alpha,\tilde{\xi}_\beta] = i\Omega_{\alpha\beta}, \tag{11}$$

where

$$\Omega = \begin{bmatrix} J & 0 \\ 0 & J\end{bmatrix}, J = \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}. \tag{12}$$

Similarly we define an variance matrix for the collective operators $\tilde{V}_{\alpha\beta} \equiv \frac{1}{2}\langle 0|\{\tilde{\xi}_\alpha,\tilde{\xi}_\beta\}|0\rangle$.

Now we show that, using the fundamental theorem of calculus[21], one can calculate the variance matrix $\tilde{V}$ for the two-mode Gaussian states with those of V in the limit $L\to 0$ (but still non-zero). For example,

$$\begin{aligned}\tilde{V}_{13} &= \lim_{L\to 0}\frac{1}{2}\langle 0|\{\Phi(\vec{x},t),\Phi(\vec{x'},t)\}|0\rangle \\ &= \lim_{L\to 0}\frac{1}{2L^6}\int_B d^3\vec{y}\int_{B'} d^3\vec{y'}\langle 0|\{\phi(\vec{x}+\vec{y},t),\phi(\vec{x'}+\vec{y'},t)\}|0\rangle \\ &= \frac{1}{2}\langle 0|\{\phi(\vec{x},t),\phi(\vec{x'},t)\}|0\rangle = V_{13} = c.\end{aligned} \tag{13}$$

This is possible since the integrand of the second line (i. e., c term in Eq. (9)) is continuous on all the region

$z > 0$. (Due to the boundary, it is enough to consider only the half-space $z > 0$.) Similarly

$$\tilde{V}_{24} = \lim_{L\to 0} L^6 \frac{1}{2L^6} \int_B d^3\vec{y} \int_{B'} d^3\vec{y}' \langle 0|\{\pi(\vec{x}+\vec{y},t) \pi(\vec{x}'+\vec{y}',t)\}|0\rangle = L^6 V_{24} = L^6 d. \quad (14)$$

Hence,

$$\tilde{V} = \begin{bmatrix} a & 0 & c & 0 \\ 0 & L^6 b & 0 & L^6 d \\ c & 0 & a' & 0 \\ 0 & L^6 d & 0 & L^6 b' \end{bmatrix} \equiv \begin{bmatrix} A & G \\ G^T & B \end{bmatrix}. \quad (15)$$

Obviously, our formalism does not work for $L = 0$, but for L infinitesimally small the above equation gives an exact result. The separability criterion we use in this paper is positive partial transpose (PPT) criterion[22, 23] for two-mode Gaussian states[24] which is equivalent to

$$F \equiv \tilde{\Sigma} - (\frac{1}{4} + 4det\tilde{V}) \le 0, \quad (16)$$

where $\tilde{\Sigma} = detA + detB - 2detG$. By inserting the components into Eq.(16) we obtain

$$\begin{aligned} F &= -\frac{1}{4} \quad (17) \\ &- \frac{L^6}{2^2\pi^4}\left[\frac{1}{2^7 z^6} + \frac{1}{2^7 z'^6} - \frac{1}{[r^2+(z+z')^2]^3}\right] \\ &- \frac{L^{12}}{2^6\pi^8}[\frac{1}{2^{12} z^6 z'^6} - \frac{1}{2^4 z^2 z'^2 (r^2+(z+z')^2)^4} \\ &- \frac{1}{2^8 z^4 z'^4 (r^2+(z+z')^2)^2} + \frac{1}{(r^2+(z+z')^2)^6}] \\ &\le -\frac{1}{4}. \end{aligned}$$

One can obtain this inequality using two absolute inequalities, $1/(X+Y)^n \le (1/X^n + 1/Y^n)/2^{n+1}$ and $X^3 + Y^3 \ge XY^2 + YX^2$ $(X, Y > 0)$ on the two square brackets, respectively. The maximum value is achieved when $\vec{x} = \vec{x}'$ $(r = 0)$. Therefore, the effective two-mode scalar field vacuum with a boundary is PPT and, hence, separable, when it is described by the variance matrix $\tilde{V}$ and the regularized Green's function G. Our results, however, do not rule out the possibility of entanglement of the vacuum for free-space scalar fields, since we have subtracted the free-space Green's function which might give entanglement. What our results really imply is that the presence of a single boundary does not change the separability or the entanglement which the vacuum for the free-space scalar fields may have, while the presence of the boundary modifies the vacuum energy to $-1/16\pi^2 z^4$[19].

Entanglement of the scalar field could be experimentally tested by the scheme with trapped ions[9] or Bose-Einstein condensates[25]. Our approach provides a new method using the Hadamard's elementary function to investigate entanglement of the quantum fields vacuum within a bounded space. It will be interesting to investigate how the number of boundaries and properties of the fields such as interactions, masses, charges, and spins change the results.

After completion of our work we found that there appears a paper about spatial entanglement of free thermal bosonic fields (quant-ph/0607069).

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