Extension Theory: the interface between set-theoretic and algebraic topology

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Abstract

Extension Theory can be defined as studying extensions of maps from topological spaces to metric simplicial complexes or CW complexes. One has a natural notion of an absolute (neighborhood) extensor $K$ of $X$. It is shown that several concepts of set-theoretic topology can be naturally introduced using ideas of Extension Theory. Also, it is shown that several results of set-theoretic topology have a natural interpretation and simple proofs in Extension Theory. Here are sample results.

**Theorem.** Suppose $X$ is a topological space. Then:

(a) $X$ is normal iff every finite partition of unity on a closed subset of $X$ extends to a finite partition of unity on $X$;

(b) $X$ is normal iff every countable partition of unity on a closed subset of $X$ extends to a countable partition of unity on $X$;

(c) $X$ is collectionwise normal iff every partition of unity on a closed subset of $X$ extends to a partition of unity on $X$;

(d) if $X$ is paracompact, then every locally finite partition of unity on a closed subset of $X$ extends to a locally finite partition of unity on $X$;

(e) if $X$ is metrizable, then every point-finite partition of unity on a closed subset of $X$ extends to a point-finite partition of unity on $X$.

**Theorem.** Suppose $X$ is a topological space. Then:

(a) finite simplicial complexes are absolute neighborhood extensors of $X$ iff every finite partition of unity on a closed subset of $X$ extends to a partition of unity on $X$;

(b) complete simplicial complexes are absolute neighborhood extensors of $X$ iff every partition of unity on a closed subset of $X$ extends to a partition of unity on $X$;

(c) simplicial complexes are absolute neighborhood extensors of $X$ iff every point-finite partition of unity on a closed subset of $X$ extends to a point-finite partition of unity on $X$;

(d) CW complexes are absolute neighborhood extensors of a first countable $X$ iff every locally finite partition of unity on a closed subset of $X$ extends to a locally finite partition of unity on $X$.

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Theorem. A complete simplicial complex $K$ is an absolute neighborhood extensor of $X$ iff its $0$-skeleton $K^0$ is an absolute neighborhood extensor of $X$.

Theorem. Suppose $X$ is a topological space and $A$ is a subset of $X$. Then:
   (a) $A$ is $C^*$-embedded in $X$ iff every finite partition of unity on $A$ extends to a finite partition of unity on $X$;
   (b) $A$ is $C$-embedded in $X$ iff every countable partition of unity on $A$ extends to a countable partition of unity on $X$;
   (c) $A$ is $P$-embedded in $X$ iff every partition of unity on $A$ extends to a partition of unity on $X$;
   (d) $A$ is $M$-embedded in $X$ iff every partition of unity on $A$ extends to a partition of unity on $X$ so that $\beta(B) = \alpha(A)$ for some zero-set $B$ of $X$ which contains $A$.

**Keywords:** Absolute extensors; ANRs; Partitions of unity

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1. Introduction

Extension Theory can be defined as studying extensions of maps from topological spaces to metric simplicial complexes or CW complexes. It emerged recently from work on cohomological dimension and its original goal was to unify covering dimension theory and cohomological dimension theory. As of now, it seems that the major results of Extension Theory of separable metric spaces deal with maps into joins of simplicial complexes (see [5,8]). However, set-theoretic topologists have done a lot of interesting research on extension of maps and it makes sense to investigate the relationship between the new theory and its predecessors. This paper will show that quite a few results of set-theoretic topology can be presented as an integral part of Extension Theory but the interesting twist is that Extension Theory can offer a new understanding of some classical results/concepts of set-theoretic topology. So the purpose of this paper is the following.

(1) Present new results regarding extension of maps into metric simplicial complexes or CW complexes.

(2) Give a unifying exposition of Basic Extension Theory which includes an account of classical work on extending maps into complete (arbitrary) ANRs or Banach spaces. One of the tools used in the paper to simplify existing proofs is the cone over a space which is an example of the most fundamental join of two spaces. In particular, a very general Homotopy Extension Theorem is proved using cones (see Theorem 13.7 and Lemma 13.8).

(3) Demonstrate that Extension Theory can be used to introduce, in a natural manner, several basic concepts of set-theoretic topology.

(4) Develop a calculus of partitions of unity and demonstrate its use.

One of the basic tasks of topology is to construct continuous functions (maps), and one of the basic ways of constructing maps is by extending existing maps. The most famous result concerning extensions of maps is:
Tietze Extension Theorem 1.1. If \( f : A \to [0, 1] \) (or \( f : A \to \mathbb{R} \)) is a map and \( A \) is a closed subset of a metric space \( X \), then \( f \) extends over \( X \). Thus, there is a map \( F : X \to [0, 1] \) with \( F(a) = f(a) \) for all \( a \in A \).

Tietze Extension Theorem gives rise to several notions. The most general is the notion of an absolute extensor:

**Definition 1.2.** \( K \) is called an absolute extensor of \( X \) (notations: \( K \in \text{AE}(X) \) or \( X, \tau K \)) provided every map \( f : A \to K \), \( A \) closed in \( X \), extends over \( X \). A related notion of an absolute neighborhood extensor is defined as follows: \( K \) is called an absolute neighborhood extensor of \( X \) (notation: \( K \in \text{ANE}(X) \)) provided every map \( f : A \to K \), \( A \) closed in \( X \), extends over a neighborhood of \( A \) in \( X \).

A natural question in Extension Theory is:

**Problem 1.3.** Given a class of spaces \( \mathcal{C} \), characterize spaces \( X \) such that \( K \in \text{AE}(X) \) (or \( K \in \text{ANE}(X) \)) for all \( K \in \mathcal{C} \).

A dual question is:

**Problem 1.4.** Given a class of spaces \( \mathcal{C} \), characterize spaces \( K \) such that \( K \in \text{AE}(X) \) (or \( K \in \text{ANE}(X) \)) for all \( X \in \mathcal{C} \).

Problem 1.4 is the starting point of the Theory of Retracts (see [1,12]).

In this paper we will address Problem 1.3 in the case of \( \mathcal{C} \) being the class of all simplicial metric complexes or the class of all CW complexes. Also, we will review work done on the following cases:

- (a) \( \mathcal{C} \) consists of the unit interval only,
- (b) \( \mathcal{C} \) consists of the reals only,
- (c) \( \mathcal{C} \) consists of all complete ARs,
- (d) \( \mathcal{C} \) consists of all ARs.

One can pose a slightly more general problem than Problem 1.3:

**Problem 1.5.** Given a class of spaces \( \mathcal{C} \), characterize all pairs \((X, A)\) of topological spaces such that every map \( f : A \to K \), \( K \in \mathcal{C} \), extends over \( X \).

**Definition 1.6.** To simplify the exposition (especially, the proofs), the fact that all maps \( f : A \to K \) extend over \( X \) will be denoted by \( K \in \text{AE}(X, A) \). One would think that \( K \in \text{ANE}(X, A) \) ought to mean that every map \( f : A \to K \) extends over a neighborhood of \( A \) in \( X \). However, as can be seen in this paper, it is much more useful to define \( K \in \text{ANE}(X, A) \) as follows: every map \( f : A \to K \) extends over a neighborhood \( U \) of \( A \) in \( X \) which is a cozero-set in \( X \), so that there is a zero-set \( B \) of \( X \) with \( A \subset B \subset U \).

It turns out that special cases of Problem 1.5 were studied by set-theoretic topologists who came up with the following notions:
**Definition 1.7.** Suppose \( X \) is a topological space and \( A \) is a subset of \( X \). Then:

(a) \( A \) is \( C^* \)-embedded in \( X \) ([10]) if every map \( f : A \to I = [0, 1] \) extends over \( X \) (i.e., \( I \in \text{AE}(X, A) \));

(b) \( A \) is \( C \)-embedded in \( X \) ([10]) if every map \( f : A \to \mathbb{R} = (-\infty, \infty) \) extends over \( X \) (i.e., \( \mathbb{R} \in \text{AE}(X, A) \));

(c) \( A \) is \( P \)-embedded in \( X \) ([22]) if every pseudo-metric \( f : A \times A \to \mathbb{R} \) extends over \( X \). Since assigning a pseudometric on \( A \) is equivalent with mapping \( A \) into a metric space, \( A \) being \( P \)-embedded in \( X \) is equivalent to every map \( f : A \to M, M \) a metric space, being extendible to \( f' : X \to M' \) for some metric space \( M' \) containing \( M \);

(d) \( A \) is \( M \)-embedded in \( X \) ([20]) if every map \( f : A \to M, M \) being a convex subset of a Banach space, extends over \( X \) (i.e., \( M \in \text{AE}(X, A) \)).

It will be shown in the paper that all of the notions in the above definition can be interpreted using the concept of extending a partition of unity. Partitions of unity play some role in set-theoretic topology but the purpose of this paper is to show that they ought to be considered as one of the fundamental tools in topology. The idea of extending partitions of unity arises naturally when discussing maps to metric simplicial complexes. The reason is that simplicial complexes are equipped with a natural partition of unity, namely the set of its barycentric coordinates. It is less obvious that all metrizable spaces have a single partition of unity determining its topology. Conceptually, partitions of unity are much easier to understand than simplicial (or CW) complexes. Therefore, we believe that the point of view presented in this paper ought to be of some value to set-theoretic topologists.

Here is the philosophical difference between classic set-theoretic topology (Hoshina’s paper [11] is a very good example of how to use its tools) and the approach used in this paper: The basic concept in set-theoretic topology is that of the family of all open covers \( \text{Covers}(X) \) of a topological space \( X \). One has the notions of refinement and star-refinement which define two partial orders on \( \text{Covers}(X) \). Also, one has the notion of the intersection of two covers which gives a structure of a semigroup on \( \text{Covers}(X) \). In this paper, the basic concept is that of the family \( \mathcal{P}(X) \) of all partitions of unity on \( X \). There is a natural transformation \( \text{Cozero} : \mathcal{P}(X) \to \text{Covers}(X) \) which assigns the family of cozero-sets to a partition of unity. One has a simple algebraic operation on \( \mathcal{P}(X) \), namely the multiplication of partitions of unity. It corresponds to the intersection of covers in the sense that \( \text{Cozero}(\alpha \ast \beta) = \text{Cozero}(\alpha) \cap \text{Cozero}(\beta) \). The interesting occurrence is that one can extend the notion of multiplication and define the join of a set of partitions of unity along a given partition of unity. This notion is closely related (but simpler) to the notion of the join of simplicial complexes. This relation is realized via the nerve functor. It is well-known that one has the nerve functor \( \mathcal{N} : \text{Covers}(X) \to \text{Complexes} \). We show that there is a more natural nerve functor \( \mathcal{N} : \mathcal{P}(X) \to \text{Complexes} \) which relates to the old one via \( \mathcal{N}(\text{Cozero}(\alpha)) = \mathcal{N}(\alpha) \). We introduce one more operation on \( \mathcal{P}(X) \), the contraction of partitions of unity. Loosely speaking, it is the partition of unity obtained from the old one by adding some of its terms together. In terms of covers, it corresponds
to taking unions of some elements of a given cover (a star of a point with respect to a given cover is a typical operation in topology).

One of the goals of this paper is to demonstrate how the above calculus of partitions of unity can be used to present a coherent view of some results/concepts of set-theoretic topology.

It should be pointed out that Dimension Theory can be defined as the area dealing with Problem 1.3 in the case of $C$ being the class of spheres (see [23]) and Cohomological Dimension Theory (see [4]) can be defined as the area dealing with Problem 1.3 in the case of $C$ being a class of Eilenberg–Mac Lane complexes. The last theory can be extremely algebraic which illustrates that Extension Theory is a natural bridge between set-theoretic and algebraic topologies.

2. Basic Extension Theory

First, let us translate some well-known notions and results into the language of the extension theory. The results given here without proofs can be found in [9]. The only difference is that we do not assume the spaces to be Hausdorff.

**Proposition 2.1.** $S^0 \in \text{ANE}(X)$ iff $X$ is a normal space (i.e., given two disjoint closed subsets $A, B$ of $X$ there exist two disjoint open subsets $U, V$ of $X$ so that $A \subseteq U$ and $B \subseteq V$).

**Proposition 2.2.** $D \in \text{ANE}(X)$ for all discrete topological spaces $D$ (of cardinality at most $m$) iff $X$ is a collectionwise normal space (an $m$-collectionwise normal space), i.e., given discrete family $\{F_s\}_{s \in S}$ of closed subsets of $X$ (with cardinality of $S$ at most $m$) there exists a discrete family $\{U_s\}_{s \in S}$ of open subsets of $X$ such that $F_s \subseteq U_s$ for all $s \in S$.

**Urysohn Lemma 2.3.** If $S^0 \in \text{ANE}(X)$ and $f : A \to \{0, 1\}$ is a map, $A$ closed in $X$, then $i \circ f$ extends over $X$, where $i : \{0, 1\} \to [0, 1]$ is the inclusion.

**Tietze–Urysohn Extension Theorem 2.4.** $S^0 \in \text{ANE}(X)$ iff $[0, 1] \in \text{AE}(X)$.

A sizable effort in set-theoretic topology is devoted to find weaker conditions which are equivalent to a given condition. A good example is the Tietze–Urysohn Extension Theorem 2.4. Here is another one due to Gillman and Jerison [10] (see also [11, Lemma 2.3]):

**Proposition 2.5.** A subset $A$ is $C^*$-embedded in $X$ iff given two disjoint zero-sets $B, C$ in $A$ there are two disjoint zero-sets $B', C'$ in $X$ such that $B \subseteq B'$ and $C \subseteq C'$.

Recall that a cozero-set $U$ (respectively zero-set $B$) in $X$ is a subset of the form $\beta^{-1}(0, 1]$ (respectively $\beta^{-1}(0)$) for some map $\beta : X \to [0, 1]$. $\beta^{-1}(0, 1]$ will be denoted by $\text{Cozero}(\beta)$. Proposition 2.5 is, in spirit, similar to the Tietze–Urysohn Extension
Theorem. Indeed, picking two disjoint zero-sets is synonymous with picking a special map \( D \to S^0 \). So what Proposition 2.5 says is that special maps on \( D \subset A \to S^0 \) can be extended to special maps on \( D' \subset X \to S^0 \).

We plan to generalize Tietze-Urysohn Extension Theorem to other spaces than the unit interval. The generalization involves cones over spaces.

**Definition 2.6.** Given a topological space \( X \) its cone \( \text{Cone}(X) \) is the quotient set \( X \times [0,1]/X \times \{1\} \) with the following topology: \( U \) is open in \( \text{Cone}(X) \) iff \( U \cap X \times [0,1) \) is open in the product topology and, if the vertex \( v \) (the point \( X \times \{1\}/X \times \{1\} \)) belongs to \( U \), then \( X \times (a,1) \subset U \) for some \( 1 > a > 0 \). The image of \( (x,t) \in X \times [0,1] \) in \( \text{Cone}(X) \) will be denoted by \( [x,t] \).

Notice that \( \text{Cone}(S^0) \) is homeomorphic to \([0,1]\). Also, notice that if \( X \) is metrizable, then so is \( \text{Cone}(X) \). Indeed, \( X \times [0,1) \) has a \( \sigma \)-discrete basis which can be used to construct a \( \sigma \)-discrete basis of \( \text{Cone}(X) \).

**Example 2.7.** The hedgehog \( J(S) \) is a useful space in set-theoretic topology. It can be defined as the cone \( \text{Cone}(S) \), where \( S \) is supplied with the discrete topology. The main importance of \( J(S) \) is that the countable product \( J(S)^{\aleph_0} \) is the universal space of all metrizable spaces of weight at most \( \text{Card}(S) \) if \( S \) is infinite (see [9] or Theorem 7.3 here).

Here is an analog of Urysohn Lemma for cones:

**Lemma 2.8.** Suppose \( X \) and \( Y \) are topological spaces, \( A \) is a subset of \( X \), and \( f : A \to Y \) is a map. The following conditions are equivalent:

1. \( f : A \to Y \) extends over a neighborhood \( U \) which is a cozero-set in \( X \) so that there is a zero-set \( B \) of \( X \) with \( A \subset B \subset U \).
2. \( f : A \to Y \) extends to \( F : X \to \text{Cone}(Y) \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose \( g : U \to Y \) is an extension of \( f \), \( U \) is a cozero-set in \( X \) so that there is a zero-set \( B \) of \( X \) with \( A \subset B \subset U \). First we need a map \( \gamma : X \to [0,1] \) such that \( \gamma^{-1}(1) = B \) and \( \gamma^{-1}(0) = X - U \). This is accomplished as follows: Choose a continuous function \( \kappa : X \to [0,1] \) so that \( \kappa^{-1}(0,1] = U \) and choose a continuous function \( \nu : X \to [0,1] \) so that \( B = \nu^{-1}(0) \). Define \( \gamma : X \to [0,1] \) by \( \gamma(x) = \kappa(x)/(\nu(x) + \kappa(x)) \). Now, define \( F : X \to \text{Cone}(Y) \) by sending \( X - U \) to the vertex and by defining \( F(x) = [g(x), \gamma(x)] \) for \( x \in U \). Notice that \( F|A = f \).

(2) \( \Rightarrow \) (1). Suppose \( F : X \to \text{Cone}(Y) \) is an extension of \( f \). Since \( Y \) is a zero-set in \( \text{Cone}(Y) \) and \( \text{Cone}(Y) - \text{vertex} \) is a cozero-set in \( \text{Cone}(Y) \) which retracts onto \( Y \), we put \( U = F^{-1}(\text{Cone}(Y) - \text{vertex}) \), \( B = F^{-1}(Y) \), and \( g : U \to Y \) is defined by \( g(x) = r(F(x)) \), where \( r : \text{Cone}(Y) - \text{vertex} \to Y \) is a retraction. \( \Box \)

Here is a generalization of Tietze-Urysohn Extension Theorem for cones:
Theorem 2.9. Suppose $Y \neq \{\text{point}\}$ is a Hausdorff space. Then, the following conditions are equivalent:

1. $\operatorname{Cone}(Y) \in \mathcal{AE}(X)$,
2. $Y \in \operatorname{ANE}(U)$ for all cozero subsets $U$ of $X$,
3. $Y \in \operatorname{ANE}(X)$.

Proof. (1) $\Rightarrow$ (2). Suppose $\operatorname{Cone}(Y) \in \mathcal{AE}(X)$ and $f : A \to Y$ is a map from a closed subset $A$ of $U = \alpha^{-1}(0, 1]$, where $\alpha : X \to [0, 1]$ is a map. Since $\operatorname{Cone}(\alpha(x)) = [0, 1]$ is a retract of $\operatorname{Cone}(Y)$ for every $x \in Y$, we infer $[0, 1] \in \mathcal{AE}(X)$. Let $v$ be the vertex of $\operatorname{Cone}(Y)$. Define $F : A \cup (X - U) \to \operatorname{Cone}(Y)$ as follows

$$F(x) = \begin{cases} v, & \text{if } x \in X - U, \\ [f(x), \alpha(x)], & \text{if } x \in A. \end{cases}$$

Let $G : X \to \operatorname{Cone}(Y)$ be an extension of $F$. Put $W = G^{-1}(\operatorname{Cone}(Y) - v)$ and notice that there is a retraction $r : \operatorname{Cone}(Y) - v \to Y$. Now, $h : W \to Y$ defined by $h(x) = r \circ G(x)$ is an extension of $f$ over a neighborhood $W$ of $A$.

(2) $\Rightarrow$ (1). Suppose $Y \in \operatorname{ANE}(U)$ for all cozero subsets $U$ of $X$. Since a two-point subset $D$ of $Y$ is a retract of its neighborhood in $Y$, we infer $S \in \operatorname{ANE}(U)$ for all cozero subsets $U$ of $X$. Suppose $f : A \to \operatorname{Cone}(Y)$ is a map, where $A$ is a closed subset of $X$. Let $\alpha : A \to [0, 1]$ be the composition of $f$ and the projection $\pi : \operatorname{Cone}(Y) \to [0, 1]$. Extend $\alpha$ to $\beta : X \to [0, 1]$ and let $U = \beta^{-1}(0, 1])$. Let $g : A \cap U \to Y$ be defined by $g(x) = r(f(x))$ for $x \in A \cap U$. Extend $g$ to $G : W \to Y$, $W$ being a cozero neighborhood of $A \cap U$ in $U$.

Now, find $\gamma : U \to [0, 1]$ which extends $\alpha|A \cap U$ and $\gamma(U - W) = \{1\}$. Finally, define $F : X \to \operatorname{Cone}(Y)$ as follows:

$$F(x) = \begin{cases} [G(x), \gamma(x)], & \text{if } x \in W, \\ v, & \text{if } x \in X - W. \end{cases}$$

Notice that $F$ extends $f$.

(3) $\Rightarrow$ (2). Let $\alpha : X \to [0, 1]$ be a map such that $U = \alpha^{-1}(0, 1]$ and suppose $f : A \to Y$ is a map, where $A$ is closed in $U$. Let $X_n = \alpha^{-1}[1/n, 1]$ and $A_n = A \cap X_n$ for $n \geq 1$. Notice that $A_n, X_n$ are closed in $X$. Let $f_1 : W_1 \to Y$ be an extension of $f|A_1$ over a closed neighborhood $W_1$ of $A_1$ in $X_1$. Suppose an extension $f_n : W_n \to Y$ of $f|A_n$ is given, where $W_n$ is a closed neighborhood of $A_n$ in $X_n$. The union $f_n \cup f|A_{n+1} : W_n \cup A_{n+1} \to Y$ is a continuous map. (Notice that traditionally one talks of pasting $f_n$ and $f|A_{n+1}$ together. However, if one knows that functions $f : X \to Y$ are certain subsets of $X \times Y$, then one is perfectly justified in taking unions of functions.) Therefore, it extends to $f_{n+1} : W_{n+1} \to Y$ for some closed neighborhood $W_{n+1}$ of $W_n \cup A_{n+1}$ in $X_{n+1}$. The union of all $f_n$ (traditionally called the direct limit of $f_n$) is a desired extension of $f$ over a neighborhood $W$ in $X$. \[\square\]

Corollary 2.10 [9, Problem 5.5.1]. A space $X$ is collectionwise normal (m-collectionwise normal) iff $J(S) \in \mathcal{AE}(X)$ for all $S$ (for an $S$ with $\operatorname{Card}(S) = m$).

Proof. Use Proposition 2.2 and Theorem 2.9. \[\square\]
3. Absolute retracts

K. Borsuk had a useful idea to enlarge the class of known absolute neighborhood extensors of metrizable spaces by using the concept of a retract:

**Definition 3.1.** $Y \subset X$ is called a retract of $X$ if there is a retraction $r : X \to Y$, i.e., a map such that $r(y) = y$ for all $y \in Y$.

**Proposition 3.2** [12, 5.1–5.2, pp. 40–41]. If $K \in \text{A}(\text{N})\text{E}(X)$ and $L$ is a retract of $K$, then $L \in \text{A}(\text{N})\text{E}(X)$.

Using Proposition 3.2 one can introduce the notion of an euclidean (neighborhood) retract:

**Definition 3.3.** $K$ is an euclidean (neighborhood) retract (notation: $K \in \text{E}(\text{N})\text{R}$) provided it is homeomorphic to a retract of (an open subset of) an euclidean space.

**Proposition 3.4** [12, 5.3, p. 42]. $S^n \in \text{ENR}$ for all $n$.

**Proposition 3.5** [12, 6.1–6.2, pp. 42–43]. If $K \in \text{E}(\text{N})\text{R}$, then $K \in \text{A}(\text{N})\text{R}(X)$ for all metrizable spaces $X$.

Euclidean (neighborhood) retracts are generalized to absolute (neighborhood) retracts:

**Definition 3.6.** A metrizable space $K$ is an absolute (neighborhood) retract (notation: $K \in \text{A}(\text{N})\text{R}$) provided it is a (neighborhood) retract of every metrizable space containing $K$ as a closed subset.

Here is the second most useful result on extending maps:

**Borsuk Homotopy Extension Theorem 3.7** [1, p. 94]. If $K \in \text{ANR}$ and $G : A \times I \cup X \times \{0\} \to K$, $A$ closed in a metrizable space $X$, then there is an extension $H : X \times I \to K$ of $G$.

The most fundamental results of the Theory of Retracts, which studies ANRs and ARs, are the following:

**Theorem 3.8** [12, 3.1–3.2, pp. 83–84]. Let $K$ be a metrizable space. $K \in \text{ANR}$ ($K \in \text{AR}$) iff $K \in \text{ANE}(X)$ ($K \in \text{AE}(X)$) for all metrizable spaces $X$.

**Theorem 3.9** (Dugundji, [1, 7.1, p. 77]). Convex subsets of normed vector spaces are ARs.

**Theorem 3.10** (Kuratowski–Wojdysławski [12, 2.1, p. 81]). Each metric space can be isometrically embedded as a closed subset of a convex subset of a Banach space of the same weight.
The following result is useful when discussing the relation between extending maps into \( K \) and maps into its cone:

**Lemma 3.11.** Suppose \( K \in \text{AR} \). If \( K \neq \{\text{point}\} \), then \( \text{Cone}(K) \) embeds in \( K^{\aleph_0} \) as its retract.

**Proof.** We may assume that \( K \) contains \( I \) as a closed subset. Notice that \( (K \times K)^{\aleph_0} \in \text{AR} \). Given \( n > 1 \), there is an extension \( f_n: \text{Cone}(K) \to K \times K \) of the inclusion \( K \times [0, 1 - 1/n] \to K \times K \). Notice that \( f: \text{Cone}(K) \to (K \times K)^{\aleph_0}, f(x) = \{f_n(x)\}_{n>1}, \) is a homeomorphic embedding of \( \text{Cone}(K) \) onto a closed subset of \( (K \times K)^{\aleph_0} \). Therefore, \( \text{Cone}(K) \) is a retract of \( K^{\aleph_0} \) (\( \text{Cone}(K) \in \text{AR} \) by Theorem 2.9). \( \square \)

**Theorem 3.12** (Przymusiński [19]). Suppose \( K \in \text{AR} \) is noncompact and \( A \) is a subset of a topological space. If every map \( f: A \to K \) extends over \( X \), then every map \( f: A \to L \) to a complete AR of weight at most \( \text{weight}(K) \) extends over \( X \).

**Proof.** If \( K \) is noncompact, then it contains a discrete set \( S \) of cardinality \( \text{weight}(K) \). Therefore, \( \text{Cone}(K) \) contains \( \text{Cone}(S) = J(S) \) as a retract. Since \( L \) is complete, it embeds as a closed subset of \( J(S)^{\aleph_0} \) (see [9, Exercise 4.4.B] or Theorem 7.4 here). If \( L \in \text{AR} \), it must be a retract of \( J(S)^{\aleph_0} \) which proves \( L \in \text{AE}(X, A) \). \( \square \)

**Theorem 3.13** (Morita [16]). Suppose \( X \) is \( m \)-collectionwise normal. Then, \( L \in \text{AE}(X) \) for every complete AR of weight at most \( m \).

**Proof.** Let \( S \) be of cardinality \( m \). By Corollary 2.10, \( J(S) \in \text{AE}(X) \) which, in view of Theorem 3.12, proves \( L \in \text{AE}(X) \). \( \square \)

**Theorem 3.14** (Przymusiński [19]). Suppose \( K \in \text{AR} \). Then every metrizable space of the same weight embeds in \( K^{\aleph_0} \). Moreover, if \( K \) is noncompact, then every completely metrizable space of the same weight embeds as a closed subspace of \( K^{\aleph_0} \).

**Proof.** If \( K = J(S) \), Theorem 3.14 follows from Theorem 4.4.9 and Problem 4.4.B of [9] (see also Theorems 7.3–7.4 here). If \( K = \{\text{point}\} \), then every space of the same weight is a one-point space. If \( \text{weight}(K) = \aleph_0 \), then \( K \) contains a copy of \( I \) and \( K^{\aleph_0} \) contains a copy of the Hilbert cube which is universal for all separable metrizable spaces. If \( \text{weight}(K) > \aleph_0 \), then \( K \) is noncompact and it contains a discrete set \( S \) of cardinality \( \text{weight}(K) \). Therefore, \( \text{Cone}(K) \) contains \( \text{Cone}(S) = J(S) \) and \( K^{\aleph_0} \) contains \( J(S)^{\aleph_0} \) as a closed subset which proves Theorem 3.14. \( \square \)

4. CW-complexes

It is time to present a subclass of ENRs, namely finite CW-complexes (see [25]). First, we need the concept of the adjoining space. Abstractly, adjoining \( Y \) to \( X \) along a
map \( f : A \rightarrow Y \), \( A \) closed in \( X \), means forming the quotient space \( Y \oplus X / \sim \), where \( y \sim x \) iff \( x \in A \) and \( y = f(x) \).

**Definition 4.1.** A finite CW-complex is a space \( K \) so that there is a finite sequence \( K_0 \subset \cdots \subset K_m = K \) so that \( K_0 \) is finite, \( K_{i+1} = K_i \cup f_i^{-1}I^{n_i+1} \) for some nondecreasing sequence \( n_0 \leq \cdots \leq n_{m-1} \) and \( f_i : S^{n_i} = \partial I^{n_i+1} \rightarrow K_i \). The subsets \( K_{i+1} - K_i \) (and points of \( K_0 \)) are called the open \((n_i+1)\)-cells of \( K \) (points of \( K_0 \) are called 0-cells), and their closures are called closed cells of \( K \).

**Proposition 4.2** [1, p. 116]. Finite CW-complexes are ENRs.

**Definition 4.3.** A CW-complex is a space \( K \) which is the union of \( \{K_s \mid s \in S\} \) such that the following conditions are satisfied:

1. each \( K_s \) is a finite CW-complex,
2. \( K_s \cap K_t \) is a subcomplex of both \( K_s \) and \( K_t \) if \( s, t \in S \),
3. \( U \) is open in \( K \) iff \( U \cap K_s \) is open in \( K_s \) for all \( s \in S \).

Maps to CW-complexes give rise to a natural example of local finiteness.

**Theorem 4.4.** Suppose \( X \) admits a perfect map onto a first countable space and \( K \) is a CW-complex. If \( f : X \rightarrow K \) is continuous, then \( \{f^{-1}(e) \mid e \text{ is an open cell of } K\} \) is locally finite.

**Proof.** Let \( p : X \rightarrow Y \) be a perfect map of \( X \) onto a first countable space. Suppose \( x_0 \in X \) is a point such that for each neighborhood \( U \) of \( x_0 \), \( U \) intersects infinitely many sets \( \{f^{-1}(e) \mid e \text{ is an open cell of } K\} \). Choose a basis \( U_n, n \geq 1 \), of neighborhoods of \( p(x_0) \) in \( Y \), and select, by induction, points \( x_n \in p^{-1}(U_n) \) so that all \( f(x_k), k \geq 1 \), belong to different open cells of \( K \). Let \( D_n = \{f(x_k) \mid k \geq n\} \) and notice that \( D_n \cap L \) is finite for every finite subcomplex \( L \) of \( K \). Therefore, \( D_n \) is closed in \( K \) and \( f^{-1}(D_n) \) is closed in \( X \). Since the intersection of all \( D_n \) is empty, there is \( m \) such that \( f^{-1}(D_m) \cap p^{-1}(p(x_0)) = \emptyset \). Hence, \( p^{-1}(U_n) \cap f^{-1}(D_m) = \emptyset \) for some \( n > m \) which contradicts \( x_n \in p^{-1}(U_n) \).

**Corollary 4.5.** Suppose \( X \) admits a perfect map onto a first countable space and \( K \) is a CW-complex. If \( f : X \rightarrow K \) is continuous, then for each point \( x \in X \) there is a neighborhood \( U \) such that \( f(U) \) is contained in a finite subcomplex of \( K \). Moreover, if \( X \) admits a perfect map onto a second countable space, then \( f(X) \) is contained in a countable subcomplex of \( K \).

**Proof.** By the previous result, there is a neighborhood \( U \) of \( x \) so that \( f(U) \) intersects only finitely many open cells of \( K \). Therefore, the minimal complex containing \( f(U) \) is finite. Suppose \( p : X \rightarrow Y \) is perfect and \( Y \) is second countable. Since \( p^{-1}(y) \) is compact, there is a neighborhood \( V_y \) of \( y \) in \( Y \) such that \( f(p^{-1}(V_y)) \) is contained in a finite subcomplex
of $K$. Pick a countable subcover of $\{V_y\}_{y \in Y}$ and notice that a countable union of finite subcomplexes is a countable subcomplex. □

The next two propositions point out the usefulness of locally finite families in constructing maps:

**Proposition 4.6.** If $\{A_s \mid s \in S\}$ is a locally finite closed family in $X$ and $f : X \to Y$ is a function such that $f|_{A_s}$ is continuous for all $s \in S$, then $f$ is continuous.

**Proposition 4.7** [9, 1.1.13]. If $\{A_s \mid s \in S\}$ is a locally finite family in $X$, then $\{\text{cl}_X(A_s) \mid s \in S\}$ is a locally finite family in $X$.

The next result presents a wide class of ANEs for metrizable spaces. It will be generalized later on.

**Theorem 4.8** (Kodama [13]). If $X$ is a metrizable space and $K$ is a CW-complex, then $K \in \text{ANE}(X)$.

5. Simplicial complexes

There are special CW-complexes called **simplicial complexes**. Simplicial complexes can be given a smaller topology via a natural metric. The advantage is that it is easier to construct maps to metric simplicial complexes, and the inclusion map $i : K_m \to K_{cw}$ (from $K$ equipped with the metric topology to $K$ with the CW-topology) is a homotopy equivalence.

We will use [15] as the reference for some results on simplicial complexes. The major difference with the approach sketched in this paper and that of [15] is that we consider simplicial complexes as subsets of a specific normed vector space which allows us to take advantage of algebraic constructions.

**Definition 5.1.** Given a set $S$ let $V\Sigma_S$ be the vector space of all functions $v : S \to \mathbb{R}$ from $S$ to the reals such that $\{v(s)\}_{s \in S}$ is absolutely summable. When equipped with the norm $|v| = \sum_{s \in S} |v(s)|$, $V\Sigma_S$ becomes a Banach space. In the future $v(s)$ will be denoted by $v_s$. The subspace of $V\Sigma_S$ consisting of all nonnegative $v$ such that $\sum_{s \in S} v(s) = 1$ is denoted by $\Sigma_S$. Notice that $\Sigma_S$ is a closed and convex subset of $V\Sigma_S$.

**Definition 5.2.** Let $S$ be a set. A family $\text{Soul}(K)$ of finite subsets of $S$ is a simplicial complex provided $A \in \text{Soul}(K)$ and $B \subseteq A$ implies $B \in \text{Soul}(K)$. The body $K$ of $\text{Soul}(K)$ (traditionally denoted by $|K|$) is the set of all $v \in \Sigma_S$ such that $\{s \in S \mid v_s \neq 0\} \in \text{Soul}(K)$.

By the set of vertices $K^0$ of $K$ we mean the set of all Kronecker delta functions $\delta_s : S \to \mathbb{R}$, where $\{s\} \in \text{Soul}(K)$. Recall that $\delta_s(t) = 0$ if $s \neq t$ and $\delta_s(s) = 1$. 

In the future we will concentrate on the body $K$ of a simplicial complex. We will assume that its soul is lurking somewhere and the name simplicial complex will be applied to $K$.

Notice that $\Sigma_S$ contains a maximal simplicial complex; its soul consists of all finite subsets of $S$. This complex will be denoted by $\Delta_S$ as it is a generalization of the $n$-simplex $\Delta_n$. Notice that $\Delta_S$ is dense in $\Sigma_S$.

There is an alternative way of defining a topology on every linear vector space $V$. The, so-called, weak topology $\nu_{\text{CW}}$ is formed by declaring $U \subset V$ to be open iff $U \cap F$ is open in every finite-dimensional subspace $F$ of $V$ (the topology on $F$ is determined by its euclidean structure). Therefore, there are two ways of defining a topology on a simplicial complex $K$; the metric $K_m$ and the weak $K_{\text{CW}}$ topologies.

**Proposition 5.3** [15, Appendix 1]. A simplicial complex $K$ with the weak topology has a natural CW structure.

Under this CW structure, open $n$-cells are sets of the form
\[
\{ v \in K \mid v_s > 0 \text{ iff } s \in A \},
\]
where $A$ is an element of $\text{Soul}(K)$ of cardinality $n + 1$. The closure of this open cell is called an $n$-simplex, and we will denote it by $\Delta_A$. Thus,
\[
\Delta_A = \{ v \in K \mid v_s = 0 \text{ if } s \notin A \}.
\]

The simplicial complex $K$ comes equipped with the set of maps $\lambda_s : K \rightarrow [0, 1]$ (the barycentric coordinates of $K$):
\[
\lambda_s(v) = v_s.
\]

In reality, the barycentric coordinates can be defined on $\Sigma_S$ by the same formula. These are obviously continuous on $\Sigma_S$ as $|\lambda_s(x) - \lambda_s(y)| \leq |x - y|$. In a sense, $\lambda_s : \Sigma_S \rightarrow [0, 1], s \in S$, form a complete set of maps:

**Proposition 5.4.** If $X$ is a topological space, then $f : X \rightarrow \Sigma_S$ is continuous iff $\lambda_s \circ f$ is continuous for all $s \in S$.

**Proof.** Suppose $\delta > 0$ and $x \in X$. Choose $s_1, \ldots, s_n \in S$ such that
\[
\sum_{i=1}^{n} \lambda_{s_i}(f(x)) > 1 - \delta.
\]
Find a neighborhood $U$ of $x$ in $X$ such that $|\lambda_{s_i}(f(x)) - \lambda_{s_i}(f(y))| < \delta/n$ for each $y \in U$ and each $i \leq n$. Notice that $|f(x) - f(y)| < 2\delta$ for each $y \in U$ which proves continuity of $f$ provided $\lambda_s \circ f$ are continuous for all $s \in S$. □

**Remark 5.5.** This proposition is well known in case of $\Sigma_S$ being replaced by a metric simplicial complex (see [15, Theorem 8, p. 301]).
Proposition 5.4 indicates that constructing maps to metric simplicial complexes is not particularly difficult. The next result points out the ease of constructing maps on simplicial complexes with the weak topology.

**Proposition 5.6** [15, Theorem 2, p. 290]. If $K$ is a simplicial complex, then $f : K_{CW} \to X$ is continuous iff $f|\Delta$ is continuous for all simplices $\Delta$ of $K$.

Propositions 5.4 and 5.6 indicate why it is beneficial to juggle both metric and CW-topologies of simplicial complexes.

One of the most useful constructions in the theory of simplicial complexes is that of the join. Typically, one defines the join of two simplicial complexes. We will define the join of an arbitrary family of simplicial complexes.

**Definition 5.7.** Suppose $K_t$ is a simplicial complex in $\Sigma_{S(t)}$ for each $t \in T$, where $\{S(t)\}_{t \in T}$ is a decomposition of a set $S$ into mutually disjoint sets. The join $\star_{t \in T} K_t$ is the simplicial complex $K$ in $\Sigma_S$ whose soul $\text{Soul}(K)$ is the family of all finite subsets $A$ of $S$ such that $A \cap S(t) \in \text{Soul}(K_t)$ for all $t \in T$.

**Example 5.8.** Notice that the join $\{v\} \star K$ of a one-point complex $\{v\}$ and a complex $K$ is homeomorphic to the cone $\text{Cone}(K)$ of $K$. In particular, $J(S)$ is the join $\{v\} \star S$. Indeed, each point of $\{v\} \star K$ can be expressed as $t \cdot v + (1 - t) \cdot k$ for some $t \in I$ (it is unique) and $k \in K$. Map $t \cdot v + (1 - t) \cdot k$ to $[k, t] \in \text{Cone}(K)$.

6. Partitions of unity

Maps into metric simplicial complexes give rise to a natural example of a partition of unity; given a map $f : X \to K_m$ one has the family of maps $\{\lambda_s \circ f : X \to [0, 1]\}_{s \in S}$ so that

$$\sum_{s \in S} \lambda_s \circ f = 1.$$ 

More generally,

**Definition 6.1.** A partition of unity on $X$ is a family of maps $\alpha = \{\alpha_s : X \to [0, 1]\}_{s \in S}$ so that

$$\sum_{s \in S} \alpha_s = 1.$$ 

By $\text{Cozero}(\alpha)$ we mean the family $\{\alpha_s^{-1}(0, 1]\}_{s \in S}$ of cozero-sets of all $\alpha_s$, $s \in S$.

**Definition 6.2.** A partition $\{\alpha_s : X \to [0, 1]\}_{s \in S}$ of unity on $X$ is called finite provided $\alpha_s = 0$ for all but finitely many $s \in S$. A partition $\alpha$ of unity on $X$ is called point-finite provided $\alpha|\{x\}$ is finite for all $x \in X$. A partition $\alpha$ of unity on $X$ is called locally-finite provided for each $x \in X$ there is a neighborhood $U$ so that $\alpha|U$ is finite.
Since the natural inclusion \( i : \mathcal{K}_\text{cw} \rightarrow \mathcal{K}_m \) is continuous, maps \( f : X \rightarrow \mathcal{K}_\text{cw} \) also give rise to a natural example of a partition of unity. The difference is that quite often this partition of unity is locally finite.

**Proposition 6.3.** Suppose \( X \) admits a perfect map onto a first countable space and \( K \) is a simplicial complex. For every map \( f : X \rightarrow \mathcal{K}_\text{cw} \), \( \{ \lambda_s \circ f \}_{s \in K^{(0)}} \) is a locally finite partition of unity on \( X \).

**Proof.** Given \( x \in X \), there is a neighborhood \( U \) of \( x \) such that \( f(U) \) is contained in a finite subcomplex \( L \) of \( K \) (see 4.5). Now, if \( U \) intersects \( \text{Cozero}(\lambda_s \circ f) \), then \( s \) must be a vertex of \( L \). \( \square \)

In view of Proposition 5.4, a partition \( \{ \alpha_s : X \rightarrow [0,1] \}_{s \in S} \) of unity on a space \( X \) is the same as a map \( \alpha : X \rightarrow \Sigma_S \). A natural question is to identify those partitions of unity which can be interpreted as maps to simplicial metric complexes. Obviously, such a partition must be point-finite. Actually, this is all we need to require.

**Definition 6.4.** Suppose \( \alpha = \{ \alpha_s : X \rightarrow [0,1] \}_{s \in S} \) is a partition of unity on \( X \). Its nerve \( \mathcal{N}(\alpha) \) is the simplicial complex \( K \) whose Soul\( (K) \) is defined as all finite subsets \( A \) of \( S \) so that there is \( x \in X \) with \( \alpha_s(x) \neq 0 \) for all \( s \in A \).

Notice that \( \mathcal{N}(\alpha) \) is the smallest subcomplex of \( \Sigma_S \) containing \( \alpha(X) \) if \( \alpha \) is a point-finite partition of unity.

The following result provides the main connection between maps and partitions of unity. It is an easy consequence of Propositions 5.4 and 6.3:

**Theorem 6.5.** Let \( X \) be a topological space.

1. There is a bijective correspondence between partitions \( \{ \alpha_s : X \rightarrow [0,1] \}_{s \in S} \) of unity on \( X \) and maps \( \alpha : X \rightarrow \Sigma_S \). Namely, \( \alpha_s = \lambda_s \circ \alpha \), where \( \lambda_s \) is the barycentric coordinate.

2. There is a bijective correspondence between point-finite partitions \( \{ \alpha_s : X \rightarrow [0,1] \}_{s \in S} \) of unity on \( X \) and maps \( \alpha : X \rightarrow (\Delta_S)_m \).

3. If \( X \) admits a perfect map onto a first countable space, then there is a bijective correspondence between locally finite partitions \( \{ \alpha_s : X \rightarrow [0,1] \}_{s \in S} \) of unity on \( X \) and maps \( \alpha : X \rightarrow (\Delta_S)_\text{cw} \).

In the future no distinction will be made between a point-finite (arbitrary) partition of unity and the associated map into its nerve (into \( \Sigma_S \)).

**Definition 6.6.** In view of Theorem 6.5 it makes sense to consider the space \( \mathcal{P}(X) \) of all partitions of unity on a given topological space \( X \). It contains natural subspaces:

1. \( \mathcal{P}(\text{finite})(X) \) of all finite partitions of unity on \( X \),

2. \( \mathcal{P}(\text{locally-finite})(X) \) of all locally-finite partitions of unity on \( X \).
(c) \( P(\text{point-finite})(X) \) of all point-finite partitions of unity on \( X \).

Also, for each cardinal number \( m \), it contains natural subspace \( P^m(X) \) (\( P^m(\text{finite})(X) \), \( P^m(\text{locally-finite})(X) \), \( P^m(\text{point-finite})(X) \)) of all (finite, locally-finite, point-finite) partitions of unity on \( X \) whose index set is of cardinality at most \( m \).

Notice that \( X \rightarrow P(X) \) is a contravariant functor: given a map \( f : X \rightarrow Y \) and given a partition \( \alpha = \{\alpha_s\}_{s \in S} \) of unity on \( Y \), one forms the partition \( f^*(\alpha) = \{\alpha_s \circ f\}_{s \in S} \). If \( X \) is a subset of \( Y \) and \( f \) is the inclusion map, it is customary to denote \( f^*(\alpha) \) by \( \alpha|_X \).

The three structures which are fundamental in our interpretation of set-theoretic topology are that of two partial orders and that of a semigroup with unit:

- \( P(X) \) has two natural partial orders: given \( \alpha = \{\alpha_s\}_{s \in S} \), \( \beta = \{\beta_r\}_{r \in T} \in P(X) \) we declare \( \alpha \leq \beta \) (\( \alpha \leq^* \beta \)) provided \( \text{Cozero}(\alpha) \) (star) refines \( \text{Cozero}(\beta) \). Notice that the trivial partition \( \{1\} \) of unity is the unique largest element for both orders.

- \( P(X) \) has a natural structure of a semigroup: given \( \alpha = \{\alpha_s\}_{s \in S} \), \( \beta = \{\beta_r\}_{r \in T} \in P(X) \) we declare
  \[
  \alpha \cdot \beta = \{\alpha_s \cdot \beta_r\}_{s \in S, r \in T}.
  \]

Notice that this multiplication is associative and the trivial partition \( \{1\} \) of unity serves as a unit of it.

Notice that each of \( P(\text{finite})(X) \), \( P(\text{locally-finite})(X) \), \( P(\text{point-finite})(X) \) is a subgroup of \( P(X) \). The same observation applies to \( P^m(X) \) as long as \( m \) is infinite.

Let us observe that the two partial order structures are preserved by the multiplication.

**Proposition 6.7.** Suppose \( \alpha = \{\alpha_s\}_{s \in S} \), \( \beta = \{\beta_r\}_{r \in T} \in P(X) \). Then

1. \( \text{Cozero}(\alpha \cdot \beta) = \text{Cozero}(\alpha) \cap \text{Cozero}(\beta) \),
2. if \( \alpha \leq \alpha' \) and \( \beta \leq \beta' \), then \( \alpha \cdot \beta \leq \alpha' \cdot \beta' \),
3. if \( \alpha \leq^* \alpha' \) and \( \beta \leq^* \beta' \), then \( \alpha \cdot \beta \leq^* \alpha' \cdot \beta' \).

The following proposition summarizes some of the results [9, 5.1]:

**Proposition 6.8.** Suppose \( \alpha \in P^m(X) \) (respectively \( \alpha \in P(\text{finite})(X) \)), where \( m \) is an infinite cardinal number. Then, there is \( \beta \in P^m(\text{locally-finite})(X) \) (respectively \( \beta \in P(\text{finite})(X) \)) such that \( \beta \leq^* \alpha \).

Notice that in the definition of \( \alpha \cdot \beta \) as \( \{\alpha_s \cdot \beta_t\}_{s \in S, t \in T} \), where \( \alpha = \{\alpha_s\}_{s \in S} \) and \( \beta = \{\beta_r\}_{r \in T} \), all we need is \( \beta_t \) to be defined and continuous on \( \text{Cozero}(\alpha_s) \) (rather than on the whole of \( X \)). This observation leads to the following generalization of multiplication of partitions of unity:

**Definition 6.9.** Suppose \( \alpha = \{\alpha_s\}_{s \in S} \) is a partition of unity on \( X \) and, for each \( s \in S \), \( \beta_s = \{\beta_{s,t}\}_{t \in S_s} \) is a partition of unity on \( \text{Cozero}(\alpha_s) \). The join \( \alpha \cdot \{\beta_s\}_{s \in S} \) of \( \{\beta_s\}_{s \in S} \) along \( \alpha \) is the partition of unity on \( X \) defined as \( \{\alpha_s \cdot \beta_{s,t}\}_{s \in S, t \in S_s} \). The convention here is that \( \alpha_s \cdot \beta_{s,t}(x) = 0 \) if \( x \notin \text{Cozero}(\alpha_s) \).
Now, we can show the connection between joins of partitions of unity and joins of simplicial complexes:

**Proposition 6.10.** Suppose \( \alpha = \{ \alpha_t \}_{t \in T} \) is a partition of unity on \( X \) and \( \{ S(t) \}_{t \in T} \) is a decomposition of \( S \) into mutually disjoint subsets. If \( \beta_t = \{ \beta_t,s \}_{s \in S(t)} \) is a partition of unity on Cozero(\(\alpha_t\)) for each \( t \in T \), then the nerve of the join of partitions \( \beta_t, t \in T \), along \( \alpha \) is a subcomplex of *\( t \in T N(\beta_t) \).

**Proof.** The join \( \gamma = \{ \gamma_s \}_{s \in S} \) of partitions \( \beta_t, t \in T \), along \( \alpha \) is given by \( \gamma_s = \alpha_t \cdot \beta_t,s \), where \( t \) is the unique element of \( T \) with \( s \in S(t) \). If \( \gamma_s(x) \neq 0 \) for all \( s \in A \subset S \), then \( A \cap S(t) \in \text{Soul}(N(\beta_t)) \) for all \( t \in T \). Thus, if \( A \) is finite, it belongs to the soul of *\( t \in T N(\beta_t) \). \( \square \)

Here is a special case of Proposition 6.10:

**Corollary 6.11.** Suppose \( f : X \rightarrow [0, 1] \) is a map and \( \alpha \) is a partition of unity on Cozero(\(f\)). Let \( \beta \) be the join of partitions \( \alpha \) and \( \{ 1 \} \) along \( \{ f, 1 - f \} \). Then, \( N(\beta) \) is isomorphic to a subcomplex of Cone(\(N(\alpha))\).

More generally,

**Corollary 6.12.** Suppose \( f : X \rightarrow [0, 1] \) is a map, \( \alpha \) is a partition of unity on Cozero(\(f\)) and \( \beta \) is a partition of unity on Cozero(\(1-f\)). Let \( \gamma \) be the join of partitions \( \alpha \) and \( \beta \) along \( \{ f, 1 - f \} \). Then, \( N(\gamma) \) is isomorphic to a subcomplex of \( N(\alpha) \star N(\beta) \).

There is another operation on partitions of unity which is useful. We will call it a contraction of a partition of unity:

**Definition 6.13.** Suppose \( \alpha = \{ \alpha_s \}_{s \in S} \) is a partition of unity on \( X \) and \( D = \{ D_t \}_{t \in T} \) is a decomposition of \( S \) into mutually disjoint subsets. The contraction \( \alpha_D \) of \( \alpha \) along \( D \) is the partition of unity on \( X \) defined as

\[
\left\{ \sum_{s \in D_t} \alpha_s \right\}_{t \in T}.
\]

In particular, if \( A \subset X \), we put

\[
D = \{ s \in S \mid \alpha_s|A \neq 0 \} \cup \bigcup \{ \{ s \} \mid \alpha_s|A = 0 \}
\]

and we define \( \alpha_A \) as \( \alpha_D \).

**Remark 6.14.** Notice that \( \alpha \leq^* \beta \) means \( \alpha_{\{ x \}} \leq \beta \) for all \( x \in X \).

The following result shows that contractions and joins of partitions of unity occur naturally:

**Theorem 6.15.** Suppose \( D = \{ D(t) \}_{t \in T} \) is a decomposition of a set \( S \) into mutually disjoint subsets. There is a correspondence between maps \( f : X \rightarrow \Sigma_S \) and joins of
partitions of unity $\beta_t$, $t \in T$, along a partition of unity $\alpha = \{\alpha_t\}_{t \in T}$ on $X$. Namely, $\alpha$ is the contraction of $f$ along $D$, and $\beta_t = \{\beta_{t,s}\}_{s \in D(t)}$, where $\beta_{t,s}(x) = \lambda_s \circ f(x)/\alpha_t(x)$ for $x \in \text{Cozero}(\alpha_t)$.

**Proof.** It is contained in the statement. □

Here is a simple way of constructing partitions of unity with values in $J(S)$:

**Proposition 6.16.** Suppose $\{\alpha_s : X \to [0, 1]\}_{s \in S}$ is a family of maps such that:
1. $\{\text{Cozero}(\alpha_s)\}_{s \in S}$ is locally-finite and consists of mutually disjoint sets in $X$,
2. $\sum_{s \in S} \alpha_s \leq 1$.

Then there is a partition of unity $\alpha = \{h\} \cup \{\alpha_s\}_{s \in S} : X \to J(S)$ on $X$ such that $1 - h = \sum_{s \in S} \alpha_s$.

**Proof.** Notice that $f = \sum_{s \in S} \alpha_s$ is continuous with values in $I$ and put $h = 1 - f$. Now, $\alpha = \{h\} \cup \{\alpha_s\}_{s \in S}$ is a partition of unity whose nerve is a subcomplex of $J(S)$. □

**Corollary 6.17** [19,9]. If $S$ is infinite, then $J(S)^2$ contains the reals as a closed subset.

**Proof.** Assume $S$ contains the integers. Given an interval $[a, b]$ in $\mathbb{R}$, let $\kappa_{a,b} : \mathbb{R} \to [0, 1]$ be the piecewise linear map which is 0 on $\mathbb{R} - (a, b)$ and maps $(a + b)/2$ to 1. Let $\alpha = \{h\} \cup \{\kappa_{n,n+1}\}_{n \in \mathbb{Z}} : \mathbb{R} \to J(\mathbb{Z})$ and $\beta = \{g\} \cup \{\kappa_{n+1/2,n+3/2}\}_{n \in \mathbb{Z}} : \mathbb{R} \to J(\mathbb{Z})$. Notice that $f : \mathbb{R} \to J(\mathbb{Z})^2$, $f(x) = (\alpha(x), \beta(x))$, embeds $\mathbb{R}$ as a closed subset of $J(\mathbb{Z})^2$. □

It makes sense to investigate which open families in $X$ are of the form $\text{Cozero}(\alpha)$ for some partition of unity $\alpha$ on $X$. Here is a partial answer:

**Proposition 6.18.** Suppose $\{U_s\}_{s \in S}$ is a locally-finite family of cozero-sets in $X$. There is a partition of unity $\alpha = \{h\} \cup \{\alpha_s\}_{s \in S}$ on $X$ such that $U_s = \text{Cozero}(\alpha_s)$ for $s \in S$.

**Proof.** Given $s \in S$, choose $f_s : X \to [0, 1]$ so that $\text{Cozero}(f_s) = U_s$. Notice that $f = \sum_{s \in S} f_s$ is continuous and put $g = f/(1 + f)$. Notice that $\beta = \{f_s/f\}_{s \in S}$ is a partition of unity on $\text{Cozero}(g)$. Let $\alpha$ be the join of $\beta$ and $\{1\}$ along $\{g, h\}$, where $h = 1 - g$. □

7. Partitions of unity and metrization

According to Proposition 5.4, the space $\Sigma_S$ (and all simplicial complexes contained in it) has a partition of unity which completely determines if a map $f : X \to \Sigma_S$ is
continuous or not. It turns out this property characterizes all metrizable spaces. In a way we have a metrization criterion in terms of partitions of unity:

Theorem 7.1. Suppose \( X \) is a \( T_0 \)-space of weight \( \text{Card}(S) \geq \aleph_0 \). The following conditions are equivalent:

1. \( X \) embeds into \( \Sigma_S \).
2. There is a partition \( \alpha = \{ \alpha_s \}_{s \in S} \) of unity on \( X \) such that \( f : Y \to X \) is continuous iff \( \alpha_s \circ f \) is continuous for all \( s \in S \).
3. \( X \) has a \( \sigma \)-locally-finite basis consisting of cozero-sets.
4. There is a partition \( \alpha = \{ \alpha_s \}_{s \in S} \) of unity on \( X \) such that \( \text{Cozero}(\alpha) \) is a basis of \( X \).

Proof. (1) \( \Leftrightarrow \) (2). If \( \alpha : X \to \Sigma_S \) is an embedding, then (by Proposition 5.4) \( f : Y \to X \) is continuous iff \( \alpha_s \circ f \) is continuous for all \( s \in S \). Conversely, assume that there is a partition \( \alpha = \{ \alpha_s \}_{s \in S} \) of unity on \( X \) such that \( f : Y \to X \) is continuous iff \( \alpha_s \circ f \) is continuous for all \( s \in S \). Notice that \( \alpha \) is one-to-one. Indeed, put the antidiscrete topology on \( \alpha^{-1}(x) \) and let \( i : \alpha^{-1}(x) \to X \) be the inclusion map. Since \( \alpha_s \circ i \) is continuous (it is constant) for all \( s \in S \), then \( i \) is continuous. Any antidiscrete space which is \( T_0 \) is either empty or a single point. Thus, \( \alpha \) is one-to-one. Let \( j : \alpha(X) \to X \) be its inverse. Now, \( \alpha_s \circ j = \lambda_s|\alpha(X) \) for each \( s \in S \). Thus, \( j \) is continuous and \( \alpha \) is an embedding.

(1) \( \Rightarrow \) (3). It suffices to show that \( \Sigma_S \) has a \( \sigma \)-locally-finite basis. Choose a countable basis \( \{ U_n \}_{n \geq 1} \) of \( [0, 1] \) and notice that \( U_n = \{ \lambda_s^{-1}(U_n) \}_{s \in S} \) is locally-finite. Given a finite subset \( C \) of natural numbers, the intersection \( \bigcap_{n \in C} U_n \) is locally-finite and \( \bigcup_{C \subset 2^C} U_C \) is a basis of \( \Sigma_S \).

(3) \( \Rightarrow \) (4). Suppose \( U_n \) is a locally finite family consisting of cozero-sets in \( X \) and \( \bigcup_{n \geq 1} U_n \) is a basis of \( X \). Let \( \alpha_n \) be a partition of unity on \( X \) such that \( \text{Cozero}(\alpha_n) \) contains \( U_n \) (see Proposition 6.18). Finally, let \( \alpha \) be the join of \( \{ \alpha_n \}_{n \geq 1} \) along \( \{ 2^{-n} \}_{n \geq 1} \).

(4) \( \Rightarrow \) (2). Notice that \( f^{-1}(\text{Cozero}(\alpha_s)) = \text{Cozero}(\alpha_s \circ f) \). Thus, continuity of all \( \alpha_s \circ f \), \( s \in S \), implies continuity of \( f \) if \( \{ \text{Cozero}(\alpha_s) \}_{s \in S} \) is a basis of \( X \). \( \square \)

Let us show the usefulness of partitions of unity in a variant of Kuratowski–Wojdysławski theorem:

Theorem 7.2. Suppose \( X \) is a \( T_0 \)-space of weight \( \text{Card}(S) \geq \aleph_0 \) and \( \alpha = \{ \alpha_s \}_{s \in S} \) is a partition of unity on \( X \) such that \( \text{Cozero}(\alpha) \) is a basis of \( X \). Then:

1. \( \alpha : X \to \Sigma_S \) embeds \( X \) as a closed subset of the convex hull of \( \alpha(X) \).
2. Points \( \{ \alpha(x) \}_{x \in X} \) are linearly independent in \( V \Sigma_S \).

Proof. Suppose \( \beta = \sum_{i=1}^n c_i \cdot \alpha(x_i) = 0 \), \( x_i \neq x_j \) if \( i \neq j \), and \( c_i \neq 0 \). Choose \( s \in S \) such that \( \text{Cozero}(\alpha_s) \) contains only \( x_1 \) among \( x_1, \ldots, x_n \). Then, \( 0 = \beta(s) = c_1 \cdot \alpha_s(x_1) \neq 0 \), a contradiction. This proves (2).

Suppose \( \alpha(X) \) is not closed in its convex hull. Thus, there is a sequence \( x_n \in X \), \( n \geq 1 \), such that \( \alpha(x_n) \) converges to \( \beta = \sum_{i=1}^n c_i \cdot \alpha(a_i) \neq \alpha(X) \) for some \( a_1, \ldots, a_n \in X \). Notice that \( Y = \{ x_n \}_{n \geq 1} \) does not contain a convergent subsequence. Therefore, we
may assume that \( Y \) and \( A = \{ a_i \}_{1 \leq i \leq m} \) are disjoint. Also, we may assume that \( c_1 \neq 0 \). Choose \( s \in S \) such that Cozero(\( \alpha_s \)) contains only \( a_1 \) among \( Y \) and \( a_1, \ldots, a_m \). Then, \( \beta(s) = c_1 \cdot \alpha_s(a_1) \neq 0 \) and \( \alpha_s(x_n) = 0 \) for all \( n \), a contradiction. This proves (1). \( \square \)

**Theorem 7.3.** Suppose \( X \) is a \( T_0 \)-space of weight \( \text{Card}(S) \geq \aleph_0 \). The following conditions are equivalent:

1. \( X \) embeds in the countable product \( J(S)^\aleph_0 \).
2. \( X \) has a \( \sigma \)-discrete basis consisting of cozero-sets.

**Proof.** (1) \( \Rightarrow \) (2). Notice that \( J(S) \) has a \( \sigma \)-discrete basis. Indeed, choose a countable basis \( \{ U_n \}_{n \geq 1} \) of \( [0, 1) \) and notice that \( \{ \{ s \} \times U_n \}_{s \in S} \) is a discrete family of cozero-sets in \( J(S) \). Add a countable basis of the vertex of \( J(S) = S \times [0, 1]/S \times \{ 1 \} \). Now, notice that \( J(S)^\aleph_0 \) has a \( \sigma \)-discrete basis consisting of cozero-sets.

(2) \( \Rightarrow \) (1). Suppose \( \mathcal{U}_n \) is a discrete family consisting of cozero-sets in \( X \) and \( \bigcup_{n=1}^{\infty} \mathcal{U}_n \) is a basis of \( X \). Each \( \mathcal{U}_n \) gives rise to a partition of unity \( \alpha_n : X \rightarrow J(S) \) (see Proposition 6.16). Now, the diagonal \( f \) of all \( \{ \alpha_n \}_{n \geq 1} \) (i.e., \( f(x) = \{ \alpha_n(x) \}_{n \geq 1} \)) is an embedding of \( X \) into the countable product \( J(S)^\aleph_0 \). \( \square \)

**Corollary 7.4.** Suppose \( X \) is a completely metrizable space of weight \( \text{Card}(S) \geq \aleph_0 \). Then \( X \) embeds in the countable product \( J(S)^\aleph_0 \) as a closed subset.

**Proof.** By Theorem 7.3, \( X \) embeds in \( J(S)^\aleph_0 \). Therefore, \( X \) embeds in \( J(S) \times \mathbb{R}^\aleph_0 \) as a closed subset. Use Corollary 6.17. \( \square \)

8. Extending partitions of unity

In this section we will discuss the problem of extending partitions of unity. The purpose of the next result is to show that quite often there is no need to extend a partition of unity over the whole \( X \).

**Theorem 8.1.** Suppose \( A \) is a subset of a topological space and \( S \) is a set. Then the following conditions are equivalent:

1. Every (point-finite, locally-finite) partition of unity \( \alpha = \{ \alpha_s \}_{s \in S} \) on \( A \) extends over \( X \).
2. Every (point-finite, locally-finite) partition of unity \( \alpha = \{ \alpha_s \}_{s \in S} \) on \( A \) extends over a neighborhood \( U \) which is a cozero-set in \( X \) so that there is a zero-set \( B \) of \( X \) with \( A \subset B \subset U \).

**Proof.** It suffices to show (1) \( \Rightarrow \) (2) (to prove (2) \( \Rightarrow \) (1) take \( U = B = X \)). Notice that \( S \neq \emptyset \). Suppose \( \alpha = \{ \alpha_s : U \rightarrow [0, 1] \}_{s \in S} \) is a (point-finite, locally-finite) partition of unity on a cozero-set neighborhood \( U \) of \( A \) of \( X \) and suppose \( B \) is a zero set in \( X \), \( A \subset B \subset U \). First we need a map \( \gamma : X \rightarrow [0, 1] \) such that \( \gamma^{-1}(1) = B \) and \( \gamma^{-1}(0) = X - U \). This is accomplished as follows. Choose a continuous function \( \kappa : X \rightarrow [0, 1] \) so
that \( \kappa^{-1}(0, 1] = U \) and choose a continuous function \( \nu: X \to [0, 1] \) so that \( B = \nu^{-1}(0) \). Define \( \gamma: X \to [0, 1] \) by \( \gamma(x) = \kappa(x)/((\nu(x) + \kappa(x)) \). If \( \alpha \) is point-finite or locally-finite, we need to improve \( \gamma \). Let \( W = \gamma^{-1}(1/2, 1] \) and pick \( \mu: X \to [0, 1] \) so that \( \mu^{-1}(1) = B \) and \( \mu^{-1}(0) = X - W \) (\( \mu = \max(2 \cdot \gamma - 1, 0) \) would do). Now, consider the join of \( \alpha \) and \( \{1\} \) along \( \{\mu, 1 - \mu\} \). Contract \( 1 - \mu \) by adding it to an arbitrary remaining term of the join. Here are the details: Pick \( s_0 \in S \) and define \( \beta_s: X \to [0, 1] \) as follows:

\[
\beta_s = \begin{cases} 
\alpha_s(x) \cdot \mu(x), & \text{if } x \in U, \\
0, & \text{if } x \in X - U,
\end{cases}
\]

if \( s \neq s_0 \), and

\[
\beta_s = \begin{cases} 
\alpha_s(x) \cdot \mu(x) + 1 - \mu(x), & \text{if } x \in U, \\
1, & \text{if } x \in X - U,
\end{cases}
\]

if \( s = s_0 \). Notice that \( \{\beta_s\}_{s \in S} \) is a (point-finite, locally-finite) partition of unity on \( X \) which extends \( \{\alpha_s | A\}_{s \in S} \). \( \Box \)

Now, let us discuss the issue of extending finite partitions of unity.

**Theorem 8.2.** Suppose \( A \) is a subset of a topological space \( X \). Then the following conditions are equivalent:

1. Every map \( f: A \to [0, 1] \) extends over \( X \).
2. Every finite partition of unity \( \alpha = \{\alpha_s\}_{s \in S} \) on \( A \) extends to a finite partition of unity on \( X \).
3. Every map \( f: A \to K \) to a compact AR extends over \( X \).
4. Every map \( g: A \to L \) to a compact ANR extends over a neighborhood \( U \) which is a cozero-set in \( X \) so that there is a zero-set \( B \) of \( X \) with \( A \subset B \subset U \).
5. Every map \( g: A \to L \) to a compact ANR extends to \( g': X \to \text{Cone}(L) \).
6. Every map \( g: A \to L \) to a compact ANR extends to \( g': X \to L' \) for some compact ANR \( L' \) containing \( L \).

**Proof.** (1) \( \Rightarrow \) (2). Notice that \( I^m \in \text{AE}(X, A) \) for all \( m \geq 1 \). A partition \( \{f_0, \ldots, f_m\} \) of unity on \( A \) is synonymous with the map \( f: A \to \Delta_m \) from \( A \) to the \( (m) \)-simplex \( \Delta_m \) so that \( f_i = \lambda_i \circ f \), \( \lambda_i \) being the barycentric coordinate of the \( i \)-th vertex. Since \( \Delta_m \) is homeomorphic to \( [0, 1]^m \), \( f \) extends over \( X \).

(2) \( \Rightarrow \) (3). Since a compact AR is a retract of the Hilbert cube, the same argument works as in (1) \( \Rightarrow \) (2).

(3) \( \Rightarrow \) (5). Suppose \( g: A \to L \) is a map to a compact ANR. Notice that \( \text{Cone}(L) \) is a compact AR (it follows from Theorem 2.9).

(4) \( \iff \) (5). This is Lemma 2.8.

(5) \( \Rightarrow \) (6). (6) is a special case of (5).

(6) \( \Rightarrow \) (1). Notice that \( I \) is a retract of every compact space containing it. \( \Box \)

**Remark 8.3.** (1) \( \iff \) (3) was proved by Morita [16].
Next we turn to the question of extending locally-finite partitions of unity:

**Theorem 8.4.** If \( X \) is a paracompact space and \( A \) is a closed subset of \( X \), then every locally-finite partition of unity \( \alpha \) on \( A \) extends to a locally-finite partition of unity on \( X \).

**Proof.** Let \( \mathcal{U} \) be an open cover of \( X \) such that for each \( U \in \mathcal{U} \), \( \alpha|U \cap A \) is finite. There is a locally-finite partition of unity \( g \) on \( X \) such that \( \operatorname{cl}_X(\operatorname{Cozero}(g)) \) refines \( \mathcal{U} \) (use [9, 5.1.9 and 5.1.11]). This partition of unity induces a map \( g: X \to L \) to a simplicial complex with the CW topology \( L \) so that \( g^{-1}(\Delta) \) is contained in an element of \( \mathcal{U} \) for each simplex \( \Delta \) of \( L \). Given a vertex \( v \) of \( L \) we can extend \( \alpha|g^{-1}(v) \cap A \) over \( g^{-1}(v) \) as \( g^{-1}(v) \) is a normal space. Suppose \( k \geq 0 \) and for each \( k \)-simplex \( \Delta \) of \( L \) there is a finite partition of unity \( \alpha_\Delta \) on \( g^{-1}(\Delta) \) such that the following conditions are satisfied:

1. \( \alpha_\Delta|g^{-1}(\Delta) \cap A = \alpha|g^{-1}(\Delta) \cap A \),
2. if \( \Delta, \Delta' \) are two \( k \)-simplices of \( L \), then \( \alpha_\Delta = \alpha_{\Delta'} \) on \( g^{-1}(\Delta \cap \Delta') \).

Given an \((k+1)\)-simplex \( \Delta \) of \( L \) one can paste partitions \( \alpha_s, s \) a \( k \)-dimensional face of \( \Delta \), and obtain a partition \( \alpha_{\partial \Delta} \) on \( g^{-1}(\partial \Delta) \). Notice that one can extend \( \alpha_{\partial \Delta} \) to \( \alpha_\Delta \) on \( g^{-1}(\Delta) \) so that \( \alpha_\Delta|g^{-1}(\Delta) \cap A = \alpha|g^{-1}(\Delta) \cap A \) (use Theorem 8.2). Now, all partitions of unity \( \alpha_\Delta \) on \( g^{-1}(\Delta) \) can be pasted to obtain a partition of unity \( \beta \) on \( X \) (use Proposition 4.6 and the fact that \( g \) is locally-finite to conclude that each term of \( \beta \) is continuous). Since \( g \) is locally-finite, each \( x \in X \) has a neighborhood \( U \) such that \( g(U) \) is contained in a finite subcomplex \( K \) of \( L \). Since, by construction, \( \beta|U \) is finite, we infer that \( \beta|U \) is finite. Thus, \( \beta \) is locally-finite. Also, by construction, \( \beta|A = \alpha \). \( \square \)

**Theorem 8.5.** If \( X \) is a metrizable space, then every (point-finite) partition of unity \( \alpha \) on a closed subset \( A \) of \( X \) extends to a (point-finite) partition of unity \( \beta \) on \( X \) so that \( \beta(X) \) is contained in the convex hull of \( \alpha(A) \).

**Proof.** First, we pick a retraction \( r: X \to A \) (not necessarily continuous) so that \( d(x, r(x)) < 2 \text{dist}(x, A) \) if \( x \in X - A \) (in this way \( r \) is continuous at points of \( A \)). If \( a \in A \) we put \( r(a) = a \) and if \( x \notin A \), then \( \text{dist}(x, A) > 0 \) so that there is \( r(x) \in A \) with \( d(x, r(x)) < 2 \text{dist}(x, A) \). Second, we pick a locally-finite partition \( \{\gamma_t\}_{t \in T} \) of unity on \( U = X - A \) so that if \( x_n, y_n \in \operatorname{Cozero}(\gamma_{t_n}) \) and \( x_n \) converges to \( x_0 \in A \), then \( y_n \) converges to \( x_0 \in A \). This is accomplished as follows: At each point \( x \in X - A \) there is an open ball \( B(x) \) so that \( |\text{dist}(x, A) - \text{dist}(y, A)| < \text{dist}(x, A)/2 \) for all \( y \in B(x) \) (this follows from the continuity of the \( \text{dist}(-, A) \) function). Now, if \( x_n, y_n \in B(t_n) \) and \( x_n \) converges to \( x_0 \in A \), then \( y_n \) converges to \( x_0 \in A \). Since \( X - A \) is paracompact, there is a locally-finite partition of unity \( \{\gamma_t\}_{t \in T} \) on \( X - A \) so that for each \( t \in T \) there is \( z_t \in X - A \) with \( \operatorname{Cozero}(\gamma_t) \subseteq B(z_t) \). Third, choose \( z_t \in \operatorname{Cozero}(\gamma_t) \) for each \( t \in T \). Finally, we define \( \beta_s(x) \) for \( x \in X - A \) as

\[
\beta_s(x) = \sum_{t \in T} \gamma_t(x) \cdot (\alpha_s \circ (r(x_t)))
\]
and we define \( \beta_s(x) \) for \( x \in A \) as \( \alpha_s(x) \). Clearly, \( \{\beta_s\}_{s \in S} \) is a partition of unity, so the difficulty lies in proving that \( \beta_s \) is continuous at \( x_0 \in A \). Suppose \( x_n \to x_0 \) as \( n \to \infty \) but \( |\beta_s(x_n) - \alpha_s(x_0)| > \varepsilon > 0 \) for all \( n \). Since

\[
\beta_s(x_n) - \alpha_s(x_0) = \sum_{t \in T} \gamma_t(x_n) \cdot \left( \alpha_s \circ (r(x_t)) - \alpha_s(x_0) \right),
\]

there is, for each \( n \), \( t_n \) so that \( \gamma_{t_n}(x_n) \neq 0 \) and \( |\alpha_s \circ (r(x_{t_n})) - \alpha_s(x_0)| > \varepsilon \). Then, \( x_{t_n} \to x_0 \) which implies \( r(x_{t_n}) \to x_0 \) and \( \alpha_s \circ (r(x_{t_n})) \to \alpha_s(x_0) \), a contradiction. \( \square \)

**Remark 8.6.** The proof of Theorem 8.5 is much simpler than the classic proof of the Dugundji Theorem 3.9 (see [11] or [12]) and, for practical purposes, Theorem 8.5 is equivalent to Theorem 3.9. Since metrizable spaces are paracompact one would like, in view of Theorem 8.4, to include locally-finite partitions of unity in the statement of Theorem 8.5. However, the author failed in efforts to modify the proof in order to make it work for the locally-finite case.

### 9. Extending maps into metric simplicial complexes

In this section we prove an analog of Theorem 8.2 for point-finite partitions of unity.

**Theorem 9.1.** Suppose \( A \) is a subset of a topological space \( X \) and \( m \) is an infinite cardinal number. Then the following conditions are equivalent:

1. Every map \( f : A \to K \) to a contractible metric simplicial complex (of weight at most \( m \)) extends over \( X \).
2. Every map \( g : A \to L \) to a metric simplicial complex (of weight at most \( m \)) extends over a neighborhood \( U \) which is a cozero-set in \( X \) so that there is a zero-set \( B \) of \( X \) with \( A \subset B \subset U \).
3. Every map \( g : A \to L \) to a metric simplicial complex (of weight at most \( m \)) extends to \( g' : X \to \text{Cone}(L) \).
4. Every map \( g : A \to L \) to a metric simplicial complex (of weight at most \( m \)) extends to \( g' : X \to L' \) for some metric simplicial complex \( L' \) (of weight at most \( m \)) containing \( L \) as a subcomplex.
5. Every point-finite partition of unity \( \alpha = \{\alpha_s\}_{s \in S} \) (with \( \text{card}(S) \leq m \)) on \( A \) extends to a point-finite partition of unity on \( X \).

**Proof.** (3) is a special case of (1).

(2) \( \iff \) (3) follows from Lemma 2.8.

(3) \( \Rightarrow \) (4). Take \( L' = \text{Cone}(L) \).

(4) \( \Rightarrow \) (5). A point-finite partition of unity \( \alpha \) on \( A \) can be considered as a map \( \alpha : A \to L = \mathcal{N}(\alpha) \) (see Theorem 6.5). Choose an extension \( \beta : X \to L' \) of \( \alpha \), where \( L' \) contains \( L \) as a subcomplex. Let \( r : V \to L \) be a retraction of a neighborhood \( V \) of \( L \) in \( L' \) onto \( L \) [15, Theorem 12, p. 305]. Now, \( B = \beta^{-1}(L) \) is a zero-set in \( X \), \( U = \beta^{-1}(V) \) is a cozero-set in \( X \) and \( r \circ \beta : U \to L \) extends \( \alpha \). By Theorem 8.1, \( \alpha \) extends to a point-finite partition of unity on \( X \).
(5) \(\Rightarrow\) (1). Suppose \(f : A \to K\) is a map to a contractible metric simplicial complex (of weight at most \(m\)). By Theorem 6.5, \(f\) is synonymous with a point-finite partition of unity \(f : A \to \Sigma_s, S\) being the set of vertices of \(K\), so that \(f(A) \subset K\). Let \(f' : X \to \Sigma_S\) be a point-finite partition of unity which extends \(f\). Since \(K\) is contractible, there is a retraction \(r : K \cup N'(f') \to K\). Notice that \(r \circ f' : X \to K\) extends \(f\). \(\Box\)

Right now we can give an answer to Problem 1.3 in the special case of complete simplicial complexes.

**Theorem 9.2.** Suppose \(K\) is a complete metric simplicial complex. For every topological space \(X\) the following conditions are equivalent:

1. \(K \in \text{ANE}(X)\).
2. \(K^0 \in \text{ANE}(X)\).

**Proof.** Since \(K^0\) is a neighborhood retract of \(K\), (1) \(\Rightarrow\) (2) for all complexes \(K\).

(2) \(\Rightarrow\) (1). If \(K = K^0\), it is trivial. If \(K\) is compact, it follows from Theorem 8.2. Finally, if \(K\) is not compact, then Theorem 2.9 says \(\text{Cone}(K^0) \in \text{AE}(X)\), Theorem 3.12 says \(\text{Cone}(K) \in \text{AE}(X)\), and Theorem 2.9 says \(K \in \text{ANE}(X)\). \(\Box\)

We can also give a generalization of Theorem 2.4:

**Theorem 9.3.** Suppose \(K\) is a contractible, complete, metric simplicial complex, and \(X\) is a topological space. Then, \(K^0 \in \text{ANE}(X)\) iff \(K \in \text{AE}(X)\).

**Proof.** Only the case of \(K \neq K^0\) is of interest. As in the proof of Theorem 9.2, we get \(K^0 \in \text{ANE}(X)\) implies \(\text{Cone}(K) \in \text{AE}(X)\). Since \(K\) is contractible, it is a retract of \(\text{Cone}(K)\) (actually, contractibility means precisely that) and \(K \in \text{AE}(X)\). \(\Box\)

10. Extending maps into metrizable spaces

In this section we prove analogs of Theorem 9.1 for arbitrary (complete) ANRs.

**Theorem 10.1.** Suppose \(A\) is a subset of a topological space \(X\) and \(m\) is an infinite cardinal number. Then the following conditions are equivalent:

1. Every map \(f : A \to K\) to an AR (of weight at most \(m\)) extends over \(X\).
2. Every map \(g : A \to L\) to an ANR (of weight at most \(m\)) extends over a neighborhood \(U\) which is a cozero-set in \(X\) so that there is a zero-set \(B\) of \(X\) with \(A \subset B \subset U\).
3. Every map \(g : A \to L\) to an ANR (of weight at most \(m\)) extends to \(g' : X \to \text{Cone}(L)\).
4. Every map \(g : A \to L\) to a metrizable space (of weight at most \(m\)) extends to \(g' : X \to L'\) for some metrizable space \(L'\) (of weight at most \(m\)) containing \(L\) as a closed subset.
(5) Every partition of unity \( \alpha = \{\alpha_s\}_{s \in S} \) (with \( \text{card}(S) \leq m \)) on \( A \) extends to a partition of unity \( \beta \) on \( X \) so that \( \alpha(A) = \beta(B) \) for some zero-set \( B \) of \( X \) containing \( A \).

**Proof.** (3) is a special case of (1) as \( \text{Cone}(L) \in \text{AR} \) if \( L \in \text{ANR} \) (see Theorem 2.9). (2) \( \iff \) (3) follows from Lemma 2.8. (3) \( \Rightarrow \) (4). Take \( L' = \text{Cone}(L) \).

(4) \( \Rightarrow \) (1). Suppose \( f : A \to K \) is a map to an AR (of weight at most \( m \)) and \( f' : X \to K' \) is an extension of \( f \) with \( K \) being a closed subset of \( K' \). Since \( K \) is an AR, there is a retraction \( r : K' \to K \). Notice that \( r \circ f' : X \to K \) extends \( f \).

(1) \( \Rightarrow \) (5). Suppose \( \alpha : A \to \Sigma_S \) is a partition of unity on \( A \). Choose \( j : \alpha(A) \to \Sigma_S \) so that \( j(\alpha(A)) \) is closed in its convex hull \( C \) and \( j : \alpha(A) \to j(\alpha(A)) \) is a homeomorphism (see Theorem 7.2). Since \( C \in \text{AR} \), there is \( \gamma : X \to C \) extending \( j \circ \alpha \). Let \( B = \gamma^{-1}(j(\alpha(A))) \) and notice that \( B \) is a zero-set in \( X \) containing \( A \). Since \( \Sigma_S \in \text{AR} \), there is an extension \( \mu : C \to \Sigma_S \) of \( j^{-1} : j(\alpha(A)) \to \alpha(A) \). Put \( \beta = \mu \circ \gamma \).

(5) \( \Rightarrow \) (1). Suppose \( f : A \to K \) maps \( A \) to an AR (of weight at most \( m \)). By Theorem 7.1 we may assume that \( K \subset \Sigma_S \) (and \( \text{Card}(S) \leq m \)). Extend \( f \) to \( f' : X \to \Sigma_S \) so that \( f'(B) = f(A) \) for some zero-set \( B \) of \( X \) containing \( A \). Let \( \gamma : X \to [0,1] \) with \( B = \gamma^{-1}(0) \). Define \( H : X \to K \times 0 \cup \Sigma_S \times (0,1] \) by \( H(x) = (f'(x), \gamma(x)) \). Notice that \( K \times 0 \) is closed in \( K \times 0 \cup \Sigma_S \times (0,1] \). Hence, there is a retraction \( r : K \times 0 \cup \Sigma_S \times (0,1] \to K \times 0 \). Notice that \( r \circ H \) extends \( f \). \( \Box \)

**Remark 10.2.** The problem of characterizing pairs \( (X, A) \) such that \( K \in \text{AE}(X, A) \) for all \( K \in \text{AR} \) (with \( \text{weight}(K) \leq m \)) is the content of [20]. In that paper Sennott found several equivalent conditions stated in terms of pseudometrics. It is highly likely that they are formally equivalent to one of conditions stated in Theorem 10.1.

**Theorem 10.3.** Suppose \( A \) is a subset of a topological space \( X \) and \( m \) is an infinite cardinal number. Then the following conditions are equivalent:

1. Every map \( f : A \to K \) to a complete AR (of weight at most \( m \)) extends over \( X \).
2. Every map \( g : A \to L \) to a complete ANR (of weight at most \( m \)) extends over a neighborhood \( U \) which is a cozero-set in \( X \) so that there is a zero-set \( B \) of \( X \) with \( A \subset B \subset U \).
3. Every map \( g : A \to L \) to a complete ANR (of weight at most \( m \)) extends to \( g' : X \to \text{Cone}(L) \).
4. Every map \( g : A \to L \) to a metric space (of weight at most \( m \)) extends to \( g' : X \to L' \) for some metric space \( L' \) (of weight at most \( m \)) containing \( L \) as an isometrically embedded subset.
5. \( A \) is \( C \)-embedded in \( X \) and every map \( g : A \to L \) to a metrizable space (of weight at most \( m \)) extends to \( g' : X \to L' \) for some metrizable space \( L' \) (of weight at most \( m \)) containing \( L \).
6. Every partition of unity \( \alpha = \{\alpha_s\}_{s \in S} \) (with \( \text{card}(S) \leq m \)) on \( A \) extends to a partition of unity \( \beta \) on \( X \).
Proof. (3) is a special case of (1) as $\text{Cone}(L)$ is a complete AR if $L$ is a complete ANR (see Theorem 2.9).

(2) $\iff$ (3) follows from Lemma 2.8.

(3) $\Rightarrow$ (1). If $L$ is a complete AR, it is a retract of its cone which is also a complete AR. Thus, any map $f : A \to L$ extends to a map $F : X \to \text{Cone}(L)$ and $r \circ F$, $r : \text{Cone}(L) \to L$ being a retraction, extends $f$.

(1) $\Rightarrow$ (3). If $L$ is a complete ANR, its cone is a complete AR (see Theorem 2.9).

(1) $\Rightarrow$ (4). Embed $L$ isometrically into a Banach space $L'$ (of the same weight).

(4) $\Rightarrow$ (6). Suppose $\alpha : A \to \Sigma_S$ is a partition of unity which extends to $f : X \to L'$ for some metric space containing $\Sigma_S$ as an isometrically embedded subspace. $\Sigma_S$ must be a closed subspace of $L'$ (it is complete). Since $\Sigma_S \in \text{AR}$, there is a retraction $r : L' \to \Sigma_S$.

Now, $r \circ f$ extends $\alpha$.

(6) $\Rightarrow$ (5). Suppose $g : A \to L$ is a map to a metrizable space (of weight at most $m$). By Theorem 7.1 we may assume $L \subset \Sigma_S$ (with $\text{Card}(S) \leq m$). By Theorem 6.5, $g$ can be viewed as a partition of unity and, as such, can be extended over $X$. If $L = \mathbb{R}$ is the reals, then $\mathbb{R}$ can be embedded in $\Sigma_S$ as a closed subset (actually, a simplicial complex). Thus, there is a retraction of $\Sigma_S$ onto $\mathbb{R}$, which proves that $A$ is $C$-embedded in $X$.

(5) $\Rightarrow$ (1). Suppose $f : A \to K$ is a map to a complete AR (of weight at most $m$) and $f' : X \to K'$ is an extension of $f$ with $K$ being a subset of $K'$. Notice that there is an inclusion $i : K \to K' \times \mathbb{R}^m$ onto a closed subset so that the first coordinate of $i(x)$ is $x$. Since $K$ is an AR, there is a retraction $r : K' \times \mathbb{R}^m \to i(K)$. Since $A$ is $C$-embedded in $X$, there is an extension $F : X \to K' \times \mathbb{R}^m$ of $i \circ f$. Notice that $i^{-1} \circ r \circ F : X \to K$ extends $f$. $\square$

Remark 10.4. (1) $\iff$ (4) was proved by Morita [16] and Przymusiński [19].

11. Extending maps into CW-complexes

In this section we prove an analog of Theorem 9.1 for CW-complexes.

Theorem 11.1. Suppose $A$ is a subset of a topological space $X$, and $m$ is an infinite cardinal number. If both $X$ and $A$ admit perfect maps onto first countable spaces, then the following conditions are equivalent:

1. Every map $f : A \to K$ to a contractible CW-complex (with $\text{Card}(K^0) \leq m$) extends over $X$.

2. Every map $g : A \to L$ to a CW-complex (with $\text{Card}(L^0) \leq m$) extends over a neighborhood $U$ which is a cozero-set in $X$ so that there is a zero-set $B$ of $X$ with $A \subset B \subset U$.

3. Every map $g : A \to L$ to a CW-complex (with $\text{Card}(L^0) \leq m$) extends to $g' : X \to \text{Cone}(L)$, where $\text{Cone}(L)$ is taken with the CW topology.

4. Every map $g : A \to L$ to a CW-complex (with $\text{Card}(L^0) \leq m$) extends to $g' : X \to L'$ for some CW-complex $L'$ (with $\text{Card}((L')^0) \leq m$) containing $L$ as a subcomplex.
(5) Every locally-finite partition of unity \( \alpha = \{ \alpha_s \}_{s \in S} \) (with \( \text{Card}(S) \leq m \)) on \( A \) extends to a locally-finite partition of unity \( \beta \) on \( X \).

**Proof.** (1) \( \Rightarrow \) (3). (3) is a special case of (1) as \( \text{Cone}(L) \) is a contractible CW-complex.

(3) \( \Rightarrow \) (2) follows from Lemma 2.8. Indeed, every map into \( \text{Cone}(L)_{\text{CW}} \) is also continuous as a map into \( \text{Cone}(L) \) as the inclusion \( i : \text{Cone}(L)_{\text{CW}} \to \text{Cone}(L) \) is continuous. \( i \) is continuous (use Proposition 5.6) in view of \( i : \Delta_{\text{CW}} \to \Delta_m \) being a homeomorphism for each simplex \( \Delta \) of \( \text{Cone}(L) \).

(2) \( \Rightarrow \) (5). Suppose \( \alpha \) is a locally finite partition of unity on \( A \). Notice that \( \alpha : A \to L = \mathcal{N}(\alpha)_{\text{CW}} \) is continuous. Suppose \( \beta : U \to L \) is an extension of \( \alpha \) over a cozero-set \( U \) in \( X \) containing a zero-set \( B \) of \( X \) with \( A \subseteq B \). Choose \( \gamma : X \to [0, 1] \) with \( B = \gamma^{-1}(0) \) and \( X - U = \gamma^{-1}(1) \). Let \( V = \gamma^{-1}[0, 1/2] \). By Theorem 6.5, \( \beta|V \) induces a locally-finite partition of unity. Now, \( \beta|\gamma^{-1}[0, 1/2] \) is locally-finite and Theorem 8.1 says that \( \alpha \) extends to a locally-finite partition of unity on \( X \).

(5) \( \Rightarrow \) (3). \( L \) is a retract of a simplicial complex \( K \) with the CW topology (see [2] or [15, Remark 6, p. 320]). Now, \( g \) as a map to \( K \) is synonymous with a locally-finite partition of unity which we extend to \( g : X \to K' \), where \( K \) is a subcomplex of \( K' \). Now, \( \text{Cone}(L) \) is a retract of \( \text{Cone}(K) \) which in turn is a retract of \( \text{Cone}(K') \) (all spaces taken with the CW topology). Thus, \( g : A \to \text{Cone}(L) \) has an extension over \( X \).

(3) \( \Rightarrow \) (4). Put \( L' = \text{Cone}(L) \).

(4) \( \Rightarrow \) (1). Suppose \( f : A \to K \) is a map to a contractible CW-complex (of weight at most \( m \)) and \( f' : X \to K' \) is an extension of \( f \), where \( K' \) is a CW-complex containing \( K \) as a subcomplex. Notice that \( K \) is a retract of \( K' \). \( \square \)

As an application, let us improve Kodama’s Theorem 4.8.

**Theorem 11.2.** If \( X \) admits a perfect map onto a first countable paracompact space, then \( K \in \mathcal{A}(X) \) for every contractible CW-complex \( K \).

**Proof.** Notice that \( X \) is paracompact and each closed subset of \( X \) admits a perfect map onto a first countable space. Use Theorems 11.1 and 6.5. \( \square \)

### 12. \( P \)-embeddings and generalizations

In view of our previous results it makes perfect sense to introduce the following concept:

**Definition 12.1.** Let \( m \) be a cardinal number. A subset \( A \) of a topological space \( X \) is \( P^m \)-embedded (\( P^m \)(point-finite)-embedded, \( P^m \)(point-finite)-embedded) in \( X \) provided every (point-finite, locally-finite) partition \( \{ \alpha_s \}_{s \in S} \) of unity on \( A \), where \( \text{Card}(S) \leq m \), extends over \( X \).

\( A \) is \( P \)-embedded (\( P \)(point-finite)-embedded, \( P \)(point-finite)-embedded) in \( X \) provided it is \( P^m \)-embedded (\( P^m \)(point-finite)-embedded, \( P^m \)(point-finite)-embedded) for all \( m \).
Remark 12.2. The notion of a $P^m$-embedding was originally introduced by Shapiro [22] using the concept of extending pseudometrics. In view of Theorem 10.3 our definition is equivalent to his.

Notice that in case of $m$ being finite there is no difference between $P^m$-embeddings, $P^m$(point-finite)-embeddings and $P^m$(locally-finite)-embeddings.

Before continuing with properties of $P$-embeddings we challenge the reader to prove the next result on her/his own.

Proposition 12.3. Every subset $A$ of $X$ is $P^1$-embedded.

Now, let us express other classical concepts using the idea of extending partitions of unity. In analogy to Definition 1.7 one can define $A$ being $M^m$-embedded in $X$ provided $M \in AE(X, A)$ for all $M \in AR$ of weight at most $m$ (see [20]).

Theorem 12.4. Suppose $X$ is a topological space and $A$ is a subset of $X$. Then:

(a) $A$ is $C^*$-embedded in $X$ iff $A$ is $P^2$-embedded in $X$;
(b) $A$ is $C$-embedded in $X$ iff $A$ is $P^0$-embedded in $X$;
(c) if $A$ is $M^m$-embedded in $X$, then $A$ is $P^m$(point-finite)-embedded in $X$;
(d) if $A$ is $P^m$-embedded in $X$ and is a zero-set in $X$, then $A$ is $M^m$-embedded in $X$.

Proof. (a). Notice that extending $f : A \to [0, 1]$ over $X$ is equivalent to extending the partition $\{f, 1 - f\}$ of unity over $X$.

(b). Use Theorem 10.3.

(c). Use Theorem 9.1.

(d). Use Theorem 10.1. \[\]

Remark 12.5. (b) and (d) were proved by Morita [16].

Now, let us establish the equivalence of all three definitions of $P^m$-embeddings. Here is a classical definition of $P^m$-embedding due to Morita and Hoshina [11]:

Definition 12.6. Suppose $m$ is an infinite cardinal number. $A$ is $P^m$-embedded in $X$ (in the sense of Morita–Hoshina) provided a normal cover $U$ of $A$ of cardinality at most $m$ gives rise to a normal cover $U'$ of $X$ such that its trace $U' \cap A$ on $A$ refines $U$.

Since normal covers $U$ are synonymous with those which admit a partition $\alpha = \{\alpha_s\}_{s \in S}$ of unity such that $\text{Cozero}(\alpha)$ refines $U$ [11, Theorem 1.2], the following result is obvious:

Proposition 12.7. $A$ is $P^m$-embedded in $X$ (in the sense of Morita–Hoshina) iff given a partition $\alpha = \{\alpha_s\}_{s \in S}$ of unity on $A$ with $\text{Card}(S) \leq m$, there is a partition $\beta$ of unity on $X$ such that $\beta|A \leq \alpha$. 
Proposition 12.8. Suppose \( m \) is an infinite cardinal number and \( A \) is a subset of a topological space \( X \). The following conditions are equivalent:

1. Every partition \( \{ \alpha_s \}_{s \in S} \) of unity on \( A \), where \( \text{Card}(S) \leq m \), extends over \( X \).
2. Every locally-finite partition \( \{ \alpha_s \}_{s \in S} \) of unity on \( A \), where \( \text{Card}(S) \leq m \), extends over \( X \) to a (not necessarily locally-finite) partition of unity.
3. Given a partition \( \{ \alpha_s \}_{s \in S} \) of unity on \( A \), where \( \text{Card}(S) \leq m \), there is a partition \( \beta \) of unity on \( X \) such that \( \beta|A \leq \alpha \).
4. Given a locally-finite partition \( \{ \alpha_s \}_{s \in S} \) of unity on \( A \), where \( \text{Card}(S) \leq m \), there is a partition \( \beta \) of unity on \( X \) such that \( \beta|A \leq \alpha \).

Proof. It suffices to prove (4) \( \Rightarrow \) (1).

Claim. Suppose \( f : A \rightarrow B = \bigvee S \) is a map so that \( |f(a)| \leq \delta \) for all \( a \in A \), and \( \varepsilon > 0 \). There is \( F : X \rightarrow B \) so that \( |F(x)| \leq \delta \) for all \( x \in X \) and \( |F(x) - f(x)| < \varepsilon \) for all \( x \in A \).

Proof. Choose a locally-finite partition \( \{ \alpha_s \}_{s \in S} \) of unity on the closed \( \delta \)-ball in \( B \) such that the diameter of \( \alpha_s^{-1}(0,1] \) is less than \( \varepsilon/2 \) for each \( s \in S \). Choose \( v_s \in \alpha_s^{-1}(0,1] \) for each \( s \in S \). Notice that \( \{ \alpha_s \circ f \}_{s \in S} \) is a locally-finite partition of unity on \( A \). Choose a partition \( \{ \beta_t \}_{t \in T} \) of unity on \( X \) such that for some function \( \gamma : T \rightarrow S \)
\[
\beta_t^{-1}(0,1] \cap A \subset (\alpha_{\gamma(t)} \circ f)^{-1}(0,1]
\]
for all \( t \in T \). Now, define \( F : X \rightarrow B \) by
\[
F(x) = \sum_{t \in T} \beta_t(x) \cdot v_{\gamma(t)}
\]
Notice that \( |F(x)| \leq \delta \) for all \( x \in X \). We need to check that \( |F(x) - f(x)| < \varepsilon \) for all \( x \in A \). Indeed,
\[
F(x) - f(x) = \sum_{t \in T} \beta_t(x) \cdot (v_{\gamma(t)} - f(x))
\]
and if \( |f(x) - v_{\gamma(t)}| \geq \varepsilon \), then \( \beta_t(x) = 0 \). Therefore \( |F(x) - f(x)| < \varepsilon \) for all \( x \in A \). \( \square \)

Now, we are in the position of the classic proof of the Tietze Extension Theorem (see [9, Theorem 2.1.8]): first extend a map approximately, then use the limit process. Thus, given a partition of unity \( \alpha : A \rightarrow \Sigma_S \), we can construct a sequence of maps \( F_n : X \rightarrow B \) so that \( F_n(X) \) is contained in the closed \( (1/2)^{n-1} \)-ball around 0, and \( |\alpha(x) - \sum_{i=1}^{n} F_i(x)| < (1/2)^{n} \) for all \( n \) and all \( x \in A \). This is done inductively with the help of Claim: given \( F_1, \ldots, F_n \) apply Claim to \( f = \alpha(x) - \sum_{i=1}^{n} F_i(x) \), \( \delta = (1/2)^{n} \) and \( \varepsilon = (1/2)^{n+1} \). This produces \( F_{n+1} \) so that \( |F_{n+1}(x)| < (1/2)^{n+1} \) and \( |\alpha(x) - \sum_{i=1}^{n+1} F_i(x)| < (1/2)^{n+1} \) for all \( x \in A \). Notice that \( \sum_{n=0}^{\infty} F_n \) extends \( f \) and notice that \( \Sigma_S \) is a retract of \( B \) (indeed, the identity partition of unity \( i : \Sigma_S \rightarrow \Sigma_S \) extends over \( B \) by Theorem 8.5). \( \square \)
Corollary 12.9 (Lisica [14]). If $X$ admits a perfect map onto a metrizable space, then $M \in \mathcal{AE}(X)$ for all $M \in \mathcal{AR}$.

**Proof.** Our idea is to use Theorem 10.1(5). Suppose $\alpha : A \to \Sigma_S$ is a partition of unity on a closed subset $A$ of $X$. Since every locally-finite partition of unity on $A$ extends over $X$ (see Theorem 8.4), there is an extension $\beta : X \to \Sigma_S$ of $\alpha$ by Proposition 12.8. Let $p : X \to Y$ be a perfect map onto a metrizable space $Y$. Consider the space $Z = \{(y, v) \in Y \times \Sigma_S \mid y = p(x) \text{ and } v = \beta(x) \text{ for some } x \in X\}$ and its subspace $T = \{(y, v) \in Y \times \Sigma_S \mid y = p(a) \text{ and } v = \beta(a) \text{ for some } a \in A\}$. Notice that $T$ is closed in $Z$ which is metrizable. Therefore, it is a zero-set in $Z$ and $B = \{x \in X \mid (p(x), \beta(x)) \in T\}$ is a zero-set in $X$ containing $A$ so that $\alpha(A) = \beta(B)$. By Theorem 10.1, $A$ is $M$-embedded in $X$ which proves $M \in \mathcal{AE}(X)$ for all $M \in \mathcal{AR}$. □

**Problem 12.10.** Suppose $A$ is $P$(point-finite)-embedded in $X$. Is it $M$-embedded in $X$? Is it $P$(locally-finite)-embedded in $X$?

**Problem 12.11.** Suppose $A$ is $M$-embedded in $X$. Is it $P$(locally-finite)-embedded in $X$?

Sennott [20] gave an example of an $P$-embedding which is not an $M$-embedding. Namely, the rationals $\mathbb{Q}$ are $P$-embedded in $\beta \mathbb{Q}$ (see below for the definition of $\beta \mathbb{Q}$) but not $M$-embedded in $\beta \mathbb{Q}$. We do not know if $\mathbb{Q}$ is $P$(point-finite)-embedded in $\beta \mathbb{Q}$, so we are going to construct our own example.

**Definition 12.12.** Given a space $X$ and its subspace $M$ one creates a new topological space $X_M$ on the set $X$ with the new topology: $U \subset X_M$ is open iff $U$ is a neighborhood (in the old topology) of $U \cap M$ in $X$.

It is known that $X_M$ is hereditarily paracompact, Hausdorff space if $X$ is metrizable [9, Example 5.1.22].

**Theorem 12.13.** Let $X$ be a metrizable space and let $M$ be its subspace. Then:

1. every closed subset $A$ of $X_M$ is $P$(locally-finite)-embedded in $X_M$;
2. if $M$ is a noncompletely metrizable metric simplicial complex and $X$ is complete, then $M$ is not $P$(point-finite)-embedded in $X_M$.

**Proof.** Since $X_M$ is a paracompact space, Theorem 8.4 says that every closed subset $A$ of $X_M$ is $P$(locally-finite)-embedded in $X_M$.

Suppose $M$ is a noncompletely metrizable metric simplicial complex and assume $M$ is $P$(point-finite)-embedded in $X_M$. Consider the inclusion $i : M \to \Delta_S$, where $S$ is the set of vertices of $M$. $i$ is a point-finite partition of unity on $M$, so there is an extension $J : X_M \to \Delta_S$ of $i$. The nerve $\mathcal{N}(J)$ of $J$ is a simplicial complex containing $M$. Therefore, there is a neighborhood $V$ of $M$ in $\mathcal{N}(J)$ which retracts onto $M$ (since $M \in \mathcal{ANR}$). Thus, there is a retraction $r : U = J^{-1}(V) \to M$ of an open neighborhood $U$ of $X$ in $X_M$. Let $d$ be the complete metric on $X$. Given an open set $V$ in $X_M$,
its interior in the metric topology on $X$ will be denoted by $\text{int}_X(V)$. Given $n > 0$ let $F_n = \{ x \in i(U) \mid d(x, r(x)) \geq 1/n \}$. Since $M$ cannot be a $G_\delta$-set in $X$, there is $m > 0$ such that the closure of $F_m$ in $X$ intersects $M$. Thus, there are points $x_n \in F_m$, $n > 1$, converging to $x_0 \in M$. Let $W$ be the open ball of radius $1/(2m)$ centered at $x_0$. Since $r$ is continuous, $r^{-1}(W)$ is open in $X$ and $\text{int}_X(r^{-1}(W))$ is a neighborhood of $x_0$ in $X$. In particular, $x_n \in r^{-1}(W)$ for some $n$ so that $x_n \in W$. Thus, $d(x_n, x_0) < 1/(2m)$ and $d(r(x_n), x_0) < 1/(2m)$ which implies $d(r(x_n), x_0) < 1/m$, a contradiction. \[ \square \]

**Remark 12.14.** Let $S$ be a countable, infinite set. Notice that the maximal simplicial complex $\Delta_S$ in $\Sigma_S$ is not completely metrizable. Indeed, $\Delta_S$ is the union of all (countably many) of its simplices, and each of them is nowhere dense in $\Delta_S$. Thus, $\Delta_S$ is $P^{\aleph_0}(\text{locally-finite})$-embedded in $(\Sigma_S)_{\Delta_S}$ but not $P^{\aleph_0}(\text{point-finite})$-embedded.

### 13. Homotopy Extension Property

This section is devoted to generalizations of the Borsuk Homotopy Extension Theorem.

**Proposition 13.1.** Suppose $\alpha : X \times 0 \cup A \times I \to C$ is a map to a convex subset $C$ of a normed vector space $V$. If $\alpha|A \times I$ extends to $\beta : X \times I \to C$, then $\alpha$ extends over $X \times I$.

**Proof.** Notice that $H : X \times I \to C$, $H(x, t) = (1 - t) \cdot \alpha(x, 0) + t \cdot \beta(x, 0)$, is a homotopy starting at $\alpha|X \times 0$ and ending at $\beta|X \times 0$. The idea is to combine $H$ and $\beta$ to produce an extension $\gamma : X \times I \to C$ of $\alpha$. Let $\mu : X \to I$ be defined by

$$\mu(x) = \min \left( \frac{1}{2}, |\alpha(x, 0) - \beta(x, 0)| \right).$$

Notice that $A \subset \mu^{-1}(0)$. Now, $\gamma$ is going to be defined so that on $\{x\} \times [0, \mu(x)]$ we use $H$ and on $\{x\} \times [\mu(x), 1]$ we use $\beta$. The linear function mapping $[0, \mu(x)]$ (respectively $[\mu(x), 1]$) onto $[0, 1]$ is $t \to t/\mu(x)$ (respectively $t \to (t - \mu(x))/(1 - \mu(x))$). Thus, the formal definition of $\gamma$ is:

$$\gamma(x, t) = \begin{cases} 
\alpha(x, 0), & \text{if } t = 0; \\
(1 - t/\mu(x)) \cdot \alpha(x, 0) + (t/\mu(x)) \cdot \beta(x, 0), & \text{if } 0 < t < \mu(x); \\
\beta(x, (t - \mu(x))/(1 - \mu(x))), & \text{if } \mu(x) \leq t < 1; \\
\beta(x, 1), & \text{if } t = 1.
\end{cases}$$

**Lemma 13.2.** Suppose $m$ is a cardinal number and $A$ is a subset of a topological space $X$. Let $\Pi$ be one of the following four possibilities: $P^m$, $P^m(\text{point-finite})$, $P^m(\text{locally-finite})$, $M^m$. Then $X \times 0 \cup A \times I$ is $\Pi$-embedded in $X \times I$ if $A \times I$ is $\Pi$-embedded in $X \times I$.

**Proof.** Let $S$ be of cardinality $m$. Suppose $\Pi = P^m(\text{locally-finite})$ (if $\Pi = P^m(\text{point-finite})$ or $\Pi = P^m$ the proof is similar). Suppose $\alpha : X \times \{0\} \cup A \times I \to \Sigma_S$ is a locally-finite partition of unity. Since $A \times I$ is $\Pi$-embedded in $X \times I$, $\alpha|A \times I$ extends
over $X \times I$. Apply the proof of Proposition 13.1 to $C = \Sigma_S$ and notice that one gets a locally-finite partition of unity on $X \times I$ extending $f$.

If $H = M^m$ and $C$ is a convex subset of a Banach space of weight at most $m$, we get $C \in \mathcal{AE}(X \times I, A \times I)$. Therefore, $C \in \mathcal{AE}(X \times I, X \times \{0\} \cup A \times I)$ by Proposition 13.1.

If $K$ is an AR of weight at most $m$, it is a retract of a convex a Banach space of weight at most $m$. Thus, $K \in \mathcal{AE}(X \times I, X \times \{0\} \cup A \times I)$ and 10.1 says that $X \times 0 \cup A \times I$ is $M^m$-embedded in $X \times I$. □

**Remark 13.3.** In the case of $H = \mathcal{P}^m$, Lemma 13.2 is due to Morita and Hoshina [17] who improved earlier work of Dowker [3]. In the case of $H = M^m$, Lemma 13.2 is due to Sennott [20].

**Theorem 13.4.** Suppose $m$ is an infinite cardinal number and $A$ is a subset of a topological space $X$. If $A$ is $\mathcal{P}^m$-embedded (respectively $M^m$-embedded) in $X$, then $X \times 0 \cup A \times I$ is $\mathcal{P}^m$-embedded (respectively $M^m$-embedded) in $X \times I$.

**Proof.** In view of Lemma 13.2 it suffices to prove that $A \times I$ is $M^m$-embedded (respectively $\mathcal{P}^m$-embedded) in $X \times I$. Thus, it suffices to show that $K \in \mathcal{AE}(X \times I, A \times I)$ for all (complete) ARs of weight at most $m$. Any such $K$ is a retract of a convex (and closed) subset $C$ of a Banach space $V$ of weight at most $m$ (use Theorem 7.2). So, it suffices to consider the case $K = C$. Suppose $C$ is a convex (and closed) subset of $V$ and $f : A \times I \to C$ is a map. $f$ induces $f' : A \to C^I$ and $C^I$ is a convex (and closed) subset of the Banach space $V^I$ of weight at most $m$. Thus, $f'$ extends over $X$ to $g' : X \to C^I$. Now, the induced map $g : X \times I \to C$ is an extension of $f$. □

**Remark 13.5.** The case of $\mathcal{P}^m$-embeddings is due to Morita and Hoshina [17]. The case of $M^m$-embeddings is due to Sennott [20]. Our proof is based on that of Przymusiński [18].

**Problem 13.6.** Suppose $m$ is an infinite cardinal number and $A$ is a subset of a topological space $X$. If $A$ is $\mathcal{P}^m$-(point-finite)-embedded (respectively $\mathcal{P}^m$-(locally-finite)-embedded) in $X$, is $A \times I$ $\mathcal{P}^m$-(point-finite)-embedded (respectively $\mathcal{P}^m$-(locally-finite)-embedded) in $X \times I$?

Recall that $(X, A)$ has the Homotopy Extension Property (HEP, in short) with respect to $Y$, provided every map $f : X \times \{0\} \cup A \times I \to Y$ extends over $X \times I$ [11, p. 75]. In our terminology this means $Y \in \mathcal{AE}(X \times I, X \times \{0\} \cup A \times I)$. Here is our version of a homotopy extension theorem. As far as we can tell it is the most general of homotopy extension theorems. We will demonstrate its generality by deriving most of results of [11, Section 5] from it. Also, Theorem 13.7 proves the usefulness of our interpretation of what $Y \in \mathcal{ANE}(X, A)$ ought to mean.

**Theorem 13.7.** Suppose $A$ is a subspace of $X$ and $Y$ is a topological space. Then, $Y \in \mathcal{AE}(X \times I, X \times \{0\} \cup A \times I)$ iff $Y \in \mathcal{ANE}(X \times I, X \times \{0\} \cup A \times I)$. 

Theorem 13.7 is a straightforward consequence of the following (in view of Lemma 2.8 and Definition 1.6):

**Lemma 13.8.** Suppose $X$ and $Y$ are topological spaces, $A$ is a subset of a $X$, and $f : X \times \{0\} \cup A \times I \to Y$ is a map. If $f$ extends to $H : X \times I \to \text{Cone}(Y)$, then $f$ extends to $G : X \times I \to Y$.

**Proof.** Suppose $f : X \times \{0\} \cup A \times I \to Y$ and let $F : U \to Y$ be an extension of $f$ over a cozero-set $U$ in $X \times I$ so that $X \times \{0\} \cup A \times I \subseteq B \subseteq U$ for some zero-set $B$ of $X \times I$ (see Lemma 2.8). Let $\gamma : X \times I \to I$ be a map such that $\gamma^{-1}(1) = B$ and $\gamma^{-1}(0) = X \times I - U$. Define $\kappa : X \to I$ by $\kappa(x) = \inf\{\gamma(x,t) \mid t \in I\}$ and notice that $\kappa$ is continuous, and $\kappa(a) = 1$ for all $a \in A$. Define $r : X \times I \to U$ by $r(x,t) = (x, \kappa(x) \cdot t)$ and notice that $F \circ r$ extends $f$. It remains to check that the range of $r$ is contained in $U$. Suppose $(x, \kappa(x) \cdot t) \notin U$ for some $(x,t) \in X \times I$. Then, $\kappa(x) > 0$ and $\kappa(x) \leq \gamma(x, \kappa(x) \cdot t) = 0$, a contradiction. □

**Corollary 13.9** (Morita and Hoshina [17]). Suppose $m$ is an infinite cardinal number, $A$ is $P^m$-embedded in $X$, and $M$ is a complete ANR of weight at most $m$. Then, $M \in \text{AE}(X \times I, X \times \{0\} \cup A \times I)$.

**Proof.** By Theorem 13.4, $X \times \{0\} \cup A \times I$ is $P^m$-embedded in $X \times I$. Hence, by Theorem 10.3(2), $M \in \text{ANE}(X \times I, X \times \{0\} \cup A \times I)$. By Theorem 13.7, $M \in \text{AE}(X \times I, X \times \{0\} \cup A \times I)$.

**Corollary 13.10** (Sennott [20]). Suppose $m$ is an infinite cardinal number, $A$ is $M^m$-embedded in $X$, and $M$ is an ANR of weight at most $m$. Then, $M \in \text{AE}(X \times I, X \times \{0\} \cup A \times I)$.

**Proof.** By Theorem 13.4, $X \times \{0\} \cup A \times I$ is $M^m$-embedded in $X \times I$. Hence, by Theorem 10.1(2), $M \in \text{ANE}(X \times I, X \times \{0\} \cup A \times I)$. By Theorem 13.7, $M \in \text{AE}(X \times I, X \times \{0\} \cup A \times I)$.

**Corollary 13.11** (Morita [16]). Suppose $m$ is an infinite cardinal number, $A$ is a zero-set in $X$ which is $P^m$-embedded in $X$, and $M$ is an ANR of weight at most $m$. Then, $M \in \text{AE}(X \times I, X \times \{0\} \cup A \times I)$.

**Proof.** By Theorem 13.4, $X \times \{0\} \cup A \times I$ is $P^m$-embedded in $X \times I$. Notice that $X \times \{0\} \cup A \times I$ is a zero-set in $X \times I$ if $A$ is a zero-set in $X$. Hence, by Theorem 10.1(2) and (5), $M \in \text{ANE}(X \times I, X \times \{0\} \cup A \times I)$. By Theorem 13.7, $M \in \text{AE}(X \times I, X \times \{0\} \cup A \times I)$.

**Remark 13.12.** Suppose $C$ is a class of spaces and $K$ is a subclass of ANRs or of CW-complexes. To be able to do extension theory according $f : A \to K$, where $A$ is closed in $X \in C$ and $K \in K$, one seems to require that $K \in \text{ANE}(X)$ and $K \in$
AE(\(X \times I, X \times \{0\} \cup A \times I\)). In view of the results of this paper the following are appropriate pairings:

(a) \(C\) consists of all normal spaces and \(\mathcal{K}\) consists of separable complete ANRs (Theorem 10.3 and Lemma 13.8 for \(m = \aleph_0\));

(b) \(C\) consists of all collectionwise normal spaces and \(\mathcal{K}\) consists of complete ANRs (Theorem 10.3 and Lemma 13.8 for all \(m\));

(c) \(C\) consists of all paracompact p-spaces and \(\mathcal{K}\) consists of ANRs (Theorem 10.1 and Corollary 12.9);

(d) \(C\) consists of all spaces admitting a perfect map onto a first countable paracompact space and \(\mathcal{K}\) consists of CW-complexes (Theorems 11.1 and 11.2).

14. Normality of products with metrizable spaces

The purpose of this section is to review results of Sennott and Waśko which indicate that there is a mysterious connection between extending of maps to ARs and the normality of product spaces.

**Theorem 14.1** (Waśko [24]). Suppose \(m\) is an infinite cardinal number and \(A\) is a \(P^m\)-embedded subset of a topological space \(X\). If \(A \times M\) is \(P^\aleph_0\)-embedded in \(X \times M\) (equivalently, \(A \times M\) is C-embedded in \(X \times M\)) for all metrizable spaces \(M\) of weight at most \(m\), then \(A\) is \(\text{M}^m\)-embedded in \(X\).

**Proof.** Suppose \(f : X \to M\) is a map into a metrizable space of weight at most \(m\) (\(M = \Sigma_S\) for some \(S\)). We need to show that there is a zero-set \(B\) containing \(A\) such that \(f(B) = f(A)\) (see Theorem 10.1(5)). Define \(\mu : A \times (M - f(A)) \to (0, \infty)\) by \(\mu(a, y) = d(f(a), y)^{-1}\) and extend \(\mu\) to \(\mu : X \times (M - f(A)) \to (0, \infty)\). The map \(\phi : X \times (M - f(A)) \to [0, \infty)\) defined by \(\phi(x, y) = d(f(x), y) \cdot \mu(x, y)\) satisfies \(\phi(A \times (M - f(A))) = \{1\}\) and \(\phi^{-1}(0) = \{ (x, f(x) \mid f(x) \in M - f(A)\}\). Let \(\{W_s\}_{s \in S}\) be a \(\sigma\)-discrete basis of \(M\) and let \(T = \{ s \in S \mid W_s \cap (M - f(A)) \neq \emptyset\}\). For each \(s \in T\) choose \(y_s \in W_s \cap (M - f(A))\) and let \(V_s = \{ x \in X \mid \phi(x, y_s) < 1/2\}\). Notice that \(U_s = f^{-1}(W_s) \cap V_s\) is a cozero set in \(X\). Also, notice that \(U = \bigcup_{s \in T} U_s\) is a cozero set in \(X\) and \(A \subset X - U\). Suppose \(f(X - U) \neq f(A)\), i.e., there is \(x \in X - U\) with \(f(x) \notin f(A)\). Since \(\phi(x, f(x)) = 0\), there is \(s \in T\) with \(f(x) \in W_s\) and \(\phi(\{x\} \times W_s) \subset [0, 1/2)\). In particular, \(\phi(x, y_s) < 1/2\) and \(x \in U_s\), a contradiction. \(\square\)

**Corollary 14.2** (Sennott [21]). If \(X\) is collectionwise normal and \(X \times M\) is normal for all metrizable spaces \(M\), then \(K \in \text{AE}(X)\) for all \(K \in \text{AR}\).

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