Stabilizing Monetary-Injection Policies

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The primary concern of this paper is understanding how prices and interest rates respond to open-market operations. In a model economy developed by Grossman and Weiss, but with more general utility functions, we establish global existence of equilibrium paths and observe that prices and interest rates rise initially, but thereafter oscillate around a new steady state. Also investigated are optimal monetary policies wherein injections take place in more than one time period and corresponding to which the equilibrium prices are most stable. Existence of such policies is established by mathematical analysis and its detailed structure explored by numerical simulation. Journal of Economic Literature Classification Numbers: E50, E52.

1 INTRODUCTION

The goal of the present work is to develop theory bearing upon the response of prices and interest rates to government intervention in the money supply. This issue is one of obvious importance that has attracted both descriptive and theoretical commentary. Conclusions have been drawn from several points of views and based on a variety of models (c.f. Baumol [2], Tobin [22], Hahn [12], Grandmont and Younes [9], Bryant and Wallace [5], Bewley [4], Hartley [13], Lucas [18], Townsend [23, 24], Sargent and Wallace [21], Grossman [10], Grossman and Weiss [11], Bona and Grossman [3],

1This work was partially supported by the National Science Foundation and by the W. M. Keck Foundation.
Jovanovic [15], Lucas and Stokey [19], Cochrane [8], and Christiano [6]). Analytically, an open-market operation is defined in these papers as a government purchase with money of some other asset together with the associated adjustments in (lump-sum) transfer payments that are needed to keep the path of government consumption unchanged. The main concern is whether the open market operation has any effect on economic quantities such as real and nominal interest rates, level of output, and prices. Most of the aforementioned works have concluded that the open-market operations have real economic effects. Not all of the studies adhere to this conclusion. Modigliani and Miller suggest that changes in the money supply brought about by open-market operations will have no effect on price level. Wallace [25] and Chamley and Polemarchakis [7] have shown that in a model in which money is held only for its rate of return, characteristic open-market operations are neutral.

The approach taken here is to develop a simplified mathematical model which maintains the primary relationships between the dependent and independent variables and to investigate this model using both analytical and numerical tools. The model upon which our study is based was put forward by Grossman and Weiss [11]. This model, described in detail in Section 2, features consumers with perfect foresight, firms owned by the consumers that produce a single, perishable good, and the government. Going beyond this pioneering model where consumers possessed only logarithmic preferences, consideration is given to a broader class of utility functions. In the case considered by Grossman and Weiss, namely consumers having a logarithmic utility function, it was shown that open-market operations can have real effects and that a monetary expansion will lead to a temporary reduction in both real and nominal interest rates and an eventual increase in prices. However, current price levels are unaffected by anticipated future monetary injections, a situation contrary to common experience.

The analysis of the class of utility functions considered here, which includes a one-parameter family of the homothetic ones, is considerably more involved than that coming to the fore in Grossman and Weiss [11]. The equilibrium prices as a function of a discrete period variable \( n \) are governed by a second-order, implicit, difference equation with but one initial condition, whereas the assumption of logarithmic utility leads to a first-order difference relation. In the second-order case that arises here, the relevant solution is obtained by solving a functional equation of the form \( g(f, f(x), x(x), x) = 0 \) for the unknown function \( f \). The equilibrium price sequence is then realized by iterating the function \( f \) on the initial price \( p_1 \), so that the price at the \( n \)th period is \( p_n \) where \( p_2 = f(p_1) \), \( p_3 = f(p_2) \), and so on. The existence of a price function \( f \) relating the price at period \( n + 1 \) to the price at period \( n \) is established in two principal steps. First the stable manifold theorem from dynamical systems theory is employed to prove the existence of a local
solution by relating $f$ to the graph of a stable manifold for an auxiliary two-dimensional mapping. The local solution is then shown to be the restriction of a globally defined solution by an extension argument. The analysis just described, which is worked out in detail in Section 3, is telling in the case of homothetic utility functions.

The outcome of this analysis is quite satisfactory as regards its economic interpretation. Indeed, the aforementioned drawbacks of the Grossman–Weiss model are seen to be special to the assumption of logarithmic utility. To understand better the detailed structure of solutions for interesting values of the model's parameters, numerical simulation seems to be indicated. Accordingly, we report in Section 4 an efficient and accurate scheme for the approximation of solutions of the model which is implemented as a computer code. After suitable testing, the computer program was used in an investigative mode. In addition to providing quantitative information about solutions, the code led to the formulation of an interesting conjecture about monetary policy.

Briefly stated, it is found both analytically and numerically that a one-period expansion of the money supply leads eventually to a new, higher, price level. However, equilibrium prices approach this new steady state in an oscillatory manner. If the injection is spread over two or more periods, the oscillatory behavior may be much less severe. Furthermore, if the injection is spread in a particular manner, then it will be demonstrated that the equilibrium price sequence may be stabilized after two time periods. This point may have some bearing on the differing market response to monetary shocks found in real observations (see Section 4).

2. THE MODEL

In this section, a detailed description is presented of the model which will be the center of attention henceforth.

2.1. Description of the Economy

The major components of this class of models are consumers, firms, and the government. Consumers are supposed to have perfect foresight and they come equipped initially with a stock of money and government bonds. Firms, owned by consumers, produce an exogenous good and sell it to other consumers for profit. Because consumers need cash to purchase goods, they sell their bonds to the firms and to the government. The firms use the profit from selling goods to purchase bonds from consumers and immediately put them into the owners' interest-bearing accounts. The government uses cash to purchase bonds, thereby injecting money into the
market. Consumers can only obtain cash from the bond market (which will be referred to as the “bank” henceforth). They use the money from their bank withdrawals to purchase the unique perishable consumption good which is exogenously supplied at a rate of $C$ units per period. Because of transaction costs, consumers tend to bunch cash withdrawals and hence there is a period between consumers’ trips to the bank. It is assumed here that the time between two consecutive trips to the bank is fixed, and that consumers will not all go to the bank at the same time. More specifically, we take it that each consumer goes to the bank every other period and withdraws enough cash to finance consumption expenditure over the ensuing two periods. The amount of withdrawal is determined by the possibility of intertemporal consumption substitution and thus is influenced by the expected prices and future nominal interest rates. We assume that there are two classes of consumers. The $a$-type consumers go to the bank every odd period, whereas $b$-type consumers appear there every even period. As usual, these two classes of consumers will be lumped, and from now on it will appear that there are exactly two consumers in the economy. The consumers are infinitely lived here, though one can also realize this sort of model in an overlapping-generations context where consumers are participating economic agents for exactly two periods each (see Bona and Grossman [2]).

2.2. Competitive Equilibria

The optimization problem faced by consumers is now analyzed in more detail. Let $N_a$ and $N_b$ denote the set of odd and even positive integers, respectively. Each consumer must choose bank withdrawals $M_i^t (t \in N_i)$ and consumption plans $C_i^t$ so as to maximize

$$
\sum_{t=1}^{\infty} \beta^{t-1} U(C_i^t),
$$

(1)

$i = a, b$, subject to the constraints that only money can purchase goods and that all the money will be spent before the next bank withdrawal. As in the original Grossman–Weiss papers, we neglect the possibility that agents will not exhaust their supply of cash before returning to the bank. The initial cash constraints are binding if, for example, the sufficient condition $U'(C_i^t) \geq \beta U'(C_{i+1}^t) p_i/p_{i+1}$ holds for both consumers. For periods $t \geq 3$, perfect foresight together with positive nominal rates of return assure the cash constraints to be binding. These constraints may be expressed as

$$
p_a C_a^t = M_0^a, \quad p_a C_a^t + p_b C_b^t = M_0^b,
$$

(2)
and

\[ p_{t+1}C_{t+1} + p_{t+2}C_{t+2} = M_t, \quad \text{for all } t \in \mathbb{N}, \quad i = a, b. \] (3)

In (1), \( U \) is a given utility function that reflects consumer preference and \( \beta \in (0, 1) \) is a “discount” factor. As usual, we assume that \( U'(x) > 0 \) and \( U''(x) < 0 \). The price of the single good in period \( t \) is \( p_t, t = 0, 1, \ldots \). The two quantities \( M'_0 \) are the consumer’s initial money holdings and the later money holdings \( M'_t \) must satisfy the overall wealth constraint

\[ \sum_{t \in \mathbb{N}_i} \frac{M'_t}{\alpha_t} \leq W^i, \quad i = a, b. \] (4)

In (4), the \( W^i \) are the nominal values of the nonmonetary wealth of consumer \( i \),

\[ \alpha_t = R_1R_2\cdots R_{t-1} \quad (\text{with } \alpha_1 = 1) \]

is the nominal growth factor between period 1 and period \( t \), and \( R_{t-1} \) is the nominal interest rate on assets held in the \( t \)th period (between \( t \) and \( t+1 \)).

In this model, the nonmonetary wealth \( W^i \) of consumer \( i \) consists of three components. The first is a claim to a fraction of the firm’s revenue which is deposited into an interest-earning account every period. If \( s^i \in (0, 1) \) denotes the fraction of the firm’s revenue claimed by consumer \( i \), then assuming there is no government ownership we must have \( s^a + s^b = 1 \). Thus the current nominal value of the total income of consumer \( i \) from the firm’s revenue is

\[ s^i \sum_{t=1}^{\infty} \frac{p_tC_t}{\alpha_t} \]

where \( C \) is the exogenously given output produced by the firm. The second component of \( W^i \) is the one-period maturity government bonds that the consumer currently holds. This component is denoted by \( B'_0 \). The third component of \( W^i \) is the per-capita tax that consumer \( i \) is obligated to pay to cover the government debt (denoted by \( B \)). In this two-consumer economy, the per-capita taxes are taken to be \( B/2 \), where \( B \) equals the value of the government bonds that the consumers currently hold minus the current nominal values of the future government money injections, which is to say

\[ B = B'_0 - \sum_{t=1}^{\infty} \frac{M'_t - M'_{t-1}}{\alpha_t}, \] (5)
where \( B_0 = B^*_b + B^*_b \) and \( M_t \) is the exogenously given money supply. Combining these three components leads to a formula for nonmonetary wealth, namely,

\[
W' = s' \sum_{t=1}^{\infty} \frac{p_t C}{s_t} + B_0' - \frac{B}{2}.
\] (6)

To study the optimization problem (1)-(3), consideration is given to the auxiliary optimization problem that concerns a consumer who has just withdrawn \( M \) units of money from the bank at the end of a given period, and will allocate it between expenditures in the next two periods. This consumer thus faces the problem

\[
\max_{C_1, C_2} U(C_1) + \beta U(C_2)
\]

subject to

\[
p_1 C_1 + p_2 C_2 = M.
\]

Under the assumptions mentioned earlier on the utility function \( U \), it is well known that the above optimization problem has a unique solution for given \( p_1, p_2, \) and \( M \). The solution for the optimal choice \( C_2 \) is denoted by

\[
C_2 = C_2(X, Y), \text{ where } X = \frac{p_1}{p_2} \text{ and } Y = \frac{M}{p_2}. \tag{7}
\]

Because of the special structure of the global optimization problem (1)-(3), the just described local optimization problem plays a central role in its resolution. Indeed, it is straightforward to verify the standard transversality condition as in Arrow and Kurz [1]. Then, using the identity

\[
\sum_{t=1}^{\infty} \beta^{s-1} U(C^s_t) = U(C^s_1) + \sum_{t \in K^s} \left[ \beta^t (U(C^s_{t+1}) + \beta U(C^s_{t+2})) \right]
\]

and the fact that the constraints (2) and (3) are decoupled with regard to the period \( t \), it is concluded that the optimal strategy for a consumer is, at each visit to the bank, to withdraw all their available cash and allocate it between the ensuing two periods according to the solution of the local optimization problem. In the notation introduced above, the optimal choices of \( C^s_t \) for the problem (1)-(3) are given by

\[
p_1 C^s_1 = M^s_0 \quad \text{and} \quad p_{t+1} C^s_{t+1} + p_{t+2} C^s_{t+2} = M^s_t, \tag{8}
\]
where
\[
C^a_{t+2} = C_2 \left( \frac{p_{t+1}}{p_{t+2}}, \frac{M^a_t}{p_{t+2}} \right), \quad \text{for } t \in N_a.
\]

Similarly, the optimal choices of \( C^b_t \) satisfy
\[
p_{t+1} C^b_{t+1} + p_{t+2} C^b_{t+2} = M^b_t,
\]
where
\[
C^b_{t+2} = C_2 \left( \frac{p_{t+1}}{p_{t+2}}, \frac{M^b_t}{p_{t+2}} \right), \quad \text{for } t \in N_b \cup \{0\}.
\]

These dynamical conditions are supplemented by the market clearing conditions
\[
C^a_t + C^b_t = C, \quad \text{for } t \geq 1,
\]
\[
M^a_t + M^b_t = M^*_t, \quad \text{for } t \geq 0.
\]

It is assumed that the money supply changes only through open-market operations. For \( t > 0 \), \( M^*_i \) represents the money holding of consumer \( i \), which is, at the start of a banking period \( t \in N_i \), equal to the amount just withdrawn from the bank, and on the period following a banking period is equal to the withdrawal in the previous period, but reduced by the previous period's spending, namely,
\[
M^*_i = M^*_{i-1} - p_i C^i_i \quad \text{for } t \notin N_i.
\]

As a consequence of (12) and (3), it follows that
\[
M^*_{i-1} = p_i C^i_i, \quad \text{for } t \in N_i.
\]

Combining (12), (10), (13), and (11), one obtains the formulas
\[
M^*_i = p_i C + M^*_i - M^*_{i-1}, \quad \text{for } t \in N_i,
\]
and
\[
M^*_i = M^*_{i-1} - p_i C, \quad \text{for } t \notin N_i.
\]

Equation (14) states that the flow of aggregate money withdrawals from the bank must equal the flow of money into the bank. The latter consists of two terms. The first is the aggregate value of the firm's receipts \( p_i C \)
which are assumed to be deposited instantly by firms into the interest-earning accounts. The second term $M^*_t - M^*_{t-1}$ is the change in aggregate nominal money engendered by the government through open-market operations. Thus, under equilibrium conditions, it appears that the bank withdrawals are determined solely by the price levels the consumer will face for the next two periods and the money supply.

Attention is now turned to a specification of the equilibrium prices. Combining (13) with (15) (with $i = b$) gives that, for $t \in \mathbb{N}_a$,

$$p_{t+2}C^a_{t+2} = M^a_{t+1} = M^*_t - p_{t+1}C,$$

which, by (8) and (14), can be written as

$$p_{t+1}C + p_{t+2}C^2 \left( \frac{p_{t+1}}{p_{t+2}} \cdot \frac{p_tC + M^*_t - M^*_{t-1}}{p_{t+2}} \right) = M^*_t \quad \text{for} \quad t \in \mathbb{N}_a.$$

A similar argument shows that the above identity also holds for $t \in \mathbb{N}_b$. In consequence, for $t \geq 1$, it transpires that

$$p_{t+1}C + p_{t+2}C^2 \left( \frac{p_{t+1}}{p_{t+2}} \cdot \frac{p_tC + M^*_t - M^*_{t-1}}{p_{t+2}} \right) = M^*_t. \quad (16)$$

Combining (8) with (9) at $t = 0$ gives

$$p_1C + p_2C^2 \left( \frac{p_1}{p_2} \cdot \frac{M^*_0}{p_2} \right) = M^*_0. \quad (17)$$

The equations in (16) comprise a system of nonlinear, second-order difference equations for the equilibrium prices $p_t$. It is worth note that once the prices are known, all the other economic quantities are determined by the preceding relations.

2.3. Homothetic Utility Functions

To gain more insight into the model, attention is focused on the class of homothetic utility functions. In Grossman and Weiss [11], where the model originated, a logarithmic utility function was proposed and the model was analyzed on the basis of this special form. The analysis was particularly simple in this case because the equilibrium prices are described by a first-order, linear difference equation with a single initial condition. However, a serious drawback was that the current price level is independent of anticipated future monetary injections. To better understand the potential of the model, we follow Grossman [10] and Grossman and Bona [3] and
introduce the one-parameter family of homothetic utility functions $U = U_A$ defined for values of consumption $c \geq 0$ by

$$U(c) = U_A(c) = \frac{c^{1-A}}{1-A},$$

where $A > 0$ is fixed and $A \neq 1$. In this case, it is easy to see that

$$C_2(X, Y) = Y\phi(X),$$

where

$$\phi(X) = \frac{\beta^{1/A}}{\beta^{1/a} + X^{A-1}/A}$$

and $\beta \in (0, 1)$ is the aforementioned discount factor. Because of the special form of $C_2$, Eqs. (16) and (17) reduce to

$$p_{t+1} C + (p_t C + M^*_t - M^*_{t-1}) \phi \left( \frac{p_{t+1}}{p_{t+2}} \right) = M^*_t, \quad \text{for} \quad t \geq 1, \quad (19)$$

and

$$p_t C + M^*_0 \phi \left( \frac{p_t}{p_2} \right) = M^*_0. \quad (20)$$

Note that $\phi$ is monotone increasing if $A < 1$, but monotone decreasing if $A > 1$. If $A < 1$, $\phi(0) = 0$ whereas if $A > 1$, $\phi(0) = 1$. In all cases, $\phi(x) \leq 1$ for all $x \geq 0$.

2.4. Basic Parameters Used in the Model

$N_o$ the set of odd positive integers,
$N_e$ the set of even positive integers,
$M^*_i$ the initial money holdings of consumer $i$,
$B^*_i$ the initial nominal holdings of government bonds by consumer $i$,
$M^s_t$ the exogenously given money supply,
$p_t$ the price level during period $t$,
$M^r_t$ the money holdings of the consumer $i$ at the end of period $t$,
$R_t$ 1 plus the interest earned from the end of period $t$ to the end of period $t + 1$,
$\alpha_i$ $\alpha_i = R_1 \cdot R_2 \cdots R_{t-1}$ with $\alpha_1 = 1$,
$C^i_t$ the consumption by consumer $i$ during period $t$, and
$W^i$ the nonmonetary wealth of consumer $i$. 

STABILIZING MONETARY-INJECTION POLICIES
3. EQUILIBRIUM PRICES

In this section, attention is given to the uniqueness, existence, and some qualitative properties of equilibrium prices in the model described in Section 2. The development here is a streamlined version of the account appearing in Grossman [10] and Bona and Grossman [3]. Going beyond these earlier works, our theory provides global existence and uniqueness rather than just a local theory in a neighborhood of an established equilibrium situation.

For simplicity of exposition, and without loss of generality, it will be assumed henceforth that

\[ C = 1 \quad \text{and} \quad M^*_0 = 1. \]

This amounts to a choice of units and it means that the single good is exogenously supplied at a rate of one unit per period and the initial money in the market is one unit. We will concentrate here on the case where \( 0 < A < 1 \) for our theoretical analysis. From the construction of the function \( C_2(X, Y) \), it is apparent that this represents the situation wherein an increase in the rate of return on savings \( X \), with the level of wealth \( Y \) held fixed, results in larger future consumption \( C_2 \). That is, the substitution effect dominates the income effect associated with a change in rates of return. At the end of this section numerically obtained approximations of solutions of equilibrium prices are provided for the case \( A > 1 \).

3.1. Single Monetary Injection

For the rest of Section 3, it is assumed that the unannounced monetary policy involves only a \( k \% \) increase in the money supply at the end of period 1. Thus money is injected into the market only once, and so

\[ M^*_t = (1 + k) M^*_0 = 1 + k, \quad t \geq 1. \]

In this case (19) can be rewritten as

\[ p_t + \phi \left( \frac{p_t}{p_{t+1}} \right) p_{t-1} = 1 \quad \text{for} \quad t \geq 2, \quad (21) \]

where \( p_t = (p_t + k)/(k + 1) \) and \( p_t = p_t/(k + 1) \) for \( t \geq 2 \) are scaled prices and the initial condition (17) becomes

\[ p_1 + M^*_0 \phi \left( \frac{p_1}{p_2} \right) = 1. \quad (22) \]
3.2. Uniqueness

It will be demonstrated in this subsection that the equilibrium prices are uniquely determined by the specifications (21)-(22). We start with some general aspects of the dynamics obtaining for the model with a homothetic utility function and \( A < 1 \).

Note that the steady-state equilibrium value associated to (21) is

\[
p^* = \frac{1}{1 + \phi(1)}.
\]

In general, as shown below, any solution of (21) will alternate below and above the steady state equilibrium value \( \bar{p} \).

**Lemma 3.1.** Let \( \{ p_t \}_{t=1}^{\infty} \) be a sequence satisfying (21). Then for any \( t \), \( p_t \leq \bar{p} \) if and only if \( p_{t+1} > \bar{p} \).

**Proof.** If for some \( t \), \( p_t, p_{t+1} < \bar{p} \), then

\[
\phi \left( \frac{p_{t+1}}{p_{t+2}} \right) = \frac{1 - p_{t+1}}{p_t} > \frac{1 - \bar{p}}{\bar{p}} = \phi(1).
\]

Because of the assumption that \( A < 1 \), \( \phi \) is an increasing function, and it follows that \( p_{t+2} < p_{t+1} < \bar{p} \). Continuing this argument inductively shows that

\[
p_{t+1} < p_s < p_t < \bar{p}, \quad \text{for all } s > t.
\]

Thus, there exists \( p^* \) such that \( p^* = \lim_{t \to \infty} p_t \leq \bar{p} \). Taking the limit as the period becomes unboundedly large in Eq. (21) gives

\[
p^* + \phi(1) p^* = 1.
\]

This means that \( p^* = \bar{p} \), which is a contradiction. Similarly, one proves there exists no \( t \) such that \( p_t, p_{t+1} > \bar{p} \).

Our next result gives a priori bounds for any possible solution of (21).

**Lemma 3.2.** Let \( \{ p_t \}_{t=1}^{\infty} \) be a sequence satisfying (21). Then there exist constants \( \sigma_1, \sigma_2 \in (0, 1) \) so that \( \sigma_1 \leq p_t \) for \( t \geq 1 \) and \( p_t \leq 1 - \sigma_2 \) for \( t \geq 2 \).

**Proof.** Let \( p^* \) (which is less than \( \bar{p} \)) be the unique solution of the equation

\[
p^* + \phi \left( \frac{p^*}{\bar{p}} \right) = 1.
\]
If $p_t < \bar{p}$, then Lemma 3.1 insures that $p_{t+1} > \bar{p}$. Therefore, again because $\phi$ is increasing,

$$1 = p_t + \phi \left( \frac{p_t}{p_{t+1}} \right) p_{t-1} < p_t + \phi \left( \frac{p_t}{\bar{p}} \right),$$

which implies that $p_t < p^*$. Thus we may choose $\sigma_1 = \min \{ \bar{p}, p^* \} = p^*$ and have $p_t > \sigma_1$ for all $t \geq 1$.

Similarly, if $p_t > \bar{p}$, then $p_{t+1} < \bar{p}$ and hence

$$p_t = 1 - \phi \left( \frac{p_t}{p_{t+1}} \right) p_{t-1} < 1 - p_{t-1} \phi(1) \leq 1 - \sigma_1 \phi(1),$$

provided that $t \geq 2$. Thus, we may take $\sigma_2 = \sigma_1 \phi(1)$.

When $t = 1$, the functional relation is

$$p_1 + M_0 \phi \left( \frac{p_1}{p_2} \right) = 1,$$

and since $M_0 < 1$ the same argument is seem to apply in this case. \[Q.E.D.\]

As regards uniqueness of the price path $\{ p_t \}_{t=1}^\infty$, the following result shows the situation to be satisfactory.

**Lemma 3.3.** For any $p_1 \in (0, 1)$, there corresponds at most one solution $\{ p_t \}_{t=1}^\infty$ of (21).

**Proof.** Suppose that $\{ p_t \}_{t=1}^\infty$ and $\{ p_t \}_{t=1}^\infty$ are both solutions of (21) with $p_1 = \hat{p}_1$ and $p_t, \hat{p}_t \in (0, 1)$ for all $t \geq 2$. If $p_2 = \hat{p}_2$, then it is obvious that $p_t = \hat{p}_t$ for all $t$. Without loss of generality, we may assume that $p_2 > \hat{p}_2$. By a direct calculation, using the fact that $\phi^{-1} = \psi$ where

$$\psi(x) = \beta^{1/\beta - 1} \left( 1 - \frac{x}{x} \right)^{A(1 - 1/\beta)},$$

it is seen that

$$\hat{p}_{t+1} = \hat{p}_t \left( \frac{g(p_t, p_{t-1})}{g(\hat{p}_t, \hat{p}_{t-1})} \right)^{A(1 - 1/\beta)}$$

with

$$g(p_t, p_{t-1}) = \frac{1 - p_t}{p_t + p_{t-1} - 1}.$$
It is clear that \( g \) is decreasing with respect to each variable. Thus successive applications of (23) yield

\[ p_t \geq \hat{p}_t, \quad \text{for all } t \geq 1. \]

Furthermore, it is elementary to see that \( p_t g(p_t, p_{t-1}) \) is decreasing with respect to the variable \( p_t \), and thus

\[ p_t g(p_t, p_{t-1}) \leq \hat{p}_t g(\hat{p}_t, \hat{p}_{t-1}) \leq \hat{p}_t g(\hat{p}_t, \hat{p}_{t-1}), \]

which, together with (23), implies that

\[ \frac{\hat{p}_{t+1}}{p_{t+1}} \leq \left( \frac{\hat{p}_t}{p_t} \right)^{A/(1-A)} = \left( \frac{\hat{p}_t}{p_t} \right)^{1 + A/(1-A)}. \]

A recursive application of the above inequality gives

\[ \frac{\hat{p}_{t+1}}{p_{t+1}} \leq \left( \frac{\hat{p}_t}{p_t} \right)^{A/(1-A)(t-1)} \rightarrow 0, \quad \text{as } t \rightarrow \infty, \]

which contradicts Lemma 3.2.

\[ \Box \]

3.3. Existence

To study the existence of solutions to the problem (21)–(22), we propose to seek the solution in terms of a price function \( f \) such that

\[ p_{t+1} = f(p_t), \quad t = 1, 2, 3, \ldots \quad (24) \]

It is easy to see that the sequence \( \{p_t\}_{t=1}^{\infty} \) given by the above formula is a solution of (21) if \( f \) satisfies the functional equation

\[ 1 - x \phi \left( \frac{f(x)}{f(x)} \right) = f(x), \quad \text{for } x \in [0, 1]. \quad (25) \]

The function \( f \) determined by the above equation will play a crucial role in our study.

**Lemma 3.4.** The functional equation (25) has a unique solution \( f \) defined and real-analytic on \([0, 1]\). This solution satisfies the properties

1. \( f(\hat{p}) = \bar{p} \), and
2. \( -1 < f'(x) < 0 \), for \( x \in [0, 1] \).

The somewhat lengthy proof of this result will be presented in Subsection 3.5 below. Combining this with the previous lemmas, the following results emerge.

**Lemma 3.4.** The functional equation (25) has a unique solution \( f \) defined and real-analytic on \([0, 1]\). This solution satisfies the properties

1. \( f(\hat{p}) = \bar{p} \), and
2. \( -1 < f'(x) < 0 \), for \( x \in [0, 1] \).

The somewhat lengthy proof of this result will be presented in Subsection 3.5 below. Combining this with the previous lemmas, the following results emerge.
Theorem 3.5. For any given normalized price \( p_1 \in (0, 1) \), the sequence \( \{ p_t \}_{t=1}^{\infty} \) satisfies (21) if and only if
\[
p_{t+1} = f(p_t), \quad t = 1, 2, 3, \ldots
\]
where \( f \) is the solution of (25).

Theorem 3.6. There exists a unique sequence \( \{ p_t \}_{t=1}^{\infty} \) that satisfies both (21) and (22). Furthermore, \( p_1 \) is the unique solution of
\[
p_1 + \phi \left( \frac{p_1}{(1+k)f \left( \frac{p_1+k}{1+k} \right)} \right) M_0^k = 1.
\]

Proof. Consider the function
\[
g_1(p) = p + \phi \left( \frac{p}{(1+k)f \left( \frac{p+k}{1+k} \right)} \right) M_0^k - 1.
\]
Note that \( g_1 \) is monotonically increasing and \( g_1(0) < 0, g_1(1) > 0 \). The desired result then follows from the intermediate-value theorem for continuous functions.

An interesting consequence is that the prices are increasing in the first three periods as a result of the unannounced money injection. After this initial increase, prices oscillate around and converge asymptotically to a value that comprises a new equilibrium for the system.

Theorem 3.7. Let \( \{ p_t \}_{t=1}^{\infty} \) be an equilibrium price path corresponding to a positive monetary injection \( k > 0 \). Then we have that
\[
p_1 < p_2 < (1+k) \bar{p},
\]
and for \( t > 2 \),
\[
p_t > (1+k) \bar{p}, \quad t \in N_a, \quad p_t < (1+k) \bar{p}, \quad t \in N_u.
\]
Moreover, it must be the case that
\[
\lim_{t \to \infty} p_t = (1+k) \bar{p}.
\]
Proof. If \( q_1 = p_1(1 + k) \), then (27) may be rewritten as

\[
H_1(q_1, k) = (1 + k) q_1 + \phi \left( \frac{q_1}{f(q_1 + k/(1 + k))} \right) M_0^* = 1. \tag{31}
\]

Recall that for all \( x \in (0, 1) \), \( \phi'(x) > 0 \) and \( f'(x) < 0 \) according to Lemma 3.4. In consequence, since \( q_1 \leq 1 \), we have

\[
\frac{\partial H_1}{\partial q_1} > 0 \quad \text{and} \quad \frac{\partial H_1}{\partial k} > 0.
\]

It follows that if \( q_1 \) is viewed as a function of \( k \), then \( dq_1/dk < 0 \). When \( k = 0 \), \( q_1 = p_1 = \bar{p} \). Hence for \( k > 0 \), \( q_1 < \bar{p} \) or, what is the same, \( p_1 < (1 + k) \bar{p} \). Now to show \( p_2 < (1 + k) \bar{p} \), we define \( q_2 = (p_1 + k)/(1 + k) \). It is easily seen that if \( p_1 \) is a solution to (27), then \( q_2 < 1 \). Write (27) as

\[
H_2(q_2, k) = q_2(1 + k) - k + \phi \left( \frac{q_2 - k/(1 + k)}{f(q_2)} \right) M_0^* = 1. \tag{32}
\]

Simple considerations establish that

\[
\frac{\partial H_2}{\partial q_2} > 0 \quad \text{and} \quad \frac{\partial H_2}{\partial k} < 0,
\]

so that \( dq_2/dk > 0 \) Thus \( q_2 \) is an increasing function of \( k \), whence \( f(q_2) \) is a decreasing function of \( k \). Since \( f(q_2) = p_2/(1 + k) \), this latter quantity decreases as a function of \( k \). At \( k = 0 \), \( p_2/(1 + k) = p_2 = \bar{p} \), and so for \( k > 0 \), \( p_2/(1 + k) < \bar{p} \), as advertised.

It remains to prove (30). By Lemma 3.4, we know that

\[
\mu = \max_{x \in (0, 1)} |f'(x)| < 1.
\]

It follows from the mean-value theorem that

\[
|f(x) - \bar{p}| = |f(x) - f(\bar{p})| = |f'(\xi)| |x - \bar{p}| \leq \mu |x - \bar{p}|.
\]

Thus, if \( p_1 \) is given, then for \( j \geq 1 \),

\[
|p_{j+1} - \bar{p}| \leq \mu^j |p_1 - \bar{p}|,
\]

and the conclusion follows. \( \blacksquare \)
3.4. *A Numerical Illustration*

In this subsection are presented some numerically-generated examples of price functions and the associated price sequences. The numerical methods used to obtain these results are somewhat technical. They are described very briefly here and are then used to demonstrate various interesting aspects of the model. A complete account of the numerical methods together with their theoretical analysis and extensive numerical testing is reported in Li [16, 17].

**Price function.** As seen from the theory developed above, the entire dynamics of the system can be obtained for all values of the initial price and monetary shock once the price function is known. The strategy proposed here is to approximate the price function directly, rather than to try to simulate individual price paths. We attempt here to approximate the price function by a finite-element discretization. Let \( C([0, 1]) \) connote the continuous, real-valued functions defined on the closed interval \([0, 1]\).

Given a positive integer \( n \), consider a partition of \([0, 1]\) as

\[
0 = x_1 < x_2 < \cdots < x_i < \cdots < x_n = 1, \quad \text{with} \quad x_i = \frac{i-1}{n-1}, \quad 1 \leq i \leq n.
\]

Let \( V_h \) connote the subspace

\[
V_h = \{ v \in C[0, 1] : v \text{ is linear on each interval } (x_{i-1}, x_i), \ i = 0, \ldots, n-1 \}.
\]

Let \( w_i \in V_h \) be the unique function satisfying

\[
w_i(x) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}
\]

Then \( \{ w_i \}_{i=1}^n \) forms a basis of \( V_h \), which is known as the nodal basis. In fact, every \( v \in V_h \) can be represented exactly as

\[
v(x) = \sum_{i=1}^{n} v(x_i) w_i(x).
\]

The so-called nodal-value interpolant

\[
I_h: C[0, 1] \to V_h
\]

plays an important role. It is defined by

\[
(I_h v)(x) = \sum_{i=1}^{n} v(x_i) w_i(x).
\]
The fully discretized version of the equilibrium equation (25) is the equation
\[ f_h(x) = F_h(f(x)) = I_h F(f)(x), \quad \text{where} \quad F(x) = 1 - x \phi \left( \frac{f(x)}{f(x)} \right) \]
for \( f_h \) in \( V_h \). A rigorous analysis of the above problem for the approximation provides several useful facts. First, for \( h \) small enough, it is inferred that the approximate problem possesses a unique solution, at least for relatively small monetary shocks. Second, if \( f \) is the solution of the continuous problem and \( f_h \) is the solution of the discrete problem, then the difference between them is bounded above as
\[ \| f - f_h \|_{C(0, 1)} \leq h^2 \| f'' \|_{C(0, 1)}, \]
Here, \( f'' \) is bounded since \( f \) is analytic on \([0, 1]\) as established in Lemma 3.4. This estimate thus indicates that the error \( f - f_h \) will be reduced roughly four times when \( h \) is reduced to \( h/2 \). As is well known in numerical analysis, this type of estimate can seldom be used directly since \( f'' \) is part of the solution, and so is not known \textit{a priori}. Salutary results may be obtained by an \textit{a posteriori} error analysis. It turns out that \( f'' \) is of order one, but we pass over this point here.

Another check on the accuracy of the numerical approximation of the price function \( f \) is to compute the residual vector \( r = (r_i) \), where
\[ r_i = \tilde{f}(x_i) - F(\tilde{f}(x_i)) \quad (33) \]
and \( x_i = (i - 1)/(n - 1), 1 \leq i \leq n, \) are as above. For \( n = 30 \), we found the maximal value of \( r_i \) to be of order \( 10^{-11} \). This shows that the computed price function very accurately satisfies Eq. (25) and, especially in light of the uniqueness result, suggests that it is a good approximation of the exact solution.

Using the computed price function \( \tilde{f} \) and the initial price \( p_1 \), the equilibrium prices are readily obtained via the iteration (26) in Theorem 3.5. Figure 1 shows an example of a typical solution. Note that \( p_1 < p_2 < (1 + k) p_1 \) and that the subsequent prices oscillate around \( (1 + k) p_1 \), a situation in agreement with the conclusions enunciated in Lemma 3.1 and Theorem 3.7. This aspect lends further assurance of the veracity of the numerical calculations.

3.5. A Proof of Lemma 3.4

This subsection provides a proof of Lemma 3.4. The proof will be divided into two parts. The first step is to establish a local existence result
and the second is to establish global existence by extending the local solution.

The proof of local existence is based on the following well-known result from dynamical systems theory as expounded in Hirsch and Pugh [14], for example. Our approach is a refinement of that appearing in Bona and Grossman [3].

**Lemma 3.8.** Assume that \( U \subset \mathbb{R}^2 \) is an open set and \( g: U \rightarrow \mathbb{R}^2 \) is a differentiable mapping satisfying

1. \( g(\bar{X}) = \bar{X} \) for some \( \bar{X} \in U \), and
2. the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( Dg(\bar{X}) \) satisfy

\[ |\lambda_1| > 1 \quad \text{and} \quad |\lambda_2| < 1. \]

Then there exists a unique curve \( \gamma \) defined in a neighborhood of \( \bar{X} \) and passing through \( \bar{X} \), such that

1. \( \gamma \) is the stable manifold for \( g \), which is to say that

\[ g(\{\gamma\}) \subset \{\gamma\}. \]
2. the eigenvector of $Dg(\bar{X})$ corresponding to $\lambda_2$ is tangent to $\gamma$ at $\bar{X}$, and

3. if $g \in C^k$ for some $1 \leq k \leq \infty$, then $\gamma \in C^k$ and $\gamma$ is real analytic if $g$ is real analytic.

The above lemma will be applied to prove local existence. It will be shown that there exists a positive constant $\delta$ and a real-analytic function $f$ defined on $\Omega_\delta = (\bar{p} - \delta, \bar{p} + \delta)$ such that for all $x \in \Omega_\delta$, $F(f) = f$, $f(\bar{p}) = \bar{p}$, and $-1 < f'(x) < 0$.

Let $\psi = \phi^{-1}$ as in the proof of Lemma 3.3 and let $g: \mathbb{R}^2 \mapsto \mathbb{R}^2$ be given by

$$g(p, q) = \left( \frac{p}{\psi\left(\frac{1}{1 - p}\right)}, p \right).$$

Obviously $g$ is real-analytic near the point $\bar{X} = (\bar{p}, \bar{q})$. Moreover, $g(\bar{X}) = \bar{X}$ and the eigenvalues of $Dg(\bar{X})$ are seen to be

$$\lambda_1 = \frac{1 + \psi'(1) + \sqrt{(1 + \psi'(1))^2 + 4\psi(1)\psi'(1)}}{2\psi'(1)},$$
$$\lambda_2 = \frac{1 + \psi'(1) - \sqrt{(1 + \psi'(1))^2 + 4\psi(1)\psi'(1)}}{2\psi'(1)},$$

with associated eigenvectors

$$Y_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix},$$

respectively. It is easily verified that $-1 < \lambda_2 < 0$ and $\lambda_1 > 1$ since $\psi'(1), \psi(1) \geq 0$. Hence, according to Lemma 3.8, there is a stable manifold for the mapping $g$ near $(\bar{p}, \bar{q})$ given by a curve $\gamma = \gamma(s)$, defined for $s$ near zero, say, such that

$$\gamma(s) = (\gamma_1(s), \gamma_2(s)) \quad \text{with} \quad \gamma_1(0) = \gamma_2(0) = \bar{p} \quad \text{and} \quad \frac{\gamma'_1(0)}{\gamma'_2(0)} = \lambda_2.$$ 

Because of these properties and the form of $g$, the curve $\gamma$ may be reparameterized in the form

$$\gamma(x) = (f(x), x),$$
for \( x \in \Omega_\delta \) for some \( \delta > 0 \), where \( f \) is an analytic function defined on \( \Omega_\delta \) such that \( f(\hat{\rho}) = \hat{\rho} \) and \( f'(\hat{\rho}) = \hat{\lambda}_2 \). In particular, we see that \( 0 > f'(\hat{\rho}) > -1 \). Because \( g \) maps \( \gamma \) into itself, if \( x \) is near \( \hat{\rho} \), then there is a \( y \) near \( \hat{\rho} \) such that

\[
g(f(x), x) = (f(y), y).
\]

By the definition of \( g \), this means that

\[
f(y) = \frac{f(x)}{\psi \left( \frac{1 - f(x)}{x} \right)} \quad \text{and} \quad y = f(x),
\]

Consequently, it transpires that

\[
f(f(x)) = \frac{f(x)}{\psi \left( \frac{1 - f(x)}{x} \right)}
\]

or equivalently

\[
f(x) = 1 - x \psi \left( \frac{f(x)}{f(f(x))} \right).
\]

This proves that \( f = F(f) \), at least when \( f \) is viewed as a mapping of \( \Omega_\delta \) into itself.

Since \( -1 < f'(\hat{\rho}) < 0 \) and \( f' \) is continuous, it follows that for sufficiently small \( \delta \), \( -1 < f'(x) < 0 \) if \( |x - \hat{\rho}| < \delta \). This result comprises the local existence theory and finishes the first step of the proof of Lemma 3.4.

Attention is now turned to the proof of existence of the price function on the whole domain \([0, 1]\). The idea is to extend the local solution defined near \( \hat{\rho} \) obtained above.

First note that the uniqueness of \( f \) is obvious. In fact, if \( \hat{f} \) is another such solution, but \( \hat{f}(p_1) \neq f(p_1) \) for some \( p_1 \in (0, 1) \), then the sequences generated by iteration, namely \( p_{t+1} = \hat{f}(p_t) \) and \( \hat{p}_{t+1} = f(p_t) \) with \( \hat{p}_1 = p_1 \), will be two different solutions of (21) with the same initial configuration. This contradicts Lemma 3.3.

Consider the auxiliary function

\[
g(t) = \frac{1 - t}{\phi \left( \frac{t}{f(t)} \right)} = (1 - t) \left( 1 + \beta^{-1/4} \left( \frac{t}{f(t)} \right)^{-1/4} \right).
\]
for all values of $t$ for which $f$ is defined. It is easy to see that $f$ is a solution of (25) if and only if

$$g(f(x)) = x$$

for all $x$ in the interval of the definition of $f$. Thus $g$ is actually the inverse function of $f$.

**Lemma 3.9.** Let $g$ be as defined above, where $f$ is a solution of 3.5 on some interval $\Omega \subseteq [0, 1]$. Then it follows that

$$g'(t) \leq -(1 + \gamma), \quad \text{for all } t \in \Omega,$$  \hspace{1cm} (34)

where

$$\gamma = \beta^{-1/4} \sigma_1^{1/4} \min \left( \frac{1}{\sigma_1}, \frac{1}{A}, 1 \right),$$  \hspace{1cm} (35)

$\beta$ is the discount factor, and $A$ is the parameter in the homothetic utility function, while $\sigma_1 \in (0, 1)$ is the parameter appearing in Lemma 3.2.

**Proof.** If $z = t/f(t)$, then by Lemma 3.2,

$$z \leq \frac{t}{\sigma_1} \leq \frac{1}{\sigma_1}.$$  

Direct calculation and Lemma 3.2 then imply that

$$-g'(t) = \frac{1}{\phi^2(z)} \left( \phi(z) + (1-t) \phi'(z) \frac{f(t) - tf'(t)}{f^2(t)} \right)$$

$$\geq \frac{1}{\phi(z)} + (1-t) \frac{\phi'(z)}{\phi^2(z)}$$

$$= 1 + \beta^{-1/4} z^{-1/4} + \left( \frac{1}{A} - 1 \right) \beta^{-1/4} (1-t)$$

$$\geq 1 + \beta^{-1/4} \sigma_1^{1/4} \left( \sigma_1^{-1} t + \left( \frac{1}{A} - 1 \right) (1-t) \right)$$

$$\geq 1 + \gamma.$$
Let $i_0$ be the smallest integer that satisfies

$$(1 + \gamma)^{i_0} \delta \geq 1.$$  

For $i \leq i_0 + 1$, define $\Omega_i = \Omega_{i_0}$ where $\delta_i = (1 + \gamma)^i \delta$. Let $f_0 = f$ and iteratively define

\begin{align*}
g_{i-1}(t) &= \frac{1 - t}{\phi \left( \frac{1}{f_{i-1}(t)} \right)} \quad \text{for} \quad x \in \Omega_{i-1}, \\
f_i(x) &= g_{i-1}^{-1}(x) \quad \text{for} \quad x \in \Omega_i.
\end{align*}

for $i = 0, 1, \ldots, i_0 + 1$. By induction, it is easy to see that $f_i$ and $g_i$ are both well defined, the estimate (34) holds for each $g_i$, and

$$\Omega_{i-1} \subseteq \Omega_i \subseteq g(\Omega_{i-1}), \quad f_i|_{\Omega_{i-1}} = f_{i-1}.$$  

By the definition of $i_0$, $1 \in \Omega_{i_0} \subseteq \Omega_{i_0 + 1}$. Thus $0 = g_0(1) \in \Omega_{i_0 + 1}$. As $\Omega_{i_0 + 1}$ must be a subinterval of $[0, 1]$, we conclude that

$$\Omega_{i_0 + 1} = [0, 1].$$

Consequently $f_{i_0 + 1}$ is defined on the whole interval $[0, 1]$ and satisfies (25).

The elementary properties of real analytic functions allow one to adduce that the extension given by (36) is analytic and hence $f$ is analytic on $[0, 1]$. That is, for all $x \in [0, 1]$, $f(x) = \sum_{n=0}^{\infty} f^{(n)}(\bar{p})(x - \bar{p})^n$. The proof of Lemma 3.4 is now complete. As a consequence of the further ruminations earlier in this section, the global existence and uniqueness of equilibrium dynamics are ascertained.

3.6. Equilibrium Prices When $A > 1$

The existence and uniqueness of equilibrium price paths have been established under the assumption $0 < A < 1$. In this section we use the algorithm developed in Subsection 3.4 and present numerical approximations of the price function and an equilibrium price path for a case where $A > 1$.

When $A > 1$, the utility function takes negative values. In Fig. 2 is shown the price function for the case $A = 4.0$. Observe that the price function $f$ is concave instead of convex as was the case when $0 < A < 1$. The equilibrium price sequence corresponding to a single monetary injection thus has behavior different from that obtaining when $A < 1$. Figure 3 presents a typical equilibrium price path corresponding to the price function in Fig. 2.
FIG. 2. The price function with $A > 1$.

FIG. 3. Plot of the price path with $A > 1$. 

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4. MULTIPLE MONETARY INJECTIONS AND STABILIZATION

This section is aimed at studying how government monetary policy can be designed to control the fluctuation of equilibrium prices that is a consequence of monetary expansion. Assume that the desired monetary policy involves a fraction \( k \) increase in the money supply. This amount of money can be injected into the market either at one time or at several, perhaps consecutive, times.

It turns out, as we shall show, that multiple injection is preferable to one-time injection in the sense that price and interest-rate oscillation can be eliminated by appropriately chosen policy. As a consequence, we conclude that the government can use the open-market operations as a tool to stimulate the economy while prices are kept relatively stable.

4.1. The Setup

In this subsection, we take it that money is injected into the market at \( n \) different, but consecutive, times \( j = 1, 2, \ldots, n \), the \( j \)th injection in the amount \( k_j M_0^s = k_j \) takes place at the end of time period \( j \), where

\[
k = \sum_{j=1}^{n} k_j,
\]

and the money supply at the end of period \( j \) is

\[
M_j^s = \begin{cases} 
1 + \sum_{i=1}^{j} k_i & 1 \leq j \leq n - 1, \\
1 + k & j \geq n.
\end{cases}
\]

In this case, (19)-(20) can be rewritten as

\[
p_t + p_{t-1} \phi \left( \frac{p_t}{p_{t+1}} \right) = 1 \quad \text{for} \quad t \geq n + 1, \tag{37}
\]

and

\[
p_t + (p_{j-1} + k_{j-1}) \phi \left( \frac{p_t}{p_{j+1}} \right) = 1 + \sum_{i=1}^{j-1} k_i, \quad 1 \leq j \leq n, \tag{38}
\]

where

\[
p_n = \frac{p_t + k_n}{k + 1}, \quad p_t = \frac{p_j}{k + 1} \quad \text{for} \quad t \geq n + 1, \quad p_0 = M_0^s, \quad \text{and} \quad k_0 = 0.
\]
Obviously we can solve (38) for \( p_{i+1} \) in terms of \( p_i \) and \( p_{i-1} \), namely
\[
p_{j+1} = S_j(p_j, p_{j-1}), \quad 1 \leq j \leq n-1, \tag{39}
\]
for certain functions \( S_j \). Applying (39) recursively for \( j = 1, 2, ..., n-1 \), gives that
\[
p_j = T_j(p_1), \quad \text{for} \quad 1 \leq j \leq n+1,
\]
for some functions \( T_j \). The explicit expression for \( T_j \) may be obtained inductively.

**Lemma 4.1.** Assume that \( p_1 \) belonging to \((0, 1)\) satisfies
\[
T_n(p_1) + (T_{n-1}(p_1) + k_{n-1}) \phi \left( \frac{T_n(p_1)}{(1+k)f(T_n(p_1) - k_n)} \right) = 1 + \sum_{j=1}^{n-1} k_j, \tag{40}
\]
Then if \( p_j, 1 \leq j \leq n, \) is determined by (39) and \( p_{t+1} = f(p_t) \), for \( t \geq n \), the path \( \{p_t\}_{t=1}^\infty \) satisfies (38) and (37).

The proof of the above lemma is straightforward. However, whether or not there exists a \( p_1 \) belonging to \((0, 1)\) satisfying (40) is a little subtle. We shall establish the existence of such values \( p_1 \) in some special and practically interesting cases.

**4.2. Two Injections**

The special case wherein money will be injected into the open market in exactly two consecutive periods will be discussed in this section. Since the total amount \( kM_0 = k \) is placed in the market in two periods, the money supply in the first two periods are \( M_0 = 1 \) and \( M_1 = (1+k_1)M_0 = 1+k_1 \), respectively, and the money supply at all other periods \( t = 2, 3, ..., \) is \( M_t = (1+k) \). According to (38) and (37), the equilibrium prices \( p_1, p_2, \) and \( p_t \) have to satisfy
\[
p_1 + M_0 \phi \left( \frac{p_1}{p_2} \right) = 1,
\]
\[
\frac{p_2}{1+k_1} + p_1 + k_1 \phi \left( \frac{p_2}{p_3} \right) = 1,
\]
\[
p_t + p_{t-1} \phi \left( \frac{p_t}{p_{t+1}} \right) = 1, \quad \text{for} \quad t \geq 3,
\]
where
\[ p_2 = \frac{p_2 + k - k_1}{k + 1} \quad \text{and} \quad p_t = \frac{p_t}{k + 1} \quad \text{for} \ t \geq 3. \]

Corresponding to (39), we have \( p_2 = S_1(p_1) \) where
\[
S_1(p_1) = \left( \frac{M^b \beta^{1/A}}{1 - p_1} - p_1^{1/A} \right)^{A(A - 1)} p_1,
\]
with \( p_2 = S_1(p_1) \), whereas (40) now becomes
\[
S_1(p_1) + p_1 + k_1 \phi \left( \frac{S_1(p_1)}{(1 + k) f \left( \frac{S_1(p_1) + k - k_1}{1 + k} \right)} \right) = 1 + k_1.
\]

Define the function \( H \) by
\[
H(x) = \frac{S_1(x)}{1 + k_1} + \frac{x + k_1}{1 + k_1} \phi \left( \frac{S_1(x)}{(1 + k) f \left( \frac{S_1(x) + k - k_1}{1 + k} \right)} \right) - 1.
\]

Evidently \( H \) is continuous and
\[
\lim_{x \to p^*} H(x) = -1, \quad \lim_{x \to 1} H(x) = \infty,
\]
where \( p^* = 1 - M_0^b \). By the intermediate-value theorem, there exists \( p_1 \in (p^*, 1) \) satisfying \( H(p_1) = 0 \), with \( p_2 = S_1(p_1) \) and \( p_t = f(p_{t-1}) \) for \( t \geq 3 \). The equilibrium price path \( \{p_t\} \) is thereby obtained.

**Comparison with One Injection.** Assume that \( \{p_t\}_{t=1}^\infty \) and \( \{q_t\}_{t=1}^\infty \) are the equilibrium price sequences for one injection and two injections, respectively, that correspond to the same total monetary expansion.

It is clear that the two sequences of prices coincide for \( t \geq 2 \) if \( k = k_1 \). As \( k_1 \) is reduced, the tail of the two-injection sequence \( \{q_t\}_{t=3}^\infty \) moves closer to the steady-state equilibrium price. However, after \( k_1 \) passes a certain critical value, the tail sequence of prices appears to drift back away from the steady state. Numerically, we were able to find an approximate value for this critical value by performing a sequence of simulations. Such a critical value \( k_1 = 0.532 \) (shown in Fig. 4), which makes the equilibrium prices most
FIG. 4. The third test, a single injection of 0.8 and double injection with \( k_1 = 0.529 \), \( k_2 = 0.271 \). Here \( A = 0.63 \), \( \beta = 0.99 \), \( M^* = 0.5 \), and \( K = 0.8 \). In the graph, the solid line corresponds to one injection, the dashed line to two injections, and the equilibrium price \( \bar{p} \) is marked by \( \cdots \).

stable in the sense of their moving most quickly and monotonically to the new steady-state price, will be called a stabilizing injection and its existence will be rigorously justified in the next subsection.

Existence of Stabilizing Injections. The stabilizing injection observed in the numerical experiments reported above is of theoretical and possible practical interest. This paragraph is devoted to a proof of its existence.

**Theorem 4.2.** Let \( 0 < A < 1 \) and \( U = U_A \) be a homothetic utility function. Let \( k > 0 \) be an overall percentage increase in the money supply and suppose that the increase is to be implemented in two consecutive times, \( k_1 \) in period 1 and \( k_2 = k - k_1 \) in period 2. Suppose that \( M^*_0 \in [\bar{p}(1 - k/\phi(1)), \bar{p}(1 + k)] \).

Then, there exists a \( k_1 \in (0, k) \) such that

\[
\forall t \geq 2, \quad \text{where}
\]

\[
p_t = \bar{p}
\]

\[
p_t = \frac{p_2 + k - k_1}{1 + k} \quad \text{and} \quad p_t = \frac{p_t}{1 + k} \quad \text{for } t \geq 3.
\]
Proof. It suffices to show (41) for \( t = 2 \). Recall that \( p_2 \) satisfies
\[
G(p_2, s) = \frac{(1 + k) p_2 + s - k}{1 + s} + \frac{p_1 + s}{1 + s} \phi \left( \frac{(1 + k) p_2 + s - k}{(1 + k) f(p_2)} \right) - 1
\]
and \( p_1 = p_1(p_2) \) is given by
\[
 p_1 + M_b \phi \left( \frac{p_1}{(1 + k) p_2 + s - k} \right) = 1.
\]
We need to show that there exists a \( k_1 \in (0, k) \) such that \( G(\bar{\rho}, k_1) = 0 \). As \( G \) is obviously continuous with respect to \( s \), it suffices to show
\[
G(\bar{\rho}, 0) < 0, \quad G(\bar{\rho}, k) > 0.
\]
It is first shown that if \( s = k \), then
\[
\frac{p_1 + k}{1 + k} > \bar{\rho}.
\]
If the latter inequality does not hold when \( s = k \), then \( p_1 \leq (1 + k) \bar{\rho} - k \) and thus
\[
0 = p_1 + M_b \phi \left( \frac{p_1}{(1 + k) p_2 + s - k} \right) - 1
\leq (1 + k) \bar{\rho} - k + M_b \phi \left( 1 - \frac{k}{(1 + k) \bar{\rho}} \right) - 1
\leq (1 + k) \bar{\rho} - 1 + M_b \phi \left( 1 - \frac{k}{(1 + k) \bar{\rho}} \right)
\leq (M_b - (1 + k) \bar{\rho}) \phi(1) < 0,
\]
which is a contradiction. Consequently,
\[
G(\bar{\rho}, k) = \frac{p_1 + k}{1 + k} \phi(1) - 1 = \left( \frac{p_1 + k}{1 + k} - \bar{\rho} \right) \phi(1) \geq 0.
\]
Next, it is asserted that if \( s = 0 \), then
\[
\frac{p_1}{1 + k} \leq \bar{\rho}.
\]
For otherwise we have \( p_1 > \hat{p}(1+k) \), and then it transpires that

\[
0 = p_1 + M_0^p \phi \left( \frac{p_1}{(1+k) \hat{p} - k} \right) - 1 \\
> (1+k) \hat{p} + M_0^p \phi \left( \frac{(1+k) \hat{p}}{(1+k) \hat{p} - k} \right) - 1 \\
> M_0^p \phi \left( \frac{(1+k) \hat{p}}{(1+k) \hat{p} - k} \right) - (\phi(1) - k) \hat{p} \\
= \phi(1) [M_0^p - (1-k/\phi(1)) \hat{p}] > 0,
\]

which is a contradiction. Consequently, it is seen that

\[
G(\hat{p}, 0) = (1+k) \hat{p} - k + p_1 \phi \left( 1 - \frac{k}{(1+k) \hat{p}} \right) - 1 \\
= \hat{p}(1+k) \left[ \phi \left( 1 - \frac{k}{(1+k) \hat{p}} \right) - \phi(1) \right] < 0.
\]

**Remark.** It follows from the proof that the restriction on \( M_0^p \) can be weakened as follows:

\[
\frac{\phi(1) - k}{\phi \left( 1 + \frac{k}{(1+k) \hat{p} - k} \right)} \hat{p} \leq M_0^p \leq \frac{(1+k) \phi(1)}{\phi \left( 1 - \frac{k}{(1+k) \hat{p}} \right)} \hat{p}.
\]

As a consequence, if \( k \geq \phi(1) \), then Theorem 4.2 is true for all \( M_0^p \in [0, 1] \).

### 5. CONCLUDING REMARKS

The principal conclusions derived on the basis of the foregoing analysis is that a monetary expansion in an economy in the presence of utility functions where the substitution effect is larger than the income effect associated with changes in the rates of return will lead to a gradual increase in price level. Moreover, in the absence of further perturbations the price path will approach a new steady state in an oscillatory manner.

The numerical experiments based on the Grossman–Weiss model show clearly that the size of the price fluctuation depends substantially on the level of government intervention in the money supply. A potentially interesting aspect of the present investigation is the existence of what we have termed ‘stabilizing’ monetary injections, in which the new equilibrium
prices and interest rates are reached monotonically, without oscillation and in a minimal length of time. This opens up the prospect of using multiple injections to move a real economy from an old equilibrium to a new equilibrium with a relatively smooth transition. It is worth mentioning that if in reality a government monetary shock takes effect over a time interval on the order of the basic period in the model or longer, then an outcome other than strict oscillation can be observed. It is possible that this point may explain some of what is observed when real data are contemplated.

Finally, it deserves remarking that the endogenous inclusion of production is needed to truly justify the appellation “optimal” to the multistep injection policies discussed in Section 4. An investigation of this aspect is currently under way and the outcome will be reported in due course.

REFERENCES