Analysis and numerical solution of Benjamin–Bona–Mahony equation with moving boundary

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A B S T R A C T

In this work we present the existence, the uniqueness and numerical solutions for a mathematical model associated with equations of Benjamin–Bona–Mahony type in a domain with moving boundary. We apply the Galerkin method, multiplier techniques, energy estimates and compactness results to obtain the existence and uniqueness. For numerical solutions, we shall employ the finite element method together with the Crank–Nicolson method. Some numerical experiments are presented to show the moving boundary for the problem.

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1. Introduction

Let \( \hat{Q} = \{(x,t) \in \mathbb{R}^2; \ x \in \Omega_t, \ 0 < t < T\} \) be the non-cylindrical domain with lateral boundary denoted by \( \hat{\Sigma} = \bigcup_{0 < t < T} \Gamma_t \times \{t\} \), where \( \Omega_t = \{x \in \mathbb{R}; \ \alpha(t) < x < \beta(t), \ 0 < t < T\} \) and \( \Gamma_t \) denotes the boundary of \( \Omega_t \). Consider the nonlinear problem:

\[
(I) \quad \begin{cases}
    u' + \frac{\partial}{\partial x}(u + u^2) - \frac{\partial^2 u}{\partial x^2} = 0, & \text{in} \ \hat{Q}, \\
    u(x, t) = 0, & \text{on} \ \hat{\Sigma}, \\
    u(x, 0) = u_0(x), & \forall x \in \Omega_0,
\end{cases}
\]

where prime denotes the time derivative and \( \Omega_0 = [\alpha(0), \beta(0)] \).

The Problem (I) in cylindrical domain is known by Benjamin–Bona–Mahony (BBM) equation and describes a mathematical model of long waves in a nonlinear dispersive system and have been considered in several papers. In [2] was proved the existence and uniqueness of solution for the Cauchy problem associated the Problem (I) in \( \mathbb{R}_x \times \mathbb{R}_t \) domain. When the spatial variable \( x \in [0, \infty) \), the BBM equation with boundary condition in \( x = 0 \), have been considered in [4] and some important properties were obtained. In [12] for \( x \in [0, 1] \) and using a general nonlinear term in the BBM the existence, uniqueness and regularity have been obtained. For \( x \in [0, 1] \) the BBM with boundary values not homogeneous the existence and uniqueness for solution was obtained in [3].

In [6] was proved, for the case unidimensional, a generalization for the BBM for limited subsets \( \Omega \subseteq \mathbb{R}^n, \ n > 1 \).

The existence of solution to the Problem (I) in noncylindrical domain was established in [11] and the existence for the \( n \)-dimensional case is studied in [8].

Numerical analysis and simulation for the string and Beam equation in the noncylindrical domain have been established in [9,14].

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In this work, we will investigate existence, uniqueness and approximate solution of the Problem (I). We will also show the influence of moving boundary employing numerical examples. Our main contribution is to obtain the uniqueness and numerical results of the Problem (I). As far as we know this is the first time that it is done for the BBM equation in noncylindrical domain.

For this we consider the following hypotheses:

**H1**: \( x, \beta \in C^2([0, T]; \mathbb{R}) \), with \( \alpha(t) < \beta(t) \) and \( \alpha'(t) \leq 0 \), \( \beta'(t) \geq 0 \), \( \forall t \in [0, T] \), where we define \( \gamma(t) = \beta(t) - \alpha(t) \), \( \forall t \in [0, T] \).

We will now consider a change of variables to transform the domain \( \hat{Q} \) into a cylindrical domain \( Q \). Observe that, when \((x, t)\) varies in \( Q \) the point \((y, t)\) of \( \mathbb{R}^2 \), with \( y = (x - \alpha(t))/\gamma(t) \) varies in the cylinder \( Q = (0, 1) \times (0, T) \). Thus, we define the application
\[
\mathcal{F} : \hat{Q} \rightarrow Q = (0, 1) \times (0, T),
\]
\((x, t) \mapsto (y, t) = \left( \frac{x - \alpha(t)}{\gamma(t)}, t \right) \).

The application \( \mathcal{F} \) belongs to \( C^2 \) and its inverse \( \mathcal{F}^{-1} \) is also \( C^2 \). The transformation of a moving boundary domain to a domain with fixed boundary has been employed elsewhere (see [10,11]).

Doing the change of variable \( v(y, t) = u(x(t) + \gamma(t)y, t) \) and applying to the Problem (I), we obtain the following equivalent problem defined in a fixed cylindrical domain:

\[
\begin{align*}
&\begin{cases}
&v' + b_1(t) \frac{\alpha'(t)}{\gamma(t)} v - b_2(t) \frac{\alpha'(t)}{\gamma(t)}^2 v - a_1(y, t) \frac{\partial v}{\partial y} + b_3(t) \frac{\partial^2 v}{\partial y^2} + a_2(y, t) \frac{\partial^2 v}{\partial y^2} = 0, \\
&v(0, t) = v(1, t) = 0, \\
v(0, y) = v_0(y),
\end{cases}
&\text{ in } Q, \\
&\forall t \geq 0, \\
&v(0, y) = v_0(y), \quad \text{ in } \Omega = (0, 1),
\end{align*}
\]

where \( Q = (0, 1) \times (0, T) \) is a cylindrical domain and
\[
a_1(y, t) = \frac{\alpha' + \gamma y}{\gamma}, \quad a_2(y, t) = \frac{\alpha' + \gamma y}{\gamma^2}, \quad b_1(t) = \frac{1}{\gamma^2}, \quad b_2(t) = \frac{1}{\gamma^2}, \quad b_3(t) = \frac{2\gamma^2}{\gamma^2}.
\]

### 2. Existence and uniqueness

Let \((\cdot, \cdot)\) and 
\( \| \cdot \| \), be respectively the scalar product and the norms in \( H^1_0(0, 1) \) and \( L^2(0, 1) \).

We shall first establish the existence of the solution for the Problem (II) in Theorem 2 as auxiliary result and then prove the existence of the solution for the original Problem (I) in the Theorem 1. The proof of the uniqueness of the solution will be made directly to the Problem (I).

**Theorem 1.** Under the hypothesis (H1) and given the initial data \( u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \) there exist functions \( u : \hat{Q} \rightarrow \mathbb{R} \), solution of Problem (I) in \( Q \), satisfying the following conditions:

\[
\begin{align*}
&u \in L^\infty(0, T; H^1_0(\Omega) \cap H^2(\Omega)), \\
&\| u' \|_{L^2(0, T; H^1_0(\Omega))} < \infty, \\
&\int_Q u \phi dx dt + \int_Q \frac{\partial}{\partial x} (u + u^2) \phi dx dt + \int_Q \frac{\partial u'}{\partial x} \phi dx dt = 0, \\
&\forall \phi \in L^2(0, T; H^1_0(\Omega)).
\end{align*}
\]

**Theorem 2.** Under the hypotheses (H1) and given the initial data \( v_0 \in H^1_0(0, 1) \cap H^2(0, 1) \) there exists functions \( v : Q \rightarrow \mathbb{R} \), solution of Problem (II) in \( Q \), \( \forall \psi \in L^2(0, T; H^1_0(\Omega)) \), satisfying the following conditions:

\[
\begin{align*}
&v \in L^\infty(0, T; H^1_0(0, 1) \cap H^2(0, 1)), \\
&v' \in L^\infty(0, T; H^1_0(0, 1)), \\
&\int_Q \left( v' \psi' + b_1(t) \frac{\partial}{\partial y} (v + v^2) \psi + b_2(t) \frac{\partial v}{\partial y} - a_1(y, t) \frac{\partial v}{\partial y} + b_3(t) \frac{\partial^2 v}{\partial y^2} + a_2(y, t) \frac{\partial^2 v}{\partial y^2} \right) \psi dx dt = 0.
\end{align*}
\]

**Proof of Theorem 2.** The demonstration of existence of solution is similar to that made in [13]. The tool of the proof is the compactness results of Lions, but to apply this theorem we need a series of bounds which we establish with a number of standard analytic techniques in the approximate solutions.

Let \( T > 0 \) and denote by \( V_m \) the subspace spanned by \( \{w_1, w_2, \ldots, w_m\} \), where \( \{w_1, \ldots, w_m\} \) are solutions of the spectral problem \((\mu, v) = \mu(w, v), \forall v \in H^1_0(0, 1) \). If \( v_m \in V_m \) then it can be represented by
\[
v_m = \sum_{i=1}^m d_{i,m} w_i(y).
\]

Let us consider \( v_m \) solutions of the system of ordinary differential equations,
\[
\begin{align*}
&\begin{cases}
&v_m' + b_1(t) \frac{\partial}{\partial y} (v_m + v_m^2), w - b_2(t) \frac{\partial v_m}{\partial y^2}, w - \left( a_1(y, t) \frac{\partial v_m}{\partial y}, w \right) + b_3(t) \left( \frac{\partial^2 v_m}{\partial y^2}, w \right) + \left( a_2(y, t) \frac{\partial^2 v_m}{\partial y^2}, w \right) = 0, \\
v_m(0) = v_m(0), \quad \text{in } H^1_0(\Omega) \cap H^2(\Omega),
\end{cases}
\end{align*}
\]
where \( w \in V_m \). The system (6) has local solution in the interval \((0, T_m)\) (see, for instance [5]). To extend the local solution to the interval \((0, T)\) independent of \( m \), the following estimates are necessary:

First Estimate

Taking \( w = v_m \) in Eq. (6), for each one term we have:

\[
(a'_m, v_m) = \frac{1}{2} \frac{d}{dt} \|v_m\|^2, \\
b_1(t) \left( \frac{\partial}{\partial y} (v_m + v'_m), v_m \right) = b_1(t) \left( \int_0^1 \frac{1}{2} \frac{\partial v'^2_m}{\partial y} + \frac{2}{3} \int_0^1 \frac{\partial v^3_m}{\partial y} \right) = 0, \\
- b_2(t) \left( \frac{\partial^2 v'_m}{\partial y^2}, v_m \right) = \frac{\gamma'_m}{\gamma} \|v_m\|^2, \\
- \left( a_1(y, t) \frac{\partial v_m}{\partial y}, v_m \right) = - b_3(t) \left( \frac{\partial^2 v_m}{\partial y^2}, \frac{\partial v'_m}{\partial y} \right) = 0.
\]

Using the definition (4), for the last term on the left side of (6), we obtain

\[
a_2(y, t) \left( \frac{\partial^2 v_m}{\partial y^2}, v_m \right) = \frac{\gamma'_m}{\gamma^2} \int_0^1 \frac{\partial^2 v_m}{\partial y^2} v_m \, dy + \int_0^1 \left( \frac{\partial^3 v_m}{\partial y^3} + \frac{\gamma y}{\gamma^2} \right) \frac{\partial^2 v_m}{\partial y^2} \, dy = 0.
\]

Substituting (7)–(12) in (6),

\[
\frac{d}{dt} \|v_m(t)\|^2 + b_2(t) \frac{\partial v_m}{\partial y} \frac{\partial v_m}{\partial y} + \frac{\gamma'_m}{\gamma} \|v_m\|^2 + 2 b_3(t) \left( \frac{\partial v_m}{\partial y} \right)^2 + 2 b_1(t) \left( \frac{\partial v_m}{\partial y} \right)^2 + \frac{2}{\gamma} \frac{\partial^2 v_m}{\partial y^2} \frac{\partial v_m}{\partial y} = 0.
\]

Let \( \gamma_2 = \max_{t \in [0, T]} |\gamma(t)| \). Since that, \( 0 < \gamma_0 \leq \gamma(t), \forall t \in [0, T] \), from hypothesis H1 then we have

\[
- \frac{\gamma'_m(t)}{\gamma^2(t)} \leq \frac{|\gamma'_m(t)|}{\gamma^2(t)} \leq \frac{\gamma_2}{\gamma_0^2 \gamma^2(t)}.
\]

Now, integrating in \( 0 \) from \( t < T \) and using again the hypothesis H1, we obtain

\[
\|v_m(t)\|^2 + b_2(t) \left( \frac{\partial v_m}{\partial y} \right)^2 \leq \|v_0\|^2 + b_2(0) \left( \frac{\partial v_0}{\partial y} \right)^2 + \frac{\gamma_2}{\gamma_0^2} \int_0^t \left( b_2(s) \left( \frac{\partial v_m}{\partial y} \right)^2 + \|v_m(s)\|^2 \right) ds.
\]

Thus, applying the Gronwall's inequality in (14), we get the first estimate

\[
\|v_m(t)\|^2 + \frac{1}{\gamma_0} \left( \frac{\partial v_m}{\partial y} \right)^2 \leq \left( \|v_0\|^2 + \frac{1}{\gamma_0} \left( \frac{\partial v_0}{\partial y} \right)^2 \right)^{\exp(\gamma_2/\gamma_0^2)} = K_0.
\]

that is,

\[
v_m \quad \text{is bounded in } L^\infty(0, T; L^2(0, 1)),
\]

\[
\frac{\partial v_m}{\partial y} \quad \text{is bounded in } L^\infty(0, T; L^2(0, 1)).
\]

Second Estimate

Taking \( w = - \frac{\partial v_m}{\partial y} \), in the approximate problem (6), we obtain:

\[
- \left( v'_m, \frac{\partial v_m}{\partial y} \right) - b_1(t) \left( \frac{\partial}{\partial y} v_m + v'_m, \frac{\partial^2 v_m}{\partial y^2} \right) + b_2(t) \left( \frac{\partial^2 v'_m}{\partial y^2}, \frac{\partial^2 v_m}{\partial y^2} \right) + \left( a_1(y, t) \frac{\partial v_m}{\partial y}, \frac{\partial^2 v_m}{\partial y^2} \right) - b_3(t) \left( \frac{\partial^2 v_m}{\partial y^2}, \frac{\partial^2 v_m}{\partial y^2} \right) = 0.
\]

We will make estimates for each of the terms of equality.
Doing the same procedure of the previous estimate and using the definition for (4), we obtain
\[
\frac{d}{dt} \left| \frac{\partial v_m}{\partial y} \right|^2 + \frac{1}{\gamma} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 - \frac{\gamma}{\gamma'} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 = \frac{2}{\gamma} \left( \frac{\partial}{\partial y} (v_m + \frac{\partial^2 v_m}{\partial y^2}) \right) - 2 \left( \frac{(x' + y'') y}{\gamma} \frac{\partial v_m}{\partial y}, \frac{\partial^2 v_m}{\partial y^2} \right).
\]
(18)

We also have the inequality of Poincaré,
\[
\| v_m(t) \| \leq \left| \frac{\partial v_m}{\partial y} (t) \right|, \quad \| \frac{\partial v_m}{\partial y} (t) \| \leq \left| \frac{\partial^2 v_m}{\partial y^2} (t) \right|, \quad \text{and} \quad \| v_m(t) \|_{L^\infty(\Omega)} \leq C_0 \left| \frac{\partial v_m}{\partial y} (t) \right|.
\]
(19)

where \( C_0 \) is a positive constant.

From (15) and (19), we obtain
\[
\left| \frac{2}{\gamma} \left( \frac{\partial}{\partial y} v_m(t) + \frac{\partial^2 v_m(t)}{\partial y^2} \right) \right| \leq (2 + 4C_0^2K_0'\gamma_0^2) \left( \left| \frac{\partial v_m}{\partial y} \right|^2 + \frac{1}{\gamma} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 \right).
\]
(20)

Since that
\[
|x' + y'|_\Omega \leq \max_{\Omega \cap [0,T]} (|x'| + |y'|) = C_1, \quad \forall \Omega \in (0, 1),
\]
then
\[
\left| \frac{2}{\gamma} \left( \frac{(x' + y') \frac{\partial v_m}{\partial y} + \frac{\partial^2 v_m}{\partial y^2} \right) \right| \leq (C^2_1 + 1) \left( \left| \frac{\partial v_m}{\partial y} \right|^2 + \frac{1}{\gamma} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 \right).
\]
(21)

So integrating (18) in 0 from \( t < T \) and using (20), (21) and hypothesis \( H_1 \), we obtain
\[
\left| \frac{\partial v_m}{\partial y} \right|^2 + \frac{1}{\gamma} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 \leq \left| \frac{\partial v_m}{\partial y} \right|^2 + \frac{1}{\gamma} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 + C_2 \int_0^t \left( \left| \frac{\partial v_m}{\partial y} \right|^2 + \frac{1}{\gamma} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 \right) ds,
\]
(22)

where \( C_2 = 2 + 4C_0^2K_0'\gamma_0^2 + C_1 \). Now, applying the inequality the Gronwall, we conclude that
\[
\left| \frac{\partial v_m}{\partial y} \right|^2 + \frac{1}{\gamma} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 \leq \left( \left| \frac{\partial v_m}{\partial y} \right|^2 + \frac{1}{\gamma} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 \right) \exp^{C_2T} = K_1.
\]
(23)

Then
\[
\frac{\partial v_m}{\partial y} \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega))
\]
\[
\frac{\partial^2 v_m}{\partial y^2} \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)).
\]
(24)

Third Estimate

Taking \( w = v_m \) in Eq. (6), for each one term we have:
\[
(v_m, v_m) + b_1(t) \left( \frac{\partial}{\partial y} (v_m + v_m^2), v_m \right) - b_2(t) \left( \frac{\partial^2 v_m}{\partial y^2}, v_m \right) - \left( a_1(y, t), \frac{\partial v_m}{\partial y}, v_m \right)
\]
\[
+ b_3(t) \left( \frac{\partial^2 v_m}{\partial y^2}, v_m \right) + \left( a_2(y, t), \frac{\partial^2 v_m}{\partial y^2}, v_m \right) = 0.
\]
(25)

Then, doing some calculus and using (4), we obtain
\[
\left| v_m \right|^2 + \frac{1}{\gamma} \left| \frac{\partial v_m}{\partial y} \right|^2 \leq \frac{1}{\gamma} \left( \frac{\partial}{\partial y} (v_m + v_m^2), v_m \right) + \left( \frac{(x' + y'') y}{\gamma} \frac{\partial v_m}{\partial y}, v_m \right) - \frac{\gamma}{\gamma'} \left( \frac{\partial^2 v_m}{\partial y^2}, v_m \right) + \left( \frac{(x' + y'') y}{\gamma} \frac{\partial^2 v_m}{\partial y^2}, v_m \right) + \left( \frac{\gamma}{\gamma'} \frac{\partial v_m}{\partial y}, v_m \right)
\]
\[
\leq \frac{1}{\gamma} \left( \left| v_m \right|^2 + 2C_0 \left| \frac{\partial v_m}{\partial y} \right|^2 \right) \left| v_m \right|^2 + \frac{C_1}{\gamma_0} \left| \frac{\partial v_m}{\partial y} \right| \left| v_m \right|^2 + \frac{1}{\gamma} \left| \frac{\partial^2 v_m}{\partial y^2} \right|^2 \left| v_m \right|^2 + \frac{C_1}{\gamma} \left| \frac{\partial v_m}{\partial y} \right|^2 \left| \frac{\partial v_m}{\partial y} \right|.
\]
(26)

From (15), (23) and (26) we get
\[
\left| v_m(t) \right|^2 + \frac{1}{\gamma} \left| \frac{\partial v_m(t)}{\partial y} \right|^2 \leq C_3 \left| v_m(t) \right| + \frac{C_4}{\gamma} \left| \frac{\partial v_m}{\partial y} (t) \right|.
\]
(27)
Then
\[
\|v_m'(t)\|^2 + \left\| \frac{\partial v_m'(t)}{\partial y} \right\|^2 \leq C_2^2 + C_4^2,
\]  
(28)
where \( C_3 = \frac{1}{\gamma_0} \sqrt{K_0} + 2C_0 \gamma_0 + C_1 \sqrt{K_0} + \frac{\gamma_2 \sqrt{K_1}}{\gamma_0}, \) and \( C_4 = \frac{C_3 \sqrt{K_1}}{\gamma_0}. \) Thus,
\[
v_m' \text{ is bounded in } L^\infty(0, T; L^2(0, 1)),
\]
\[
\frac{\partial v_m'}{\partial y} \text{ is bounded in } L^\infty(0, T; L^2(0, 1)).
\]  
(29)

The estimates obtained in (16), (24) and (29) permit us to pass the limits in the approximate system (6) in the Galerkin method and hence, we have proved the existence of solutions \( \{v\} \) in the sense defined in Theorem 2. \( \Box \)

### 2.1. Uniqueness of solution for the Problem (I)

Unlike the proof of existence of solution for the Problem (I), through the equivalent Problem (II), we shall prove the uniqueness of Problem (I) directly, by using basically the Leibniz formula.

Let \( u_1 \) and \( u_2 \) solutions of the Problem (I) in the sense defined in Theorem 2. Considerer \( w = u_1 - u_2 \) and \( \theta \in L^2(0, T; H^1_0(\Omega)). \) Then \( w(0) = 0 \) and
\[
\int_0^t \int_{\Omega} w'(s,x)\theta(s,x)dxds + \int_0^t \int_{\Omega} \left( \frac{\partial w'(s,x)}{\partial x} (u_1(s,x) + u_2(s,x)) \right) w(s,x)dxds + \int_0^t \int_{\Omega} \frac{\partial w'(s,x)}{\partial x} (s,x) \frac{\partial \theta}{\partial x} (s,x)dxds = 0.
\]

Taking \( \theta = w \) and substituting in the last equality, we have
\[
\int_0^t \int_{\Omega} w'(s,x)w(s,x)dxds + \int_0^t \int_{\Omega} \left( w'(s,x) + w(s,x) \frac{\partial}{\partial x} (u_1(s,x) + u_2(s,x)) \right) w(s,x)dxds
\]  
\[
+ \int_0^t \int_{\Omega} \frac{\partial w'(s,x)}{\partial x} \frac{\partial w}{\partial x} (s,x)dxds = 0.
\]  
(30)

Using the Leibniz’s formula, we get
\[
\int_{\Omega} w'(s,x)w(s,x)dx = \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} (w(s,x))^2 dx = \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} (w(s,x))^2 dx - \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial s} (w(s,x))^2 dx + \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial s} (w(s,x))^2 dx - \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial x} (w(s,x))^2 dx + \int_{\Omega} \frac{\partial w'(s,x)}{\partial x} \frac{\partial w}{\partial x} (s,x)dx,
\]  
(31)
\[
\int_{\Omega} \frac{\partial w'(s,x)}{\partial x} (s,x)dx = \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial s} \left( \frac{\partial w'(s,x)}{\partial x} (s,x)dx \right) dx = \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial s} \left( \frac{\partial w'(s,x)}{\partial x} (s,x)dx \right) dx - \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial x} \left( \frac{\partial w'(s,x)}{\partial x} (s,x)dx \right) dx
\]  
\[
+ \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial x} \left( \frac{\partial w'(s,x)}{\partial x} (s,x)dx \right) dx.
\]  
(32)

\[
\int_{\Omega} \frac{\partial w'(s,x)}{\partial x} (s,x)dx = \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial s} \left( \frac{\partial w'(s,x)}{\partial x} (s,x)dx \right) dx - \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial x} \left( \frac{\partial w'(s,x)}{\partial x} (s,x)dx \right) dx
\]  
\[
+ \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial x} \left( \frac{\partial w'(s,x)}{\partial x} (s,x)dx \right) dx.
\]  
(33)

where \( C_5 = \frac{1}{2} (1 + \gamma_1^2 (\|u_1\|_{L^\infty(0,T;H^1_0(\Omega))} + \|u_2\|_{L^\infty(0,T;H^1_0(\Omega))})). \)
Then, from hypothesis (H1), taking the equation (31)–(33) and substituting in (30), we obtain
\[
\|w(t)\|_2^2 + \|\frac{\partial w}{\partial x}(t)\|_2^2 \leq 2C_5 \int_0^t \left( \|w(s)\|_2^2 + \|\frac{\partial w}{\partial x}(s)\|_2^2 \right) ds
\]
(34)

From last inequality and using the Gronwall inequality, we have
\[
\|w(t)\|_2^2 + \|\frac{\partial w}{\partial x}(t)\|_2^2 = 0, \quad \forall t \in [0, T].
\]
(35)

Hence, \(w(t) = 0, \quad \forall t \in [0, T]\), i.e., is proved that the solution of Problem (I) is unique, \(u_1(x, t) = u_2(x, t)\).

3. Approximate solution

Our goal in this section is the numerical implementation of approximate solutions. To obtain the numerical approximate solutions we will use both finite element method and finite difference method. Moreover, some numerical experiments will be presented to analyze the effect of the moving boundary in the Benjamin–Bona–Mahony equation.

For convenience, our numerical analysis using finite element method approximation will be based on the equivalent Problem (II) in the rectangular domain, instead of the Problem (I), for which the domain depends on time.

3.1. Variational formulation

Multiplying Eq. (3) by \(w \in H^1_0(\Omega)\), integrating in \(\Omega\) and doing the integration by parts, we obtain the following variational formulation,
\[
\int_0^1 \frac{\partial \varphi}{\partial y} w dy + b_1(t) \int_0^1 \frac{\partial \varphi}{\partial y} w dy + 2b_1(t) \int_0^1 \varphi \frac{\partial w}{\partial y} w dy - b_2(t) \int_0^1 \varphi \frac{\partial^2 w}{\partial y^2} w dy - \int_0^1 a_1(y, t) \frac{\partial \varphi}{\partial y} w dy
\]
\[+ b_2(t) \int_0^1 \frac{\partial^2 \varphi}{\partial y^2} w dy + \int_0^1 a_2(y, t) \frac{\partial^2 \varphi}{\partial y^2} w dy = 0, \quad \forall w \in H^1_0(\Omega),
\]
where we replaced by numerical convenience, the term \(\left(\frac{\partial \varphi}{\partial y}\right)^2\) by \(2 \frac{\partial \varphi}{\partial y}\).

3.2. Galerkin method and approximation

Let \(V_m\) the subspace spanned by \(\{\varphi_1, \varphi_2, \ldots, \varphi_m\}\), where \(\{\varphi_i; \quad i = 1, \ldots, m\}\) are the first \(m\) bases vectors of the space \(V = H^1_0(\Omega)\). If \(v^h(y, t) \in V_m\), then it can be represented by
\[
v^h(y, t) = \sum_{i=1}^m d_i(t) \varphi_i(y), \quad \varphi_i(y) \in V_m.
\]
(37)

Restricting Eq. (36) in the subspace \(V_m\) and taking \(v^h\) in (37) and \(w = \varphi_k(y) \in V^m\), we obtain
\[
\sum_{i=1}^m \int_0^1 \varphi_i \frac{\partial \varphi_k}{\partial y} dy - b_1(t) \sum_{i=1}^m \int_0^1 \varphi_i \frac{\partial \varphi_k}{\partial y} dy + 2b_1(t) \sum_{i=1}^m d_i(t) \frac{\partial \varphi_k}{\partial y} \int_0^1 \varphi_i \frac{\partial \varphi_k}{\partial y} \varphi_k dy
\]
\[+ b_2(t) \sum_{i=1}^m d_i(t) \frac{\partial^2 \varphi_i}{\partial y^2} \varphi_k dy + \sum_{i=1}^m \int_0^1 a_1(y, t) \varphi_i \frac{\partial \varphi_k}{\partial y} dy + \int_0^1 a_2(y, t) \varphi_k \frac{\partial^2 \varphi_k}{\partial y^2} dy = 0
\]
(38)

We define
\[
A_{ik} = \int_0^1 \varphi_i \varphi_k dy, \quad B_{ik} = \int_0^1 \varphi_i \frac{\partial \varphi_k}{\partial y} dy, \quad C_{ik}(t) = \int_0^1 a_1(y, t) \varphi_i \frac{\partial \varphi_k}{\partial y} dy
\]
\[D_{ik}(t) = \int_0^1 a_2(y, t) \varphi_k \frac{\partial^2 \varphi_k}{\partial y^2} dy, \quad E_{ik} = \int_0^1 \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_k}{\partial y} dy, \quad M_{ik} = \int_0^1 \varphi_i \frac{\partial \varphi_k}{\partial y} \varphi_k dy.
\]
(39)

where the term \(M_{ik}\) is a tensor of third order.

Substituting the matrices and a third-order ordinary tensor in (38), we obtain the following nonlinear ordinary differential system,
\[
\begin{cases}
L_k d^2(t) + N_k d(t) + 2b_1(t)M_{ik} d^2(t) = 0, \\
d(0) = d_0,
\end{cases}
\]
(40)

where \(L_k = (A_{ik} + b_2(t)E_{ik})\) and \(N_k = \left( -b_1(t)B_{ik} + D_{ik}(t) + \frac{\partial a_2}{\partial y}(y, t)A_{ik} - \frac{\partial a_1}{\partial y}(y, t)E_{ik} - C_{ik}(t) \right)\).
3.3. Finite difference method

The nonlinear ordinary differential system (40) with the matrices characteristics (dependent on the variables $y$ and $t$) of system, obtaining the solution is not always possible in continuous time. So, we will apply a numerical method to obtain the approximated solution for the system (40), using the approximate Crank–Nicolson method (see, for instance, [7]).

Let $d^n = d(t_n)$ be the approximate solution of the exact solution $d(t)$ of (40), where we denote the discrete time in the interval $[0, T]$ by $t_n = n\Delta t$. $n = 0, 1, \ldots, N$, and the values of $W$ at the discrete time $t^n$ by $W^n$.

Before applying the Crank–Nicolson method, we will do the following linearization procedure of the tensor $M_{ijk}$ and then we can solve the system of equations, i.e., we will transform the system of nonlinear equations differential in a system of approximated linear equation.

For this, consider family of matrices $\tilde{M}_{ij}(t_l) = (M_{ijk}d_j(t_l))$ of order $m \times m$, for each $j = 1, 2, \ldots, m$. In this form, we can take from (38) and (39) the tensor by

$$M_{ijk}d^j(t) = (M_{ijk}d_j(t_l))d_l(t) = \tilde{M}_{ij}(t)\xi_l(t).$$

So, for the discrete time $t = t_n$, we have the following approximation

$$\tilde{M}_{ki}d^i = \frac{1}{2}\tilde{M}_{ki}(d^{i+1} + d^i).$$

(41)

Setting $t = t_n$ in (40), using (41) and the approximation $d^i(t_n) = \frac{1}{2\Delta t}(d^{i+1} - d^i)$, we obtain

$$L_{ki}\left(\frac{d^{i+1} - d^i}{\Delta t}\right) + (N_{ki} + 2\beta_i\tilde{M}_{ki})\left(\frac{d^{i+1} + d^i}{2}\right) = 0,$$

for each $k = 1, 2, \ldots, m$. (42)

Hence using a linearization procedure (41) of the tensor together with the Crank–Nicolson method in (40), we obtain the following iterative method, after we multiply by $\Delta t$,

$$\left(L_{ki} + \frac{\Delta t}{2}J_{ki}\right)d^{i+1} = \left(L_{ki} - \frac{\Delta t}{2}J_{ki}\right)d^i$$

for $n = 0, 1, \ldots$. (43)

where $J_{ki} = (N_{ki} + 2\beta_i\tilde{B}_{ki})$.

Suppose that the matrices, $J = J_{ki}$ and $L = L_{ki}$ which are dependent on $y$ and $t$, are known, then the iterative method (43) can be easily implemented. From initial conditions, $d_0$ is known. Then taking $t = 0$ into (43) yields, for each $k = 1, 2, \ldots, m$

$$\left(L_{ki} + \frac{\Delta t}{2}J_{ki}\right)d^i = \left(L_{ki} - \frac{\Delta t}{2}J_{ki}\right)d^i,$$

for $i = 1, 2, \ldots, m$. (44)

Solving the linear system then get the vector $d^i = (d^i_1, d^i_2, \ldots, d^i_m)$. Then for $n = 1, 2, \ldots$ we have the same procedure and the values of $d^n = (d^n_1, d^n_2, \ldots, d^n_m)$, are determined with an iterative process in each instant of time $t_n = n\Delta t$.

3.4. Finite element method

To calculate the matrices of linear system (43), we need to introduce the basis function $\varphi_i \in V_m$. In finite element method, the bases functions are piecewise polynomials of some degree in $\Omega$ and vanish on $\partial \Omega$. More specifically, in this work, mainly because of the matrix $C_{ik}(t)$ defined in (39), we have used the B-Splines as the basis function of $V_m$ subspace, that are cubic splines defined in the following way

$$B_i(y) = \begin{cases} \frac{1}{4h}(y - y_{i-2})^3, & \text{with } y_{i-2} \leq y \leq y_{i-1}, \\ \frac{1}{4h}(y - y_{i-1}) + \frac{1}{4h}(y - y_{i-2})^2 - \frac{1}{4h}(y - y_{i-1})^2, & \text{with } y_{i-1} \leq y \leq y_i, \\ \frac{1}{4h}(y - y_{i+1}) - \frac{1}{4h}(y - y_{i-1})^2, & \text{with } y_{i} \leq y \leq y_{i+1}, \\ \frac{1}{4h}(y - y_{i+2}), & \text{with } y_{i+1} \leq y \leq y_{i+2}, \end{cases}$$

where the node points are equally spaced with spacing $h = y_{i+1} - y_i$. $y_0 = 0$ and $y_m = 1$. It is easy to verify that $B_i(y_i) = 1$, $B_i(y_{i-1}) = B_i(y_{i+1}) = 1/4$ and zero at the other nodes.

We wish to take the bases functions $\varphi_1, \ldots, \varphi_m$ to be the B-splines $B_1, \ldots, B_m$. However, although $B_3, \ldots, B_{m-2}$ satisfy the zero boundary conditions, $B_1, B_2, B_{m-1}$ and $B_m$ do not. Therefore we define the another base function $\varphi_i$ to be

$$\begin{align*}
\varphi_1(y) &= B_1(y), & i = 3, \ldots, m - 2, \\
\varphi_2(y) &= B_1(y) - 4B_0(y), \\
\varphi_m(y) &= B_{m-1}(y) - B_{m+1}(y), \\
\varphi_m(y) &= B_{m}(y) - 4B_{m-1}(y),
\end{align*}$$

(45)
where $B_0(y)$ and $B_{m,1}(y)$ are the B-splines in the nodes points $y_0 = y_1 - h$ and $y_{m,1} = y_m + h$, respectively. So $\varphi_i(y)$ is a cubic spline that satisfies the boundary condition, i.e., $\varphi_1(y_1) = \varphi_2(y_1) = \varphi_m(y_m) = \varphi_{m,1}(y_{m,1}) = 0$.

Using the basis function $\varphi_i$, all matrices of (39) can be calculated and hence the linear system (43) can be resolved. Note that, each matrix is heptagonal and so does the matrix of linear system $(L + (\Delta t/2)J)$. Furthermore, it is not difficult to see that matrix is not singular, for each instant of discrete time $t_n = n\Delta t$. Note that, from (37) we have

$$v_n(y, t) = \sum_{i=1}^{m} d_i(t)\varphi_i(y), \quad \varphi_i(y) \in V_m,$$

and the B-Spline, do not have the property of interpolation, i.e., $v_n(y, t_n) \neq d_i(t_n)$, but we can obtain the approximated solution in the node $y_i$, using the relation

$$v_n(y, t_n) = v_n^i(y_i) = d_i + \frac{1}{4}(d_{i-1}^n + d_{i+1}^n), \quad \text{for } i = 2, 3, \ldots, m. \tag{46}$$

Substituting the values of $d^n_i = (d_1, d_2, \ldots, d_m)$ in Eq. (46) we obtain the approximate solutions $v_n(y, t_n)$ of the problem (3) on the subspace $V_m$. As the application $\mathcal{S}$ defined in (2) is an isomorphism then the solutions of problem (3) leads to the solution of problem (1), by the inverse change of variable $x = \alpha(t) + \gamma(t)y$.

4. Numerical simulation

Our goal in this section is the numerical implementation of approximate solutions. In order to obtain an exact solution to compare with the approximate numerical solution obtained by the iterative method (43), we will introduce an appropriate external forces $f(x, t)$ sufficiently regular on the right side of the original problem (1) instead of external force zero $f \equiv 0$, so that the exact solution is known for such convenient choice of $f(x, t)$. Since, as far as we know that, there is no error estimates (continuous or discrete time) for the problem (1) or (3), then this will be the procedure for analysis of convergence. Of course, for the problem (3) the equivalent external forces is $g(y, t)$ by use the application $\mathcal{S}$ in (2). Some numerical experiments will be presented to analyze the effect of the moving boundary in the system Benjamin–Bona–Mahony type.

The greatest difficulty in finding the numerical solution is the term $(\partial^2 v/\partial y^2)$, which is associated with matrix $C_{vl}(t)$ defined in (39). To control this term, the coefficient $a_2(y, t)$ should be taken sufficiently small, that means $\gamma(t)$, the length of the interval, should grow very slowly when $t$ grows. So, in these numerical experiments we are taking the following functions

$$\alpha(t) = -1 + 0.01e^{-(t-1)}, \quad \beta(t) = 1 - 0.01e^{-(t-1)}. \tag{47}$$

Hence $\gamma(t) = 2 - 0.02e^{-(t-1)}$ with $\gamma(0) = 1.9926$ and $\lim_{t \to \infty} \gamma(t) = 2$.

In this particular case, the function $a_2(y, t) = (\alpha' + \gamma'y)/\gamma^2$ satisfies $|a_2(y, t)| \leq 4.6 \times 10^{-4}$ and then the matrix $C_{vl}(t)$ generates the stable coefficients matrix $(L_{vl} + (\Delta t/2)J_{vl})$ in the iterative method (43).

Figs. 1 and 2 represents the movement of extremes $\alpha(t)$ and $\beta(t)$.

4.1. Example 1

Let $v(y, t) = \sin(\pi y) \cos(\pi t)$ the exact solution of the problem

$$\begin{align*}
\frac{\partial v}{\partial t} + b_1(t) \frac{\partial (v^2/2)}{\partial y} - b_2(t) \frac{\partial^2 v}{\partial y^2} - a_1(y, t) \frac{\partial v}{\partial y} + b_3(t) \frac{\partial v}{\partial t} + a_2(y, t) \frac{\partial^2 v}{\partial y^2} &= g(y, t), \quad \text{in } Q, \\
v(0, t) &= v(1, t) = 0, \quad \forall \ t \geq 0, \\
v(y, 0) &= v_0(y) = \sin(\pi y) \quad \text{in } \Omega = (0, 1),
\end{align*} \tag{48}$$

which is similar to the problem (3), replacing $g(y, t)$ by zero. Considerer the boundary function $\alpha(t)$ and $\beta(t)$ defined in (47). Then the coefficients $b_1(t), b_2(t), b_3(t), a_1(y, t), a_2(y, t)$ can be calculated explicitly and the external force $g(y, t)$ can be
obtained by replacing the exact solution \( v(y, t) = \sin(\pi y) \cos(\pi t) \) in Eq. (48)1. In the norm \( L^1(0, T; L^2(\Omega)) \) the numerical error between the approximate \( u_h(x, t) \) and exact \( u(x, t) \) solution is calculated by (see, for instance [15]):

\[
E_{L^1(0,T,L^2(\Omega))} = \max_{t_n \in [0, T]} \left( \int_\Omega |u(x, t_n) - u_h(x, t_n)|^2 \, dx \right)^{1/2}, \quad i = 1, \ldots, m \text{ and } n = 1, \ldots, N.
\]

The tables that follows, are shown the errors obtained between exact and approximate solutions to the problem (48). In Tables 1 and 2, the incremental time is fixed \( \Delta t = 0.01 \) and \( \Delta t = 0.001 \), for various sizes of mesh \( h = 1/m = 0.1; 0.05; 0.02; \)

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>( h )</th>
<th>( E_{L^1(0,T,L^2(\Omega))} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.1</td>
<td>0.007648</td>
</tr>
<tr>
<td>0.01</td>
<td>0.05</td>
<td>0.007495</td>
</tr>
<tr>
<td>0.01</td>
<td>0.02</td>
<td>0.005016</td>
</tr>
<tr>
<td>0.01</td>
<td>0.01</td>
<td>0.003383</td>
</tr>
<tr>
<td>0.01</td>
<td>0.002</td>
<td>0.001888</td>
</tr>
<tr>
<td>0.01</td>
<td>0.001</td>
<td>0.001768</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>( h )</th>
<th>( E_{L^1(0,T,L^2(\Omega))} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.01</td>
<td>0.006902</td>
</tr>
<tr>
<td>0.001</td>
<td>0.05</td>
<td>0.006780</td>
</tr>
<tr>
<td>0.001</td>
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</tr>
<tr>
<td>0.001</td>
<td>0.01</td>
<td>0.002586</td>
</tr>
<tr>
<td>0.001</td>
<td>0.002</td>
<td>0.001282</td>
</tr>
<tr>
<td>0.001</td>
<td>0.001</td>
<td>0.001198</td>
</tr>
</tbody>
</table>
0.01; 0.002; 0.001 that represent respectively $m = 10, 20, 50, 100, 500, 1000$ nodes of the interval $[\alpha(t), \beta(t)]$. In Tables 1 and 2 were made 100 and 1000 iterations respectively, in the iterative method (43).

We can verify in both tables, that each error depends algebraically on $m = \frac{1}{h}$, i.e., $E_{L^1(0,T;L^2(\Omega))} \approx a(m)^{-3}$, for some $a > 0$, $a = \tan(\theta)$. This conclusion is based observing the data of the table for $h = 0.1; 0.01; 0.001$ for $\Delta t = 0.01$ or $\Delta t = 0.001$, i.e., if the discretization divided up by 10 then the error reduces in half, $E_{L^1(0,T;L^2(\Omega))} \approx 0.015(m)^{-0.318}$.

4.2. Example 2

The results in the tables of error, encourages us that the numerical method used is good, thus, employing the same procedure to determine the numerical solution of the problem with zero external force, $g(y, t) = 0$ in the problem (48), i.e., the original problem (3).

In this example, the initial position and boundary is the same for the example 1. We will use 20 finite elements, e.g., $h = 0.05$, and the incremental discrete time is $\Delta t = 0.05$ making 20 iterations in the iterative method (43). In all the figures the space variable is the axis-x, by change of variable.

In Fig. 3 the evolution of the displacement function $u_b(x, t)$ is plotted, showing the profile of the displacement, where time varies at $\Delta t = h = 0.05$ interval. In Fig. 4, we have the same figure, using the fixed point $y = 0.5$ and too in Fig. 5 the same approximate solution $u_b(x, t)$ for some time fixed at $t = 0.0, 0.25, 0.5$, and $t = 1.0$.

References