The Complexity of Colouring by Locally Semicomplete Digraphs

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Dedicated to Carsten Thomassen on the occasion of his 60th birthday.

Abstract

In this paper we establish a dichotomy theorem for the complexity of homomorphisms to fixed locally semicomplete digraphs. It is also shown that the same dichotomy holds for list homomorphisms. The polynomial algorithms follow from a different, shorter proof of a result by Gutjahr, Welzl and Woeginger.

Keywords: digraph homomorphism, complexity, polynomial algorithm, NP-completeness, locally semicomplete digraphs.

1 Introduction

Let $H$ be a fixed directed graph. A homomorphism from a digraph $D$ to $H$ is a mapping $f : V(D) \to V(H)$ such that $xy \in A(D)$ implies that $f(x)f(y) \in A(H)$. The existence of such a homomorphism is denoted by $D \rightarrow H$.

The $H$-colouring problem ($\text{HOM}_H$) is the problem of deciding whether a homomorphism exists from an input digraph $D$ to the target digraph $H$. 
Problem 1.1 $HOM_H$

**Instance:** A digraph $D$.

**Question:** Does there exist a homomorphism $f : D \rightarrow H$?

In the case where $H$ is an undirected graph we have the following result by Hell and Nešetřil.

**Theorem 1.1** (Hell and Nešetřil [9, 10]). Let $H$ be a graph with loops allowed.

- If $H$ is bipartite or contains a loop, then the $H$-colouring problem has a polynomial time algorithm.
- Otherwise the $H$-colouring problem is NP-complete.

It has been a goal of researchers to try and extend the result by Hell and Nešetřil to the directed case. This seems to be a very hard problem and only partial results are known. One example of this is the theorem by Bang-Jensen, Hell and MacGillivray [4] on the complexity of colouring by semicomplete digraphs. A *semicomplete* digraph has the property that between every pair of vertices there is at least one arc; parallel arcs and loops are not allowed, but a pair of symmetric arcs is allowed.

**Theorem 1.2** (Bang-Jensen, Hell and MacGillivray [4]). Let $H$ be a semicomplete digraph.

- If $H$ contains at most one directed cycle, then $H$-colouring is polynomial time solvable.
- Otherwise $H$-colouring is NP-complete.

There is the related notion of a locally semicomplete digraph. A digraph $H$, is said to be *locally semicomplete* if for every vertex $v$ of $H$, both $N^+(v)$ and $N^-(v)$ induce semicomplete digraphs. A special case of this is that of a local tournament. A *local tournament*, $H$, is a digraph such that between every pair of vertices there is at most one arc and that for every vertex $v$ of $H$ both $N^+(v)$ and $N^-(v)$ induce tournaments.

Bang-Jensen introduced the notion of locally semicomplete digraphs in [1] where it was shown that many of the known results on tournaments generalize to this family of digraphs. Since their introduction locally semicomplete digraphs have been studied by many authors, see [2] for a large collection of results on these digraphs.
We will see that Theorem 1.2 doesn’t generalize in the way one would expect. In particular there are unicyclic locally semicomplete digraphs such that the corresponding colouring problem is NP-complete.

If \( H' \) is a subdigraph of \( H \), then a **retraction** of \( H \) to \( H' \) is a homomorphism \( \rho : H \to H' \) such that \( \rho(x) = x \) for every \( x \in V(H') \). In this case we say that \( H \) **retracts to** \( H' \) or that \( H' \) is a **retract** of \( H \). A digraph \( H \) is said to be a **core** if \( H \) does not retract to a proper subdigraph. It turns out that a digraph is a core if and only if it is not homomorphic to a proper subdigraph and that every digraph \( H \) has a unique retract that is also a core [10]. This retract is called the **core** of \( H \). If \( H' \) is the core of \( H \), then \( H \) and \( H' \) have equivalent homomorphism problems: \( G \to H \) if and only if \( G \to H' \).

A digraph \( H \) is said to be **smooth** if there are no sources or sinks present in \( H \). The complexity of \( \text{HOM}_H \) in this case was conjectured by Bang-Jensen and Hell in [3] and proved recently in [6].

**Theorem 1.3** (Barto, Kozik, and Niven [6]). Let \( H \) be a smooth digraph. If the core of \( H \) is a directed cycle, then \( H \)-colouring is in \( P \). Otherwise \( H \)-colouring is NP-complete.

In the case where the locally semicomplete digraph has at least two directed cycles, NP-completeness of \( \text{HOM}_H \) follows from Theorem 1.3. For acyclic locally semicomplete digraphs \( H \), polynomial algorithms for \( \text{HOM}_H \) follow from [8] and [11]. When \( H \) has a unique directed cycle we determine the dividing line and show that the polynomial algorithms again follow from results in [8]. We give a different, shorter, proof of the results in [8] that allows the algorithms to be used in the list-homomorphism context. In a personal communication, Pavol Hell mentioned that this proof has also been found by other researchers.

We would also like to point out that we have a graph theoretic proof of the NP-completeness of \( \text{HOM}_H \) in the case where \( H \) is a local tournament with at least two directed cycles. These results will be appearing elsewhere [5].

For terms not defined in this paper, the reader may consult [2] for digraphs, [10] for homomorphisms and [7] for complexity theory.

2 Some Tools

Hell and Nešetřil [9] introduced a number of powerful tools for proving that a given digraph has an NP-complete homomorphism problem. The aim of this section is to introduce these and other tools that will be useful in our proof.
2.1 The (Vertex) Sub-indicator Construction

Let \( J \) be a fixed digraph with specified vertices \( k_1, k_2, \ldots, k_t \) and \( j \). The sub-indicator construction (with respect to the sub-indicator \( J, k_1, k_2, \ldots, k_t, j \)) transforms a digraph \( H \) with specified vertices \( x_1, x_2, \ldots, x_t \) to an induced subdigraph \( H^+ \) (of \( H \)) defined as follows. Let \( W \) be the digraph obtained from a copy \( H \) and a copy of \( J \) by identifying each \( k_i \) with the corresponding \( x_i \) for \( i = 1, 2, \ldots, t \). Then \( H^+ \) is the subdigraph of \( H \) induced by those vertices \( u \) for which some retraction of \( W \) to \( H \) maps \( j \) to \( u \).

**Lemma 2.1** (Hell and Nešetřil [9, 10]). Let \( H \) be a digraph that is a core. If the \( H^+ \)-colouring problem is NP-complete, then the \( H \)-colouring problem is also NP-complete.

Often, when using the sub-indicator construction, one takes the vertices \( k_1, k_2, \ldots, k_t \) above to be a set of isolated vertices in \( J \). This has the effect that the digraph \( W \) above is \( H \cup (J - \{k_1, k_2, \ldots, k_t\}) \). In considering retractions of \( W \) to \( H \), we see that we are actually considering homomorphisms of \( J - \{k_1, k_2, \ldots, k_t\} \) to \( H \).

2.2 The Consistency Check

Suppose that \( H \) is a fixed digraph that will act as the target in a homomorphism problem. As input to the problem we have a digraph \( G \). In trying to find a homomorphism \( G \to H \), we may start the process by assigning a list \( L(v) = V(H) \) to each vertex \( v \) of \( G \). These lists record possible images for the vertices of \( G \) and initially every vertex of \( H \) is a possible image for any given vertex of \( G \). The algorithm we describe in this section processes each list \( L(v), v \in V(G) \), by removing any vertices from \( L(v) \) that cannot possibly be images of \( v \).

The lists attached to each vertex of \( G \) are said to be consistent if for any arc \( uv \) of \( G \) the following two properties hold:

- for any \( x \in L(u) \), there exists \( y \in L(v) \) such that \( xy \) is an arc of \( H \) and
- for any \( b \in L(v) \), there exists \( a \in L(u) \) such that \( ab \) is an arc of \( H \).

The goal of the consistency check (Algorithm 2.1) is to reduce the initial lists to ones that are consistent. Our presentation here follows [10].

The consistency check is often used as a building block in designing polynomial time algorithms for the digraph homomorphism problem [10]. It is also known as the arc-consistency check since it checks for consistency across arcs of \( G \). Higher order consistency checks are also possible [10].
Algorithm 2.1 The Consistency Check

**INPUT:** A digraph $G$ with lists $L(v) = V(H)$, $v \in V(G)$.

**TASK:** Reduce the lists to $L^*(v) \subseteq V(H)$, $v \in V(G)$, that are consistent.

**ACTION:** Initially set all lists $L^*(v) = L(v)$, and then, as long as changes occur, process each arc $uv$ of $G$ repeatedly as follows: remove from $L^*(u)$ any $x$ for which no element $y \in L(v)$ has $xy$ an arc in $H$, and remove from $L^*(v)$ any $b$ for which no $a \in L^*(u)$ has $ab$ an arc in $H$.

2.3 The $X$-enumeration and the Graft Extension

An enumeration $\{h_1, h_2, \ldots, h_n\}$ of the vertices of a digraph $H$ is called an $X$-enumeration if the following property holds: if $h_i h_j$ and $h_k h_l$ are arcs of $H$, then $h_{\min\{i,k\}} h_{\min\{j,l\}}$ is also an arc of $H$.

**Theorem 2.2** (Gutjahr, Woeginger and Welzl [8]). Let $H$ be a digraph such that $H$ admits an $X$-enumeration. Then the $H$-colouring problem is solvable in polynomial time.

This result follows by running the consistency check on the input digraph (this is not the original algorithm presented in [8]). If a list becomes empty at any point during the consistency check, then there is no homomorphism to the target. If, on the other hand, the resulting lists are nonempty the minimum element (with respect to the $X$-enumeration) in each list defines a homomorphism to the target [10].

There is one more result from [8] that we need, the so-called graft extension. We will consider a slightly more general problem than the one presented in [8]. Gutjahr, Woeginger and Welzl [8] only considered the $H$-colouring problem in their paper, $\text{HOM}_H$. We will show here that their result actually applies to a more general problem, namely that of list homomorphisms.

The list homomorphism problem with target $H$ is the following decision problem ($\text{LIST-HOM}_H$).

**Problem 2.1** $\text{LIST-HOM}_H$

| INSTANCE: | $(G, L)$: A digraph $G$ with lists $L(v) \subseteq V(H)$, $v \in V(G)$. |
|QUESTION: | Does there exist a homomorphism $f : G \rightarrow H$ such that $f(v) \in L(v)$ for every $v \in V(G)$? |

Let $H_1$ be a loop-free digraph that has an $X$-enumeration, say $\{h_1, h_2, \ldots, h_n\}$. Let $H_2$ be a digraph such that $H_2$-colouring is polynomial. We form a new digraph $H$
by deleting the vertex $h_n$ from $H_1$ and replacing it by the digraph $H_2$: every vertex $h_i \in V(H_1)$ that is adjacent to (from) $h_n$ is now adjacent to (from) every vertex in $H_2$. The digraph $H$ is called the $X$-graft($H_1, H_2$).

**Theorem 2.3.** Let $H = \text{X-graft}(H_1, H_2)$ such that $\text{LIST-HOM}_{H_2}$ is polynomial. Then $\text{LIST-HOM}_H$ is polynomial.

**Proof.** Let $(G, \mathcal{L})$ be an instance of $\text{LIST-HOM}_H$. Modify $H_1$ by adding a loop at $h_n$. Now clearly, the existence of a homomorphism of $G$ to (the new) $H_1$ is a necessary condition for the existence of a homomorphism of $G$ to $H$.

First, alter the lists by replacing any vertices of $H_2$ by $h_n$ in any list where they occur. Now, apply the arc-consistency check. If it fails, then $G$ is a NO-instance. Hence assume the consistency check succeeds. By the discussion earlier, the mapping $f(x) = \min L(x)$, where the minimum is with respect to the $X$-enumeration, is a list-homomorphism $G \to H_1$.

Since the minimum element in each list is chosen, the vertices that map to $h_n$ in this list-homomorphism of $G$ to $H_1$ map to vertices of $H_2$ in any list-homomorphism of $G$ to $H$. By the construction of $H$ and the consistency check, there is now a list-homomorphism of $G$ to $H$ if and only if the subdigraph of $G$ induced by $f^{-1}(h_n)$ admits a list-homomorphism to $H_2$, where the lists are the intersections of the initially given lists with $V(H_2)$. As $\text{LIST-HOM}_{H_2}$ is polynomial, the result follows.

**Corollary 2.4** (Gutjahr, Woeginger and Welzl [8]). Let $H = \text{X-graft}(H_1, H_2)$ such that $\text{HOM}_{H_2}$ is polynomial. Then $\text{HOM}_H$ is polynomial.

This result follows since $\text{HOM}_H$ is a special case of $\text{LIST-HOM}_H$ where each list $L(v) = V(H)$ for each $v \in V(G)$.

The following result of Bang-Jensen is useful in proving that connected locally semicomplete digraphs are cores.

**Lemma 2.5** (Bang-Jensen [1, 2]). A locally semicomplete digraph has a hamilton path if and only if its underlying graph is connected.

**Proposition 2.6.** A connected locally semicomplete digraph $D$ is a core.

**Proof.** By Lemma 2.5, $D$ has a hamilton path, say $v_1, v_2, \ldots, v_n$.

Let $f : D \to D'$ be a retraction of $D$ where $D'$ is a subdigraph of $D$. Let $v$ be a vertex in $D'$. By the enumeration above $v = v_i$ for some $i \in \{1, 2, \ldots, n\}$ and by the retraction $f(v_i) = v_i$.

Consider the vertices $v_{i-1}$ and $v_{i+1}$ (at least one of these exists), say $v_{i-1}$. Let $f(v_{i-1}) = v_j$, $j \in \{1, 2, \ldots, n\}$. Clearly $j \neq i$. Since $v_{i-1}v_i \in A(D)$, $f(v_{i-1})f(v_i) =
\(v_j v_i \in A(D)\). That is, \(v_{i-1}, v_j \in N^-(v_i)\). Therefore by the locally semicomplete property of \(D\), \(v_{i-1}\) and \(v_j\) are adjacent or \(j = i - 1\). If \(i - 1 \neq j\), the arc(s) between \(v_{i-1}\) and \(v_j\) are not preserved by \(f\). So the image of \(v_{i-1}\) is exactly \(v_{i-1}\) under the retraction. If \(v_{i+1}\) exists, the same reasoning shows that \(f(v_{i+1}) = v_{i+1}\).

Repeating the argument along the hamilton path eventually shows that every vertex has to map to itself. Therefore \(D\) is a core.

3 Connected vs. Disconnected Locally Semicomplete Digraphs

When \(H\) is a disconnected locally semicomplete digraph such that each component of \(H\) (all of which are locally semicomplete digraphs themselves) is polynomial time solvable, then \(\text{HOM}_H\) is also polynomial time solvable. NP-completeness results for disconnected locally semicomplete digraphs are much harder to obtain.

The NP-completeness results that follow are all for connected locally semicomplete digraphs since we only give polynomial time transformations from NP-complete problems to \(\text{HOM}_H\) when \(H\) is connected. A natural conjecture would be: if a disconnected locally semicomplete digraph \(H\) contains at least one component that is NP-complete, then \(\text{HOM}_H\) is NP-complete. The difficulty in proving this lies with constructing a polynomial transformation from some NP-complete problem. In general, it is hard to set up the transformation without forcing certain vertices of the transformed instance to map to a specific component of \(H\). This may have the unintended consequence of restricting the images of one or more vertices of the transformed instance too severely.

On the other hand one can easily obtain a polynomial time Turing reduction. Here, an instance of some NP-complete problem \(Q\) is transformed into many different instances of \(\text{HOM}_H\). The transformation has to run in polynomial time and furthermore an instance \(I\) of \(Q\) is a yes instance if and only if at least one the transformed instances is a yes instance of \(\text{HOM}_H\). This shows that for a disconnected locally semicomplete digraph with at least one NP-complete component, \(\text{HOM}_H\) is polynomial time solvable if and only if \(P=NP\). Therefore solving \(\text{HOM}_H\) in polynomial time for a disconnected locally semicomplete digraph \(H\) with at least one NP-complete component, is highly unlikely.

Let \(H\) be a disconnected locally semicomplete digraph that is also a core and let \(H'\) be a component of \(H\) such that \(\text{HOM}_{H'}\) is NP-complete. The polynomial time Turing reduction is from \(\text{HOM}_{H'}\) to \(\text{HOM}_H\). Let \(G\) be an instance of \(\text{HOM}_{H'}\). We now form \(|V(H')|\) instances of \(\text{HOM}_H\) as follows: take \(|V(H')|\) copies each of \(G\) and \(H'\), let \(v \in V(G)\) and \(V(H') = \{h_1, h_2, \ldots, h_n\}\). Denote by \(F_i\) the graph obtained by
identifying the vertex \( v \) in \( G \) with the vertex \( h_i \) in \( H' \), \( 1 \leq i \leq n \).

It is now easy to see that \( G \rightarrow H' \) if and only if there exists at least one \( i \in \{1, 2, \ldots, n\} \) such that \( F_i \rightarrow H \).

From now on we assume that all locally semicomplete digraphs are connected.

4 Unicyclic Locally Semicomplete Digraphs

Let \( T \) be a connected unicyclic locally semicomplete digraph and let \( C \) be the cycle in \( T \). Then \( C \) is induced and forms the unique non-trivial strong component in \( T \). It is not difficult to check (see e.g. [2]) that \( T = H[D_1, D_2, \ldots, D_l] \), where \( D_j = C \) for some \( j \) and \( |D_i| = 1 \) for all \( i \neq j \) and \( H \) is an acyclic local tournament. In particular, if \( l \geq 2 \) we must have that \( C \) is either a 2-cycle or a 3-cycle as every vertex of \( C \) either dominates or is dominated by some other vertex. If \( T = C \) then \( T \)-colouring is polynomial so we may assume that \( C \) is either a 2-cycle or a 3-cycle.

The unicyclic locally semicomplete digraph \( T \) may also be viewed as follows. Let \( S \) be the set of neighbours (in-and out-neighbours) of the cycle in \( T \). Then \( V(T) \setminus S \) is the union of two disjoint sets of vertices: those that come before \( S \) in the ordering shown above, call these \( A \), and those that come after \( S \) in the ordering above, call these \( B \). Define the following three induced sub-digraphs: \( T_1 = T[A] \), \( T_2 = T[S] \) and \( T_3 = T[B] \). Each \( T_i \) is locally semicomplete digraph and \( T_1 \) and \( T_3 \) are acyclic as well. Note that \( T_1 \) or \( T_3 \) may be empty. This general structure is illustrated in Figure 1 (with a 3-cycle) where we have written \( D_t = \{y_t\} \) for \( t \in \{1, 2, \ldots, l\} \setminus \{j\} \).

Note that the arc \( y_{j-1}y_{j+1} \) may or may not be present depending on whether \( T_2 \) is or isn’t semicomplete. Furthermore, since \( T \) is a locally semicomplete digraph, \( y_{j-1} \) dominates the cycle and \( y_{j+1} \) is dominated by the cycle.
Figure 1: The structure of a unicyclic locally semicomplete digraph that is not a directed cycle.
In discussing the complexity of colouring by a unicyclic locally semicomplete digraph where \( T_1 = \emptyset \) and \( T_3 \neq \emptyset \), or where \( T_1 \neq \emptyset \) and \( T_3 = \emptyset \), we only need to consider one of these situations since the digraphs are converses of one another.

In the proofs to follow we only consider the case where the directed cycle has length 3. We would like to point out that all of these proofs carry over to the case where the cycle is a 2-cycle. The reasons for this are: (i) in the polynomial cases, we are still dealing with a graft extension and (ii) for the NP-complete cases, we will be relying on oriented cycles of net-length one not mapping to the 3-cycle. These same oriented cycles do not map to a directed 2-cycle either.

**Lemma 4.1.** If \( T \) is a unicyclic locally semicomplete digraph in which \( T_2 \) is semicomplete and \( T_1 \neq \emptyset \) and \( T_3 = \emptyset \), then \( T \)-colouring is polynomial.

**Proof.** We show that \( T \) is an instance of the graft extension.

Let \( V(T_1) = \{y_1, y_2, \ldots, y_l\} \) and \( V(T_2) = \{y_{i+1}, y_{i+2}, \ldots, y_{j-1}, 0, 1, 2, y_{j+1}, \ldots, y_k\} \) (the 3-cycle’s vertices are labeled 0, 1 and 2). Define \( T' \) to be the acyclic local tournament obtained by contracting the vertices \( \{0, 1, 2, y_{j+1}, \ldots, y_k\} \) to a single vertex (in \( T \)) and deleting the resulting loop. Let \( T'' \) be the unicyclic tournament induced by the vertices \( \{0, 1, 2, y_{j+1}, \ldots, y_k\} \). Since \( T' \) is an acyclic local tournament, it has an \( X \)-enumeration (order the vertices according to the unique hamiltonian path in \( T' \)). Also, \( T'' \) is a unicyclic tournament and so \( T'' \)-colouring is polynomial. It can now be seen that \( T = X \)-graft\((T', T'')\) and thus the claim follows from Corollary 2.4.

It is worth noting at this point that if \( T_2 \) is a unicyclic semicomplete digraph and both \( T_1 \) and \( T_3 \) are empty, then \( T \)-colouring is polynomial by Theorem 1.2.

In determining the complexity of colouring by a unicyclic locally semicomplete digraph, the local tournament \( LT_5 \) shown below in Figure 2 plays a special role.

**Lemma 4.2.** Let \( T \) be a unicyclic locally semicomplete digraph. If \( T \) contains \( LT_5 \) as an induced subdigraph, then \( T \)-colouring is NP-complete.

**Proof.** The proof of NP-completeness is via a reduction from monotone one-in-three 3-SAT (i.e. no negations).

Let \( T = A[D_1, D_2, \ldots, D_l] \), with \( D_j = C_3, \ j \in \{1, 2, \ldots, l\} \) and \( |D_i| = 1 \) for \( i \in \{1, 2, \ldots, l\} \), \( i \neq j \). Label the vertices in \( D_i \) with \( 1 \leq i \leq j - 1 \) as \( t_1, t_2, \ldots, t_{j-1} \), the vertices in \( D_i \) with \( j + 1 \leq i \leq l \) as \( b_1, b_2, \ldots, b_{l-j} \) and the vertices on the \( C_3 \) as 0, 1, 2. Let \( c = \min\{i \mid t_i \text{ dominates the } C_3\} \). Since \( T \) is a local tournament and \( LT_5 \) is an induced subdigraph, \( t_c \) is not adjacent to any \( b_i, 1 \leq i \leq l - j \). Let \( d \) be equal to the maximum \( i \) such that \( t_i \) dominates the \( C_3 \) and \( t_i \) is not adjacent to any \( b_s, 1 \leq s \leq l - j \). Denote the vertices \( \{t_c, t_{c+1}, \ldots, t_d\} \) by \( F \), the vertices
Figure 2: The special unicyclic local tournament, $LT_5$, on five vertices.

$$\{t_{d+1}, t_{d+2}, \ldots, t_{j-1}\}$$ by $M$, the vertices $\{0, 1, 2\}$ by $C$, and the length of the path $t_c t_{c+1} \ldots t_{j-1}$ by $k$. In a similar way we are able to find an index $e$, such that the set $B = \{b_1, b_2, \ldots, b_e\}$ is the set of vertices being dominated by the 3-cycle.

The set $F$ will be associated with “False” and $M \cup C$ with “True.”

Consider the digraph $K$ shown below in Figure 3.

In any homomorphism from $K$ to $T$, the vertices $\ell_i$ and $\ell'_i$, for $i = 1, 2, 3$, must map into $F \cup M \cup C$. The reason for this is that each $\ell_i$ ($\ell'_i$) dominates a vertex on a 3-cycle, and a 3-cycle in $K$ has to map to the 3-cycle in $T$.

The vertices $v_1, v_4$ and $v_6$ have to map in such a way that they are being dominated by a vertex of the 3-cycle in $T$. Therefore each of $v_1, v_4$ and $v_6$ map into $C \cup B$. If $\ell_1$ maps into $F$, then $v_6$ has to map into $C$. The same is true for $\ell_2$ and $v_1$ and for $\ell_3$ and $v_4$. Therefore if $\ell_1, \ell_2$ and $\ell_3$ all map into $F$, then the oriented 7-cycle on $v_0, v_1, \ldots, v_6$ has to map into the $C_3$ in $T$. This is not possible since this oriented cycle has net-length one. Thus at most two of $\ell_1, \ell_2$ and $\ell_3$ can map into $F$.

If $\ell_i$, $1 \leq i \leq 3$, maps into $M$, then $\ell'_i$ has to map into $C$: $\ell'_i$ has to map into $M \cup C$ in this case and $\ell'_i$ cannot map into $M$ since the images of $\ell_i$ and $\ell'_i$ are joined by a directed walk of length $k$ (the image of the $P_k$ joining them) in $T$, the distance between two vertices in $M$ is less than $k$ and the subdigraph induced by $M$ is acyclic. On the other hand when $\ell_i$, $1 \leq i \leq 3$ maps into $C$, then again $\ell'_i$ maps into $C$. Therefore if $\ell_i \mapsto M \cup C$, then $\ell'_i \mapsto C$ for $i = 1, 2, 3$. Furthermore if any two of
\{\ell_1',\ell_2',\ell_3'\} are mapped into \(C\), then in fact these two vertices are mapped to the same vertex in \(C\). The reason for this is that each pair of vertices from \{\ell_1',\ell_2',\ell_3'\} have a common out-neighbour that is on a \(C_3\) in \(K\). If for instance \(\ell_1'\) and \(\ell_2'\) both mapped to vertex \(y \in C\), then vertices \(u_0\) and \(u_1\) both have to map to vertex \(y^+\), the successor of \(y\) on the \(C_3\) in \(T\). This is clearly not a homomorphism. The other two pairs from \{\ell_1',\ell_2',\ell_3'\} follow in a similar way. Therefore in a homomorphism from \(K\) to \(T\), at most one of \{\ell_1',\ell_2',\ell_3'\} can map into \(C\). This implies that at most one of \{\ell_1,\ell_2,\ell_3\} can map into \(M \cup C\).

From before we know that at most two of \(\ell_1,\ell_2\) and \(\ell_3\) can map into \(F\). Therefore exactly one of \(\ell_1,\ell_2\) and \(\ell_3\) must map into \(M \cup C\) and exactly two of \(\ell_1,\ell_2\) and \(\ell_3\) must map into \(F\).
If there is a homomorphism \( f : K \to T \) such that \( f(\ell_i) \in M \cup C, f(\ell_{i+1}) \in F \) and \( f(\ell_{i+2}) \in F \) (subscripts taken mod 3), then \( f(\ell_i) \in C \), so that \( f(\ell_{i+1}), f(\ell_{i+2}) \notin C \). This implies that \( f(\ell_{i+1}) = f(\ell_{i+2}) = x \) and that \( f(\ell_{i+1}) = f(\ell_{i+2}) = y \).

We now show that as long as exactly one of \( \{\ell_1, \ell_2, \ell_3\} \) map to \( M \cup C \), then there exists a homomorphism from \( K \) to \( T \). We denote by \( y \) a vertex in \( M \cup C \) and by \( z \) a vertex in \( C \). These homomorphisms are shown in Table 1.

<table>
<thead>
<tr>
<th>( \ell_1 )</th>
<th>( \ell_2 )</th>
<th>( \ell_3 )</th>
<th>( \ell'_1 )</th>
<th>( \ell'_2 )</th>
<th>( \ell'_3 )</th>
<th>( v_0 )</th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_4 )</th>
<th>( v_5 )</th>
<th>( u_0 )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( t_c )</td>
<td>( t_c )</td>
<td>( z )</td>
<td>( t_{j-1} )</td>
<td>( t_{j-1} )</td>
<td>( z )</td>
<td>( z^+ )</td>
<td>( z )</td>
<td>( z^{++} )</td>
<td>( z )</td>
<td>( z^{++} )</td>
<td>( b_1 )</td>
<td>( z^+ )</td>
<td>( z^{++} )</td>
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<tr>
<td>( t_c )</td>
<td>( y )</td>
<td>( t_c )</td>
<td>( z )</td>
<td>( t_{j-1} )</td>
<td>( t_{j-1} )</td>
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<td>( z )</td>
<td>( z )</td>
<td>( z^+ )</td>
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<td>( b_1 )</td>
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<td>( z^{++} )</td>
<td>( z^{++} )</td>
<td>( z )</td>
<td>( z^+ )</td>
</tr>
</tbody>
</table>

Given an instance of monotone one-in-three 3-SAT with variables \( x_1, x_2, \ldots, x_s \) and clauses \( C_1, C_2, \ldots, C_t \), we construct a digraph \( D \) as follows. For each variable \( x_i \) there a corresponding vertex \( x_i \) and for each clause \( \{\ell_1, \ell_2, \ell_3\} \), we use a copy of \( K \). Identify each vertex \( x_i \) with the corresponding vertex in each of the clauses in which it appears. This forms the digraph \( D \).

If there is a homomorphism from \( D \) to \( T \), then by what was shown above we see that in a clause \( \{\ell_1, \ell_2, \ell_3\} \), exactly one of the \( \ell_i \)'s map to \( M \cup C \), the others have to map to \( F \). We can recover a truth assignment by declaring that a variable is “True” if its corresponding vertex \( x_i \) maps to \( M \cup C \) declaring it to be “False” if it maps to \( F \).

On the other hand if there is a satisfying truth assignment, then in a clause \( \{\ell_1, \ell_2, \ell_3\} \), exactly one \( \ell_i \) is “True.” We can set up a homomorphism from \( K \) to \( T \) by mapping a vertex \( x_i \) to vertex 0 in \( T \) if the corresponding variable is “True” and mapping it to \( t_c \) if the variable is “False.” As was shown above these pre-colourings extend to all of \( D \).

Therefore \( D \to T \) if and only if there is a satisfying truth assignment. This implies that \( T \)-colouring is NP-complete.

Lemma 4.2 covers the case where \( T_2 \) is not semicomplete since the local tournament in Figure 2 must be an induced subdigraph in this case. Lemma 4.1 covers the case where \( T_2 \) is semicomplete, \( T_1 \neq \emptyset \) and \( T_3 = \emptyset \) (or \( T_1 = \emptyset \) and \( T_3 \neq \emptyset \)). All that remains therefore is the case where \( T_2 \) is semicomplete and both \( T_1 \) and \( T_2 \) are not empty.
Lemma 4.3. Let $T$ be a unicyclic locally semicomplete digraph with $T_1, T_2$ and $T_3$ defined as above (Figure 1). If both $T_1$ and $T_3$ are non-empty and $T_2$ is semicomplete, then $T$-colouring is NP-complete.

Proof. The proof of NP-completeness is by a reduction from not-all-equal 3-SAT without negated variables. We label the vertices of $T$ as in Figure 1 (note that the arc $y_{j-1}y_{j+1}$ is present in $T$ since $T_2$ is a tournament). The vertices on the 3-cycle are labeled as 0, 1 and 2.

In describing the reduction we use a number of gadgets. The first of which, $G_1$, is shown in Figure 4.

![Figure 4](image)

Figure 4: The first gadget $G_1$ for the last unicyclic case.

The property of $G_1$ of interest to us is that in any homomorphism $G_1 \rightarrow T$, the vertex labeled $v$ cannot map to any vertex of the three cycle in $T$. If this happens, the oriented 5-cycle in $G_1$ is forced to map to the 3-cycle in $T$ which is not possible since the oriented 5-cycle has net-length one.

Consider the directed path $y_{i+1}y_{i+2}\cdots y_{j-1}0$ in $T$ (recall Figure 1). Suppose that this path has length $t$. The next gadget, $G_2$, is constructed from a directed path of length $t$ and $t$ copies of $G_1$. Form $G_2$ by identifying each vertex on the directed path, except the last vertex, with the vertex $v$ in one copy of $G_1$. This is shown in Figure 5.

The third gadget, $G_3$, is shown in Figure 6. It is constructed from three copies of $G_2$ in which the three respective terminal vertices of the directed paths of length $t$ are identified with particular vertices on an oriented 7-cycle. This oriented 7-cycle has net-length one, and some of its vertices are on 3-cycles. These vertices have to map to the 3-cycle in $T$ under any homomorphism $G_3 \rightarrow T$. The gadget $G_3$ represents half of the clause gadget that will be used in the reduction from monotone not-all-equal 3-SAT.
Figure 5: The second gadget $G_2$ for the last unicyclic case.

Let $S_1$ be the set of vertices in $T_1$ that are connected to the 3-cycle in $T$ by a path of length two (these are exactly those vertices which have an arc to $y_{i+1}$). Consider a homomorphism $G_3 \to T$. Under any such homomorphism $\ell_1, \ell_2$ and $\ell_3$ have to map to vertices in $T$ that can reach the 3-cycle in $T$ along a path of length two. This follows from the fact each of $\ell_1, \ell_2$ and $\ell_3$ has been identified with the vertex $v$ in $G_1$. Therefore, the possible images of $\ell_1, \ell_2$ and $\ell_3$ are $S_1 \cup \{y_{i+1}, y_{i+2}, \ldots, y_{j-1}\}$.

On the other hand, the presence of the directed path of length $t$ in each copy of $G_2$ in $G_3$ prevents $\ell_1, \ell_2$ and $\ell_3$ from mapping to any of $\{y_{i+2}, y_{i+3}, \ldots, y_{j-1}\}$ (they are “too close” to the 3-cycle). Thus the possible images of $\ell_1, \ell_2$ and $\ell_3$ are $S_1 \cup \{y_{i+1}\}$. If $\ell_j \mapsto y_{i+1}$ (for $j \in \{1, 2, 3\}$), then $w_j$ maps to one of $\{0, 1, 2\}$. So if $\ell_1, \ell_2$ and $\ell_3$ all mapped to $y_{i+1}$, then each $w_j$ (for $j = 1, 2, 3$) maps into the 3-cycle. As a consequence of this, the oriented 7-cycle is forced to map to the 3-cycle in $T$, but this is not possible since the oriented 7-cycle has net-length one. Therefore $\ell_1, \ell_2$ and $\ell_3$ cannot all map to $y_{i+1}$. It is easy to check that if at least one of $\ell_1, \ell_2$ and $\ell_3$ map into $S_1$, then this may be extended to a homomorphism $G_3 \to T$. 

15
Let the length of the path \(0y_{j+1}y_{j+2} \ldots y_k\) (Figure 1) be \(t'\). The other half of the clause gadget, \(G_4\), is formed by taking the converse of \(G_3\), changing the lengths of the paths in \(G_2\) to \(t'\), and re-labeling the vertices \(\ell_1, \ell_2, \ell_3\) as \(\ell_1', \ell_2', \ell_3'\) respectively.

Define \(S_3\) to be the vertices in \(T_3\) that are reachable by a path of length two from the 3-cycle in \(T\) (i.e. those vertices that are dominated by \(y_k\)). It now follows in a similar manner as before that \(\ell_1', \ell_2', \ell_3'\) have to map into the set \(S_3 \cup \{y_k\}\) under any homomorphism \(G_4 \rightarrow T\). As before at least one of \(\ell_1', \ell_2', \ell_3'\) has to map to \(S_3\).

The clause gadget, \(K\), can now (finally) be constructed. \(K\) is the disjoint union of \(G_3\) and \(G_4\). The idea behind the clause gadget is that “\(True\)” and “\(False\)” are to be encoded based on how each pair \((\ell_j, \ell_j')\), \(j = 1, 2, 3\), is mapped.

We are now ready to give the reduction. Given an instance of not all equal 3-SAT (without negations) with variables \(x_1, x_2, \ldots, x_s\) and clauses \(C_1, C_2, \ldots, C_t\) we construct a digraph \(D\) as follows. For each variable \(x_i\), we have a directed path of length two, \(P^i\), with initial vertex (also) labeled \(x_i\) and terminal vertex labeled \(x'_i\).

For every clause \(\{\ell_1, \ell_2, \ell_3\}\) take a copy of \(K\). Each time a variable \(x_i\) appears in a clause \(\{\ell_1, \ell_2, \ell_3\}\) (say \(x_i = \ell_2\)), identify the vertex \(x_i\) in \(P^i\) with the vertex \(\ell_2\) and the vertex \(x'_i\) in \(P^i\) with \(\ell'_2\) in the corresponding clause gadget. This is illustrated in Figure 7.

Let \(D \rightarrow T\) be a homomorphism. Based on the discussion before the possible
images for the pair $(\ell_j, \ell'_j), j = 1, 2, 3,$ is the set $[S_1 \cup \{y_{i+1}\}] \times [S_3 \cup \{y_k\}]$. Since no vertex in $S_1$ is joined to a vertex in $S_3$ by a path of length two, we are able to refine the images somewhat. We define two subsets of the set $[S_1 \cup \{y_{i+1}\}] \times [S_3 \cup \{y_k\}]$: $T = \{(y_{i+1}, y_k) \cup \{y_{i+1}\} \times S_3\}$ and $F = S_1 \times \{y_k\}$. It can now be seen that the images of each pair $(\ell_j, \ell'_j), j = 1, 2, 3,$ is the set $T \cup F$. The three pairs are not allowed to map only to $T$ (as this forces each $\ell_j$ to map to $y_{i+1}$ which is not possible). Also, the three pairs are not allowed to map only to $F$ (as this forces all $\ell'_j$s to map to $y_k$ which is not possible). Thus each pair (representing a variable) maps in such a way that there is at least one image in $T$ and at least one image in $F$. A satisfying truth assignment can now be recovered by examining the images of the end-vertices of the path $P^i$, representing the variable $x_i$: into $T$ corresponds to “True” and into $F$ corresponds to “False.”

Given a satisfying truth assignment to not-all-equal 3-SAT (without negations) we construct a homomorphism as follows. Let $y \in S_1$. If the variable $x_i$ is assigned the value “True,” map the end-vertices of the path $P^i$ to the pair $(y_{i+1}, y_k)$. If the variable $x_i$ is assigned the value “False,” map the ends of $P^i$ to the pair $(y, y_k)$. This pre-colouring can be extended to a homomorphism on each clause gadget.

Therefore there is a homomorphism $D \to T$ if and only if there is a satisfying
truth assignment to not-all-equal 3-SAT without negations.

For unicyclic locally semicomplete digraphs, we have the following dichotomy.

**Theorem 4.4.** Let $T$ be a unicyclic locally semicomplete digraph. If $T$ is a directed cycle or $T$ has the structure shown in Figure 1 with $T_2$ semicomplete and at least one of $T_1$ and $T_3$ is empty, then $T$-colouring is polynomial. Otherwise $T$-colouring is NP-complete.

5 Locally Semicomplete Digraphs With At Least Two Cycles

The NP-completeness result in this section follows from Theorem 1.3.

**Theorem 5.1.** Let $D$ be a locally semicomplete digraph containing at least two directed cycles. Then $D$-colouring is NP-complete.

**Proof.** If $D$ is smooth, then since $D$ is a core (Proposition 2.6) and contains at least two cycles, $D$-colouring is NP-complete by Theorem 1.3.

If $D$ is not smooth, we apply a sub-indicator to $D$ first. Let $J$ be a directed path of length two and let $j$ be the middle vertex of this path. Applying this sub-indicator to $D$, we obtain an induced subdigraph of $D$, say $D^+$, that is smooth, contains at least two directed cycles and is also a locally semicomplete digraph (since it is induced). $D^+$-colouring is NP-complete from before, therefore $D$-colouring is also NP-complete.

6 The Dichotomy for Connected Locally Semicomplete Digraphs

Summarizing the results of the previous sections, we obtain the following dichotomy for connected locally semicomplete digraphs.

**Theorem 6.1.** Let $T$ be a connected locally semicomplete digraph.

- If $T$ is acyclic, then $T$-colouring is polynomial.
- If $T$ is unicyclic and $T$ is a directed cycle or $T$ has the structure shown in Figure 1 with $T_2$ semicomplete and at least one of $T_1$ and $T_3$ is empty, then $T$-colouring is polynomial. Otherwise $T$-colouring is NP-complete.
- If $T$ contains at least two cycles, then $T$-colouring is NP-complete.

We also have the following theorem.
Theorem 6.2. The dichotomy for list homomorphisms to connected locally semicomplete digraphs is exactly the same as the dichotomy for homomorphisms to connected locally semicomplete digraphs.

Proof. Let $T$ be a connected locally semicomplete digraph. If $\text{HOM}_T$ is NP-complete, then $\text{LIST-HOM}_T$ will also be NP-complete (set all lists of the input equal to $V(T)$).

On the other hand if $\text{HOM}_T$ is polynomial, then $\text{LIST-HOM}_T$ is also polynomial since our algorithm in the polynomial case is a list processing algorithm.

7 Acknowledgements

The authors would like to thank the referees for a very thorough reading of the paper.

References


