Interconnection and damping assignment passivity–based control of mechanical systems with underactuation degree one

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Abstract

Interconnection and damping assignment passivity-based control is a new controller design methodology developed for (asymptotic) stabilization of nonlinear systems that does not rely on, sometimes unnatural and technique–driven, linearization or decoupling procedures but instead endows the closed–loop system with a Hamiltonian structure with a desired energy function—that qualifies as Lyapunov function for the desired equilibrium. Its application relies on the possibility of solving a set of partial differential equations that identify the energy functions that can be assigned to the closed–loop. We prove in this paper that for a class of mechanical systems with underactuation degree one the partial differential equations can be explicitly solved. Furthermore, we introduce a suitable parametrization of assignable energy functions that provides the designer with a handle to address transient performance and robustness issues. Finally, we develop a speed estimator that allows the implementation of position–feedback controllers. The new result is applied to obtain an (almost) globally stabilizing scheme for the vertical takeoff and landing aircraft with strong input coupling, and a controller for the pendulum in a cart that can swing–up the pendulum from any position in the upper half plane and stop the cart at any desired location. In both cases we obtain very simple and intuitive position–feedback solutions.

1 Introduction

In [25] we introduced a controller design technique, called interconnection and damping assignment passivity–based control (IDA–PBC), that achieves stabilization for underactuated mechanical systems invoking the physically motivated principles of energy shaping and damping injection. IDA–PBC endows the closed–loop system with a Hamiltonian structure where the kinetic and potential energy functions have some desirable features, a minimal requirement being to have a minimum at the desired operating point to ensure its stability. Similar techniques have been reported for general port–controlled Hamiltonian and Lagrangian systems in [24, 34] and

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[26], respectively; see also [10, 11] for the case of Lagrangian mechanical systems and [22] which contains an extensive list of references on this topic. The success of these methods relies on the possibility of solving a set of partial differential equations (PDEs) that identify the energy functions that can be assigned to the closed–loop. The PDE associated to the kinetic energy defines the admissible closed–loop inertia matrices and is nonlinear, while the PDE of assignable potential energy functions is linear. In [10] the authors identify a series of conditions on the system and the assignable inertia matrices such that the PDEs can be solved. Also, techniques to solve the PDEs have been reported in [6, 9]. In [14] it is shown that the kinetic energy PDE reduces to an ordinary differential equation (ODE) if the system is of underactuation degree one, that is, if the difference between the number of degrees of freedom and the number of control actions is one—see also [7] for a detailed study of this case for the Controlled Lagrangian Method. In spite of all these developments the need to solve the PDEs remains the main stumbling block for a wider applicability of these methods.

In this paper we are interested in the application of IDA–PBC to mechanical systems with underactuation degree one. The main contributions of the paper are:

1. Identification of a class of underactuation degree one mechanical systems for which the PDEs of IDA–PBC can be explicitly solved.

2. Derivation of conditions to effectively assign a minimum to the energy function at the desired operating point—providing in this way a complete constructive procedure for stabilization. The conditions are given in terms of some algebraic inequalities involving the system equations and the solutions of the PDEs.

3. Development, using the recently introduced method of Immersion and Invariance [4, 18], of a speed estimator that allows the implementation of the proposed controllers measuring position only. To the best of our knowledge, this is the first position–feedback solutions reported for these systems—at this level of generality.

4. Last, but not least, the introduction of a suitable parametrization of assignable energy functions—via two free functions and a gain matrix—giving the designer the possibility to address transient performance and robustness issues. In spite of their great practical importance these issues are rarely studied in the literature. Indeed, most of the controllers reported for this class of systems rely on the rather unnatural, technique–driven and fragile operations of linearization and decoupling. Other existing schemes give very little freedom to the designer to tune the controller—basically only the selection of saturation and domination functions or the adjustment of high–gain injections or dampings.

Another feature of our developments is that, in contrast to [24, 25], the open–loop system need not be described by a port–controlled Hamiltonian (or Lagrangian) model, a situation that arises often in applications due to model reductions or preliminary feedbacks that destroy the structure. The new result is applied to obtain an (almost) globally stabilizing scheme for the vertical takeoff and landing aircraft with strong input coupling, and a controller for the pendulum in a cart that can swing–up the pendulum from any position in the (open) upper half plane and stop the cart at any desired location. In both cases we obtain very simple and intuitive position–feedback solutions that endow the closed–loop system with a Hamiltonian structure with desired potential and kinetic energy functions.

2 The IDA–PBC method for general nonlinear systems

In this section we show that IDA–PBC, introduced in [24] for static state feedback stabilization of equilibria of port–controlled Hamiltonian systems, can be easily adapted for nonlinear systems of the form

\[ \dot{x} = f(x) + g(x)u \]
where \( x \in \mathbb{R}^n \) is the state vector and \( u \in \mathbb{R}^m, m < n \), is the control action. Stabilization is achieved assigning to the closed-loop system the form
\[
\dot{x} = [J_d(x) - \mathcal{R}_d(x)]\nabla_x H_d
\]
(2)
where the matrices \( J_d(x) = -J_d^\top(x) \) and \( \mathcal{R}_d(x) = \mathcal{R}_d^\top(x) \geq 0 \), which represent the desired interconnection structure and dissipation, respectively, are fixed by the designer—hence the name IDA—and \( H_d(x) : \mathbb{R}^n \to \mathbb{R} \) is the desired total stored energy, that should satisfy
\[
x_* = \arg\min H_d(x)
\]
(3)
with \( x_* \in \mathbb{R}^n \) the equilibrium to be stabilized. As stated in the simple lemma below the assignable energy functions are characterized by a set of (parameterized) PDEs.

**Lemma 1** Consider the system (1). Assume there are matrices \( g^\perp(x) \), \( J_d(x) = -J_d^\top(x) \), \( \mathcal{R}_d(x) = \mathcal{R}_d^\top(x) \geq 0 \), and a function \( H_d : \mathbb{R}^n \to \mathbb{R} \) that verify the PDE
\[
g^\perp(x)f(x) = g^\perp(x)[J_d(x) - \mathcal{R}_d(x)]\nabla_x H_d
\]
(4)
where \( g^\perp(x) \) is a full rank left annihilator of \( g(x) \), i.e., \( g^\perp(x)g(x) = 0 \) and \( H_d(x) \) satisfies (3). Then the system (1) in closed-loop with the control \( u = \beta(x) \), where
\[
\beta(x) = [g^\top(x)g(x)]^{-1}g^\top(x)\{[J_d(x) - \mathcal{R}_d(x)]\nabla_x H_d - f(x)\},
\]
(5)
will take the form (2) with \( x_* \) a (locally) stable equilibrium. It will be (locally) asymptotically stable if, in addition, the largest invariant set under the closed-loop dynamics (2) contained in
\[
\{ x \in \mathbb{R}^n \mid [\nabla_x H_d]^\top \mathcal{R}_d(x)\nabla_x H_d = 0 \}
\]
equals \( \{ x_* \} \). An estimate of its domain of attraction is given by the largest bounded level set \( \{ x \in \mathbb{R}^n \mid H_d(x) \leq c \} \).

**Proof.** Equating the right hand side of (1), with \( u = \beta(x) \), with the right hand side of (2) we obtain the matching equation
\[
f(x) + g(x)\beta(x) = [J_d(x) - \mathcal{R}_d(x)]\nabla_x H_d.
\]
Multiplying on the left by \( g^\perp(x) \) we obtain the PDE (4). The expression of the control is obtained multiplying on the left by the pseudo-inverse of \( g(x) \).

Stability of \( x_* \) is established noting that, along the trajectories of (2), we have
\[
H_d = -[\nabla_x H_d]^\top \mathcal{R}_d(x)\nabla_x H_d \leq 0.
\]
Hence, \( H_d(x) \) qualifies as a Lyapunov function. Local asymptotic stability follows immediately invoking LaSalle’s invariance principle. Finally, to ensure the solutions remain bounded, we give the estimate of the domain of attraction as the largest bounded level set.

**Remark 1** Besides the interconnection and damping matrices, \( g^\perp(x) \) appears in Lemma 1 as a degree of freedom for the solution of the PDE. Indeed, although the space spanned by \( g^\perp(x) \) is univocally defined by \( g(x) \)—it is actually \( \text{ker}\{g^\top(x)\} \)—we can select an arbitrary basis that spans this space. This feature will be instrumental for the solution of the problem at hand.

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1We note that all vectors in the paper are column vectors, even the gradient of a scalar function \( \nabla (\cdot) = \frac{\partial}{\partial \gamma} \). The left annihilator of a matrix, denoted \((\cdot)^\perp\), whenever is a vector, will be a row vector. We will also assume throughout the paper that all functions are sufficiently smooth and, whenever rank conditions are imposed, we assume that they hold uniformly with respect to their arguments.
3 The PDEs for a class of mechanical systems with underactuation degree one

In this section we derive the PDEs of the IDA–PBC method for the class of mechanical systems of interest in the paper—the class contains several practically relevant examples, with two of them given in Section 7. (A complete characterization of all underactuation degree one mechanical systems which are feedback–equivalent to this class is given in [2]. See also [21].) Introducing a suitable parametrization of the desired interconnection matrix we express the PDEs in a form that clearly reveals their structure and suggests a parametrization for the desired energy function that allows us, in the next section, to explicitly solve the PDEs.

The class of systems that we consider is given by

$$
\dot{q} = M(q_r)^{-1}(q_r)p \\
\dot{p} = s(q_r) + G(q_r)u,
$$

where $q_r$, with $r$ an integer taking values in the set $\{1, \ldots, n\}$, is an element of $q \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $u \in \mathbb{R}^{n-1}$ are the control inputs, the matrix $M(q_r)$ is symmetric positive definite and bounded, and $s(q_r), G(q_r)$ are “sufficiently smooth” functions of $q_r$, and we assume that $G(q_r)$ is full column rank. Notice that the system has underactuation degree one. Structures of the form (6) result from the reduction, via singular perturbations or a preliminary feedback action, of certain classes of mechanical systems. See the examples in Section 7.\(^2\) The control objective is to stabilize an equilibrium $(q_r, 0)$.

We make two simple, but important, observations at this point.

O.1 The admissible equilibria for the coordinate $q_r$ should satisfy

$$
G^\perp(q_r)s(q_r) = 0
$$

where $G^\perp(q_r)$ is a full rank left annihilator of $G(q_r)$, i.e., $G^\perp(q_r)G(q_r) = 0$. The left annihilator is well defined in view of our assumption that $G(q_r)$ is full column rank. In the statement of our main stabilization result an assumption on the function $G^\perp(q_r)s(q_r)$ will be imposed.

O.2 $G^\perp(q_r)$ is a row vector. This follows from the facts that

$$
G^\perp(q_r) G(q_r) = 0 \iff \left[G^\perp(q_r)\right]^T \in \ker\{G^\top(q_r)\},
$$

and that $\dim \ker\{G^\top(q_r)\} = 1$.

In [25] we have shown that for mechanical systems a sensible choice of the desired dynamics (2) is given by

$$
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & M^{-1}(q_r)M_d(q_r) \\
-M_d(q_r)M^{-1}(q_r) & J_2(q_r, p) - G(q_r)K_vG^\top(q_r)
\end{bmatrix}
\begin{bmatrix}
\nabla_q H_d \\
\nabla_p H_d
\end{bmatrix},
$$

that is

$$
J_d(q_r, p) =
\begin{bmatrix}
0 & M^{-1}(q_r)M_d(q_r) \\
-M_d(q_r)M^{-1}(q_r) & J_2(q_r, p)
\end{bmatrix}, \quad \mathcal{R}(q_r) =
\begin{bmatrix}
0 & 0 \\
0 & G(q_r)K_vG^\top(q_r)
\end{bmatrix},
$$

with a total energy function of the form

$$
H_d(q, p) = \frac{1}{2} p^\top M^{-1}(q_r) p + V_d(q)
$$

\(^2\)The notation is motivated by mechanical systems where $q, p$ represent positions and momenta, respectively, and $M(q_r)$ is the inertia matrix.
where $M_d(q_r) = M_d^T(q_r) > 0$ and $V_d(q)$ represent the (to be defined) closed-loop inertia matrix and potential energy function, respectively, and to ensure stability we require that $V_d(q)$ satisfies

$$q_* = \arg \min V_d(q). \quad (11)$$

$K_v = K_v^\top > 0$ is a damping injection matrix and $J_2(q_r, p)$ is a skew–symmetric matrix which is a free parameter of our design.\(^3\) We remark at this point that $J_2(q_r, p)$ as well as the desired kinetic energy are functions only of $(q_r, p)$—as suggested by the class of systems under consideration.

As suggested in [25] we will concentrate first on the energy–shaping stage, hence set $K_v = 0$. The design is completed with a second stage of damping injection feeding back the passive output $G^\top(q_r)\nabla_p H_d$. The PDEs of IDA–PBC can then be decomposed into a nonlinear PDE for the elements of the desired inertia matrix, plus a linear PDE for the potential energy function. For the case in point we proceed as follows.

From (6), (9) and (10) it is clear that (4) becomes

$$G^\perp(q_r) \left[ s(q_r) + M_d(q_r)M^{-1}(q_r)\nabla_q V_d(q) \right] = G^\perp(q_r) \left[ \frac{1}{2} M_d(q_r)M^{-1}(q_r)e_r p^\top \frac{dM_d}{dq_r}(q_r) + J_2(q_r, \tilde{p}) \right] \tilde{p} \quad (12)$$

where $e_r \in \mathbb{R}^n$ is the $r$–th vector of the standard Euclidian basis, to simplify the expressions we have introduced the (partial) coordinate

$$\tilde{p} \triangleq M_d^{-1}(q_r)p, \quad (13)$$

used the relation

$$\frac{d}{dq_r} \left( M_d^{-1}(q_r) \right) = -M_d^{-1}(q_r) \frac{dM_d}{dq_r}(q_r)M_d^{-1}(q_r),$$

and swapped a scalar term.

The left hand terms in (12) are independent of $\tilde{p}$ and setting them equal to zero yields the following linear PDE for the desired potential energy function

$$G^\perp(q_r) \left[ s(q_r) + M_d(q_r)M^{-1}(q_r)\nabla_q V_d(q) \right] = 0. \quad (14)$$

Equating the right hand terms to zero defines the PDE for the desired inertia matrix, and it is clear from the expressions that the free matrix $J_2(q_r, \tilde{p})$ should be linear in $\tilde{p}$. We make now the important observation that, without loss of generality, $J_2(q_r, \tilde{p})$ can be parameterized in the form

$$J_2(q_r, \tilde{p}) = \begin{bmatrix}
0 & \tilde{p}^\top \alpha_1(q_r) & \tilde{p}^\top \alpha_2(q_r) & \ldots & \tilde{p}^\top \alpha_{n-1}(q_r) \\
-\tilde{p}^\top \alpha_1(q_r) & 0 & \tilde{p}^\top \alpha_2(q_r) & \ldots & \tilde{p}^\top \alpha_{n-1}(q_r) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\tilde{p}^\top \alpha_{n-1}(q_r) & -\tilde{p}^\top \alpha_{2n-3}(q_r) & \cdots & 0
\end{bmatrix}$$

where the vector functions $\alpha_i(q_r) \in \mathbb{R}^n$, $i = 1, \ldots, n_o$, $n_o = \frac{3}{2}(n - 1)$, are free parameters. Alternatively, we can write

$$J_2(q_r, \tilde{p}) = \sum_{i=1}^{n_o} \tilde{p}^\top \alpha_i(q_r) W_i,$$

with the $W_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, n_o$, defined as follows. First, we construct $n^2$ matrices of dimension $n \times n$, that we denote $F^{kl} = \{ f_{ij}^{kl} \}$, $k, l \in \{ 1, 2, \ldots, n \}$, according to the rule

$$f_{ij}^{kl} = \begin{cases}
1 & \text{if } j > i, \ i = k \text{ and } j = l \\
0 & \text{otherwise.}
\end{cases}$$

\(^3\)See [25, 9, 12] for a thorough discussion on the role of this parameter and its interpretation in the Lagrangian framework in terms of gyroscopic forces.
Notice that only \( n_o \) matrices are different from zero. Then, we define \( W^{kl} = F^{kl} - (F^{kl})^\top \). Finally, we set (in an obvious way)

\[
W_1 = W^{12}, W_2 = W^{13}, \ldots, W_n = W^{1n}, W_{n+1} = W^{23}, \ldots, W_{n_o} = W^{(n-1)n}.
\]

For instance, for the case \( n = 3 \), for which also \( n_o = 3 \), we get

\[
W_1 \triangleq \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ W_2 \triangleq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \ W_3 \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.
\]

Using this parameterization some simple calculations establish that

\[
G^\perp(q_r)J_2(q_r, \tilde{p}) = \tilde{p}^\top \mathcal{J}(q_r)A^\top(q_r)
\]

where we defined

\[
\mathcal{J}(q_r) \triangleq \begin{bmatrix} \alpha_1(q_r) : \alpha_2(q_r) : \cdots : \alpha_{n_o}(q_r) \end{bmatrix} \in \mathbb{R}^{n \times n_o},
\]

which is a free matrix, and

\[
A(q_r) \triangleq \begin{bmatrix} W_1 \left( G^\perp(q_r) \right)^\top, W_2 \left( G^\perp(q_r) \right)^\top, \ldots, W_{n_o} \left( G^\perp(q_r) \right)^\top \end{bmatrix} \in \mathbb{R}^{n \times n_o}.
\]

Replacing (15) in the right hand side of (12) and factoring \( \tilde{p} \) on both sides we get

\[
\tilde{p}^\top \left[ \frac{1}{2} G^\perp(q_r) M_d(q_r) M^{-1}(q_r) e_r \frac{dM_d}{dq_r}(q_r) + \mathcal{J}(q_r) A^\top(q_r) \right] \tilde{p} = 0.
\]

Setting the expression in brackets to zero finally yields the kinetic energy ODE

\[
G^\perp(q_r) M_d(q_r) M^{-1}(q_r) e_r \frac{dM_d}{dq_r}(q_r) = -2 \mathcal{J}(q_r) A^\top(q_r).
\]

**Remark 2** An \( n \times n \) skew–symmetric matrix contains at most \( n_o \) non–zero different terms. Hence, the proposed \( J_2(q_r, \tilde{p}) \) contains all skew–symmetric matrices which are linear in \( \tilde{p} \), that is, all matrices of the form \( \sum_{i=1}^{n_o} \Omega_i(q_r) \tilde{p}_i \), \( \Omega_i(q_r) = -\Omega_i^\top(q_r) \), and the parametrization is done without loss of generality as claimed above.\(^4\)

**Remark 3** The solutions of (14) and (17) characterize a set of assignable energy functions of the form (10). Typically in IDA–PBC we start with (17), which is a set of nonlinear ODEs in the unknown matrix \( M_d(q_r) \), with \( \mathcal{J}(q_r) \) a free matrix to be chosen by the designer. Then, plugging \( M_d(q_r) \) in (14), we solve the PDE for \( V_d(q) \). It is important to recall that, to comply with the stability requirements, we also have to satisfy the additional constraints of positivity of \( M_d(q_r) \) and the minimum condition (11).

## 4 A parametrization of \( M_d(q_r) \) that solves the PDEs

In this section we present a parametrization of the desired inertia matrix for which there exists a \( \mathcal{J}(q_r) \) that solves (17); with the additional advantage that (14) becomes a trivial linear PDE that we can explicitly solve, e.g., with the techniques of [8]. This result will be used in the next section to give stabilization conditions in terms of a set of algebraic inequalities

\(^4\)The space of skew–symmetric matrices, usually denoted \( so(n) \), can be alternatively defined noting that \( so(n) \) is isomorphic to \( \mathbb{R}^{n_o} \) via the hat operator \( \hat{\cdot} : \mathbb{R}^{n_o} \to so(n) \), and then use the basis \( \{ \hat{e}_1, \ldots, \hat{e}_{n_o} \} \). We thank the reviewer for this remark.
Even though we have full freedom in the selection of $\mathcal{J}(q_r)$ finding a solution of (17) is nontrivial because, on one hand, the matrix $\mathcal{A}(q_r)$ is not full rank—in particular, from (15) and $J_2(q_r, \tilde{p}) = -J_2^T(q_r, \tilde{p})$—we have that

$$G^\perp(q_r)\mathcal{A}(q_r) = 0. \quad (18)$$

On the other hand, there are no systematic methods for the solution of nonlinear ODEs. One key property, that stems from Observation O.2 in Section 3, is that the term $G^\perp(q_r)M_d(q_r)M^{-1}(q_r)e_r$ is a scalar, and as shown below we will be able to select a suitable vector $G^\perp(q_r)$ to make this term a constant. In this way, (17) becomes a linear ODE.

Before introducing the parametrization we present three results that will be instrumental for further developments. First, we make the basic observation that left annihilators of $G(q_r)$ are “invariant” under multiplication by scalar functions. More precisely, we have:

**Fact 1** For all $1 \times n$ vectors $\tilde{G}^\perp(q_r)$ such that $\tilde{G}^\perp(q_r)G(q_r) = 0$, we have that $G^\perp(q_r) \triangleq \eta(q_r)\tilde{G}^\perp(q_r)$, with $\eta(q_r)$ an arbitrary scalar function, is also a left annihilator of $G(q_r)$.

Second, we prove a simple linear algebra lemma.

**Lemma 2** Consider a matrix $A \in \mathbb{R}^{n \times n_0}$ with $n_0 \geq n$, rank $A = n - 1$, and such that $w^\top A = 0$ for some $w \in \mathbb{R}^n$. Then, for all vectors $x \in \mathbb{R}^n$ such that $w^\top x = 0$ there exists a vector $y \in \mathbb{R}^{n_0}$ such that $x = Ay$.

*Proof.* First, recall that given $A$ and $x$, there exists $y$ such that $x = Ay$ if and only if

$$\text{rank } A = \text{rank } [A \mid x].$$

Let us denote with $S \in \mathbb{R}^n$ the space of all $n$-dimensional vectors orthogonal to $w$, which is an $n - 1$-dimensional space. Now, $w^\top A = 0$ implies that all columns of $A$ are in $S$. Also, from $w^\top x = 0$ we have that $x \in S$. Since the rank of $A$ is $n - 1$ there are $n - 1$ linearly independent columns that span the whole space $S$. Therefore, the rank of $A$ cannot be increased by adding another vector in the same $(n - 1)$-dimensional space and the rank identity above holds.

Third, we establish that $\mathcal{A}(q_r)$ satisfies the rank condition required by the lemma above.

**Lemma 3** For the matrix $\mathcal{A}(q_r)$, defined in (16), we have

$$\text{rank } \mathcal{A}(q_r) = n - 1.$$

*Proof.* We first recall (18). To establish the proof we will show that $G^\perp(q_r)$ spans the left kernel of $\mathcal{A}(q_r)$. To simplify the notation we define $v \triangleq \text{col}(v_1, \ldots, v_n) = (G^\perp(q_r))^\top$ and assume, without loss of generality, that $v_1 \neq 0$—see below.

From the construction of the matrices $W^{kl}$ given in the previous section we have

$$e_r^\top W^{kl} = \begin{cases} \ e_l^\top & \text{if } r = k \\ -e_k^\top & \text{if } r = l \\ 0 & \text{otherwise} \end{cases}$$

Consider now a vector $w = \sum_{i=1}^n a_ie_i$ in the left kernel of $\mathcal{A}(q_r)$. We thus have

$$w^\top \mathcal{A}(q_r) = 0 \Rightarrow \sum_{i=1}^n a_i e_i^\top W^{1j} v = 0, \quad j = 2, \ldots, n$$

$$\Leftrightarrow a_1 e_j^\top v - a_j e_1^\top v, \quad j = 2, \ldots, n$$

$$\Leftrightarrow a_1 e_j^\top v - a_j v_1, \quad j = 2, \ldots, n$$
The latter is a set of $n - 1$ equations with $n$ unknowns (the coefficients $a_j$) that, invoking the assumption of $v_1 \neq 0$, has the form
\[
\begin{bmatrix}
\frac{v_2}{v_1} \\
\vdots \\
\frac{v_n}{v_1}
\end{bmatrix} a_1 = \begin{bmatrix}
a_2 \\
\vdots \\
a_n
\end{bmatrix}.
\]
Clearly, all solutions of this equation are co-linear with $v$, completing the proof.

We are in a position to present the main result of this section—a parametrization of $M_d(q_r)$ such that the PDE’s can be explicitly solved. For, we make the following assumption.

**Assumption A.1** Given $r \in \{1, \ldots, n\}$, $G(q_r)$, $M(q_r)$ and $q_{r^*}$. There exists an $n \times n$ symmetric positive definite matrix $M^0_d$ such that
\[
|e_i^\top M^{-1}(q_{r^*}) M^0_d w| \geq \epsilon,
\]
for some vector $w \in \mathbb{R}^n$, $w \in \ker\{G^\top(q_{r^*})\}$ and some $\epsilon > 0$.

**Proposition 1** For all desired inertia matrices of the form
\[
M_d(q_r) = \int_{q_{r^*}}^{q_r} G(\mu)\Psi(\mu)G^\top(\mu)d\mu + M^0_d
\]
where the matrix $\Psi(q_r) = \Psi^\top(q_r) \in \mathbb{R}^{(n-1) \times (n-1)}$ may be arbitrarily chosen, and $M^0_d$ satisfies Assumption A.1, there exists a matrix $\mathcal{J}(q_r)$ such that the kinetic energy ODE (17) holds in a neighborhood of $q_{r^*}$.

Furthermore, the solution of the potential energy PDE (14) is given by
\[
V_d(q) = -\frac{1}{\rho} \int_{0}^{q_r} G^\perp(\mu) s(\mu)d\mu + \Phi(z(q)),
\]
where $z(q)$ is an $n - 1$ dimensional vector whose elements are of the form
\[
z_i(q_i, q_r) \triangleq q_i - \frac{1}{\rho} \int_{0}^{q_r} G^\perp(\mu) M_d(\mu)M^{-1}(\mu)e_i d\mu,
\quad i = 1, \ldots, n,
\]
with $\Phi(z)$ is an arbitrary differentiable function, and we define below the constant $\rho$ and $G^\perp(q_r)$.

**Proof.** First, note that the integration limits have been chosen such that $M_d(q_{r^*}) > 0$. Therefore, $M_d(q_r) > 0$ on some neighborhood of $q_{r^*}$.

Now, fix a constant $\rho$, an arbitrary left annihilator of $G(q_r)$, say $G^\perp(q_r)$, and $M_d(q_r)$ as specified in the proposition and define
\[
\eta(q_r) = \frac{\rho}{G^\perp(q_r) M_d(q_r) M^{-1}(q_r)e_r}.
\]
We will prove that $G^\perp(q_r) M_d(q_r) M^{-1}(q_r)e_r$ evaluated at $q_{r^*}$ is bounded away from zero, and consequently in view of the smoothness of all the functions, the function $\eta(q_r)$ is well defined on some neighborhood of $q_{r^*}$. This follows, indeed, taking into account (20)—notice the integration limits—and Assumption A.1, and recalling (8). Now, choose $G^\perp(q_r) = \eta(q_r) G^\perp(q_r)$, which in the light of Fact 1 is also a left–annihilator of $G(q_r)$, then
\[
G^\perp(q_r) M_d(q_r) M^{-1}(q_r)e_r = \rho,
\]

---

$^5$ $z(q)$ is the, so–called, characteristic of the homogeneous part of the PDE [8].
and the kinetic energy equation (17) becomes the linear ODE
\[ \frac{dM_d(q_r)}{dq_r} = -\frac{2}{\rho} A(q_r) \mathcal{J}^\top(q_r). \] (24)

We have to prove now that, for the proposed inertia matrix (20), there exists \( \mathcal{J}(q_r) \) such that (24) holds. Toward this end, we recall that \( A(q_r) \) is not full rank and in particular (18) holds. Consider first the case of \( n > 2 \), then \( n_0 \geq n \). Viewing (24) and (18) as a (matrix) set of algebraic equations and invoking Lemma 2 and Lemma 3 we have that for all \( M_d(q_r) \) verifying
\[ G^\perp(q_r) \frac{dM_d}{dq_r}(q_r) = 0, \] (25)
there exists a matrix \( \mathcal{J}(q_r) \) such that (24) holds. Clearly, the proposed inertia matrices (20) verify, by construction, the condition (25). For \( n = 2 \) we have that \( n_0 = 1 \), hence \( A(q_r) \) is a vector and we can give an explicit expression for \( \mathcal{J}(q_r) \) in terms of the pseudoinverse of \( A(q_r) \).

We will now prove that thanks to (23) the determination of the potential energy function is trivialized. Indeed, in this case the PDE (14) becomes
\[ \rho \nabla q_V d + \sum_{i=1,i\neq r} G^\perp(q_r) M_d(q_r) M^{-1}(q_r) e_i \nabla q_V d = -G^\perp(q_r) s(q_r) \]
whose solution, as can be easily verified, is given by (21), (22).

\[ \triangleright \]

**Remark 4** A comment regarding Assumption A.1 is in order. As seen from the proof of the proposition, we have constructed a particular \( G^\perp(q_r) \) so that the key (linearizing) condition (23) holds. For an \( M_d(q_r) \) given by (20), condition (19) ensures, because of smoothness of all functions, the existence of a neighborhood of \( q_{rs} \), such that \( G^\perp(q_r) \) is well-defined, and consequently (23) holds. If we are interested in actually quantifying the domain we need to check the interval, around \( q_{rs} \), where
\[ |G^\perp(q_r) M_d(q_r) M^{-1}(q_r) e_r| \geq \epsilon > 0 \] (26)
holds. This verification is important because, as will be shown below, \( G^\perp(q_r) \) is needed in the controller expression. See Subsection 7.1 for an example. As shown in Subsection 7.2 this problem may be avoided if, instead of fixing \( M_d(q_r) \) and then computing \( G^\perp(q_r) \) as done in Proposition 1, we restrict from the outset some of the elements of \( M_d(q_r) \) to satisfy (23) and then complete the matrix so that (24) holds. See also Remark 8.

**Remark 5** The choice of the free parameter \( \Psi(q_r) \) is limited by the possibility of obtaining an explicit expression for the integral \( \int_0^{q_r} G^\perp(\mu) s(\mu) d\mu \)—that is needed for the computation of \( V_d(q) \) (21), and consequently for the definition of the control law. Although we know that the integral, which is the area under the graph of \( G^\perp(q_r) s(q_r) \), is well defined it is not always possible to express it with an explicit formula. As discussed in Subsection 7.1 this is the only obstacle that hampers us from obtaining an (almost) globally stabilizer of the upward position of the pendulum in the cart. (Of course, for a computer implementation we do not need to give explicit expressions for the control laws and the controller may be given in integral form. This possibility is currently under investigation.)

## 5 Main stability result

In the previous section we proposed a parametrization of the assignable energy functions in terms of the triple \( \{ \Psi, M_d^0, \Phi \} \). Here we will impose some additional constraints on these parameters to ensure asymptotic stability of the closed-loop. As expected, for stability (besides positivity of \( M_d(q_r) \)) we will require assignment of the
desired minimum to \( V_d(q) \). To articulate this condition we note first that the change of coordinates \( q \sim (z, q_r) \) is a **diffeomorphism** that preserves the extrema—hence we analyze the potential energy function in these new coordinates. Now, from (21), and the fact that \( \Phi(z) \) is arbitrary, it is clear that restrictions will only be imposed on the term \(-\frac{1}{\rho} \int_{0}^{\infty} G^{-1}(\mu)s(\mu)d\mu\). Recalling (7) we note that this function already has an extremum at \( q_{rs} \). To ensure that it is a minimum we impose:

**Assumption A.2** 
\[
-\frac{1}{\rho} \frac{d}{dq_r} [G^{-1}(q_{rs})] > 0,
\]
where \( \rho \) is given by (23).\(^6\)

Interestingly, we will show in the proposition that no additional condition is imposed for **asymptotic stability**. Furthermore, for the particular case of quadratic \( \Phi(z) \), a very simple **explicit** expression for the control law is given.

**Proposition 2** Consider the system (6). Assume there exists matrices \( \Psi(q_r) \) and \( M_d \) such that Assumptions A.1 and A.2 hold. Under these conditions, for all differentiable functions \( \Phi(z) \) the IDA–PBC (5) ensures that the closed-loop dynamics is a Hamiltonian system of the form (9) with total energy function (10), where \( M_d(q_r) \) and \( V_d(q) \) are given in Proposition 1. Moreover, if

\[
z(q_*) = \arg \min \Phi(z), \tag{27}
\]

then \((q_*, 0)\) is a locally asymptotically stable equilibrium with Lyapunov function \( H_d(q, p) \). In particular, if we select

\[
\Phi(z(q)) = \frac{1}{2} [z(q) - z(q_*)]^T P [z(q) - z(q_*)]
\]

with \( P = P^T > 0 \), the control law is of the form

\[
u = A_1(q)PS(q - q_*) + \begin{bmatrix} p^T A_2(q_r)p \\ \vdots \\ p^T A_n(q_r)p \end{bmatrix} + A_{n+1}(q_r) - K_v A_{n+2}(q_r)p \tag{28}
\]

where \( K_v = K_v^T > 0 \) is free, \( S \in \mathbb{R}^{(n-1)\times n} \) is obtained removing the \( r \)-th row from the \( n \)-dimensional identity matrix, and the matrices \( A_i, i = 1, \ldots, n+1 \), are of dimensions

\[
A_1(q) \in \mathbb{R}^{(n-1)\times (n-1)}, \quad A_2(q_r), \ldots, A_n(q_r) \in \mathbb{R}^{n\times n}, \quad A_{n+1}(q_r) \in \mathbb{R}^{(n-1)\times 1}.
\]

**Proof.**\(^{7}\) The first matching claim follows immediately from Proposition 1 and our previous derivations. To prove stability of the equilibrium we note that \( M_d(q_*) > 0 \) ensures \( H_d(q, p) \) is (locally) positive definite in \( p \), therefore to qualify as a Lyapunov function candidate we only need to prove that \( V_d(q) \) satisfies the minimum condition (11). As discussed above, we analyze the potential energy function in the coordinates \((z, q_r)\). Condition (27) assures the required property for the \( z \) coordinates. In view of (7) and Assumption A.2—which pertain, respectively, to the first and second derivative of \( V_d(q) \) with respect to \( q_r \) evaluated at \( q_{rs} \)—the minimum is also at the desired equilibrium for \( q_r \).

The control expression (28) is obtained, after some lengthy but straightforward calculation, from (5) selecting the free function \( \Phi(z(q)) \) in (21) as indicated in the proposition.

It only remains to establish **asymptotic stability**.\(^8\) The derivative of \( H_d(q, p) \) along the dynamics (9) is given as

\[
\dot{H}_d = -\tilde{p}^T G(q_r) K_v G^T(q_r) \tilde{p} \leq 0,
\]

\(^{6}\)It is important to underscore that \( \rho \) is not a free parameter, as it disappears in the computation of \( \frac{G^{-1}(q_r)}{\rho} \).

\(^{7}\)For the sake of brevity, some of the steps of the proof are omitted here, but they may be found in [1].

\(^{8}\)The authors would like to thank the anonymous reviewer for this elegant and simple proof.
where we have used the partial coordinate \( \tilde{p} \) defined in (13). From positivity of \( H_d(q, p) \) and the expression above we conclude boundedness of all solutions starting sufficiently close to the equilibrium. We study then the dynamics restricted to the residual set \( \{ (q, \tilde{p}) | G^\top(q, \tilde{p}) \tilde{p} = 0 \} \). We have the following chain of implications

\[
G^\top(q, \tilde{p}) \tilde{p} = 0 \Rightarrow \exists \nu \in \mathbb{R} : \tilde{p}^\top = \nu G^\perp(q, \tilde{p})
\]

\[
\Rightarrow \dot{q} = M^{-1}(q) M_d(q, \tilde{p}) = \nu M^{-1}(q) M_d(q, (G^\perp(q, \tilde{p}))^\top)
\]

\[
\Rightarrow \dot{q}_r = e_r^\top \dot{q} = \nu e_r^\top M^{-1}(q) M_d(q, (G^\perp(q, \tilde{p}))^\top) = \nu \rho
\]

\[
\Leftrightarrow \nu = \frac{1}{\rho} \dot{q}_r.
\]

where we have used (23) to obtain the third implication. From the last identity and boundedness of solutions, we conclude that \( \nu = 0 \)—otherwise \( q_r \) would grow unbounded. Now, from the second implication we have that \( \dot{q} = 0 \). Therefore, the only possible points in the residual set are equilibria, but the desired equilibrium \((q_*, 0)\) is (locally) isolated, so the residual set is only \((q_*, 0)\).

\[
\begin{align*}
\text{Remark 6} & \quad \text{To quantify the domain of attraction, e.g., obtain an (almost) global version of the asymptotic stability claim, we need to rule out the existence of stable equilibria, different from the desired one. This can be done imposing} \\
\text{Assumption A.3} & \quad \text{Fix } a > 0 \text{ (possibly } a = +\infty) \text{. For all points } \bar{q}_r \in [q_{rs} - a, q_{rs} + a], \bar{q}_r \neq q_{rs} \text{ such that } G^\perp(\bar{q}_r)s(\bar{q}_r) = 0 \text{ we have that} \\
& \quad \frac{1}{\rho} \frac{d}{dq_r} \{G^\perp s\}(\bar{q}_r) < 0.
\end{align*}
\]

This ensures that all other equilibria correspond to maximum points of the desired potential energy function, and are henceforth unstable.

\[
\text{Remark 7} & \quad \text{There is an interesting connection between Assumption A.2 and smooth stabilizability of the system (6). On one hand, the celebrated Brockett’s condition, see e.g. Theorem 4.5.2 of [11], establishes that injectivity of } G^\perp(q_r)s(q_r) \text{ is necessary for stabilization. (For instance, if } s(q_r) \equiv 0 \text{ the system is not smoothly stabilizable.)} \\
& \quad \text{On the other hand, the condition of Assumption A.2 is clearly sufficient for injectivity of the function in question.}
\]

\[
\text{Remark 8} & \quad \text{The set of assignable energy functions of the form (10) that lead to a stabilizing controller is parameterized by all triplets } \{ \Psi, M^0_d, \Phi \} \text{ that satisfy the conditions of Proposition 2. The first two parameters shape the kinetic energy according to (20) while the last one modifies the potential energy as indicated in (21).}
\]

\section{Implementation of the controller via position feedback}

In this section we prove that, using the recently introduced method of Immersion and Invariance [4, 18], we can design a speed estimator that allows the implementation of the proposed controllers measuring only position—stability being ensured imposing the (rather weak) additional assumption that the matrix \( \Psi(q_r) \) (that defines \( M_d(q_r) \)) is bounded.

\[
\text{Proposition 3} & \quad \text{Consider the system (6) together with a constant matrix } M^0_d \text{ such that Assumptions A.1 and A.2 of Proposition 1 hold. Select the matrix } \Psi(q_r) \text{ in (20) to be bounded and assume, without loss of generality, that}
\]

\[
\begin{align*}
\text{...}
\end{align*}
\]
G(q_r) is bounded. Define the position feedback controller

\[ u = A_1(q)PS(q - q_s) + \begin{bmatrix} (\hat{p} + \lambda q)\mathsf{T}A_2(q_r)(\hat{p} + \lambda q) \\ \vdots \\ (\hat{p} + \lambda q)\mathsf{T}A_n(q_r)(\hat{p} + \lambda q) \end{bmatrix} + A_{n+1}(q_r) - K_vG^\mathsf{T}(q_r)M_d^{-1}(q_r)(\hat{p} + \lambda q) \]  

where \( \lambda > 0 \), and \( \hat{p} \) is an estimate of \( p - \lambda q \) generated via

\[ \hat{p} = s(q_r) + G(q_r)u - \lambda M^{-1}(q_r)(\hat{p} + \lambda q). \]

Then, there exists a neighborhood of the point \((q_s, 0, -\lambda q_s)\) such that all trajectories of the closed-loop system starting in this neighborhood are bounded and satisfy

\[ \lim_{t \to \infty} (q(t), p(t), \hat{p}(t)) = (q_s, 0, -\lambda q_s). \]

Furthermore, if Assumption A.3 holds and the full state feedback controller (28) ensures global asymptotic stability then the neighborhood is the whole space \( \mathbb{R}^{3n} \), thus boundedness and convergence are global.

Proof. To carry out the proof we follow verbatim the Immersion and Invariance procedure of [4, 18]. For, we define the partial coordinate

\[ \zeta = \hat{p} - p + \lambda q, \]

whose derivative, upon replacement of the system dynamics (6) and the estimator above, takes the simple form

\[ \dot{\zeta} = -\lambda M^{-1}(q_r)\zeta. \]

From boundedness and positivity of \( M(q_r) \) we immediately conclude that \( |\zeta(t)| \to 0 \) exponentially fast—for instance, evaluating the derivative of \(|\zeta|^2\), where |·| is the Euclidean norm.

We will show now that the proposed position-feedback control law can be expressed as the sum of the full-state feedback control plus a perturbation term that depends on \( \zeta \), that as shown above, exponentially goes to zero. Indeed, using \( \hat{p} + \lambda q = \zeta + p \), the controller (29) can be written as \( u = u_s(q, p) + \chi(q_r, p, \zeta) \), where we use \( u_s(q, p) \) to denote the full state feedback controller (28), and we have defined

\[ \chi(q_r, p, \zeta) \triangleq \begin{bmatrix} \zeta\mathsf{T}A_2(q_r)\zeta + \zeta\mathsf{T}[A_2(q_r) + A_2^\mathsf{T}(q_r)]p \\ \vdots \\ \zeta\mathsf{T}A_n(q_r)\zeta + \zeta\mathsf{T}[A_n(q_r) + A_n^\mathsf{T}(q_r)]p \end{bmatrix} - K_vG^\mathsf{T}(q_r)M_d^{-1}(q_r)\zeta. \]  

Replacing (30) in (6) and denoting \( x = [q^\mathsf{T}, p^\mathsf{T}]^\mathsf{T} \), we have that the closed-loop system can be written in the perturbed form

\[ \dot{x} = f_s(x) + \begin{bmatrix} 0 \\ G(q_r)\chi(q_r, p, \zeta) \end{bmatrix}, \]

where \( \dot{x} = f_s(x) \) are the dynamics of the system in closed-loop with the full state feedback controller. From Proposition 2 we have that the latter is asymptotically stable. Furthermore, the disturbance term is such that \( G(q_r)\chi(q_r, p, 0) = 0 \). Therefore, invoking the recent result of [32], the proof will be completed if we can establish boundedness of the trajectories \( x(t) \).

Towards this end, we proceed as follows. As shown in Proposition 2 the desired total energy qualifies as Lyapunov function for the unperturbed system \( \dot{x} = f_s(x) \). Computing its time derivative for the complete system we get the bounds

\[ \dot{H}_d \leq -\lambda_{\text{min}}\{K_v\} \left[ G^\mathsf{T}(q_r)\hat{p}\right]^2 + \hat{p}^\mathsf{T}G(q_r)\chi(q_r, p, \zeta) \]

\[ \leq -\kappa_1||\hat{p}||^2 + \kappa_2||\hat{p}||\chi(q_r, p, \zeta) \]

This assumption is indeed done without loss of generality, because we can always redefine the control signal with a scalar normalizing factor without affecting the stability properties.
where the second bound has been obtained using the assumption of bounded $G(q_r)$, $\kappa_1, \kappa_2$ are some positive constants, and we recall that $\tilde{p}$ is defined in (13). From the expression above it is clear that the key step to prove boundedness of trajectories is to establish a suitable bound for $\chi(q_r, p, \zeta)$. The third right hand term of (30) is an exponentially decaying disturbance whose effect on the inequality above can be dominated invoking standard (Young’s inequality) arguments. The second right hand term stems from the quadratic term in $p$ of $\nabla_q H_d$ in (5), more precisely from the term $-\frac{1}{2}\tilde{p}^T M'_d(q_r)\tilde{p} e_r$. Replacing (20) we see, after some simple calculations, that it has the form

$$-\frac{1}{2} \left( G^T G \right)^{-1} G^T \left[ \zeta^T M_d^{-1} G \Psi G^T M_d^{-1} \zeta + 2 \zeta^T M_d^{-1} G \Psi G^T \tilde{p} \right] e_r,$$

where we have omitted the arguments for brevity. If $\Psi(q_r)$ is bounded—hence the need for the additional assumption—this term is (linearly) bounded by $(|G^T(q_r)\tilde{p}| + 1)e_t$ where $e_t$ is an exponentially decaying term.

From the bound $H_d \geq \frac{1}{2}\tilde{p}^T M_d(q_r)\tilde{p}$ and the remarks made above we can prove the existence of an integrable function $k(t)$ such that $\dot{H}_d \leq k(t) H_d$, from which, invoking the Comparison Lemma [34] we immediately conclude boundedness of trajectories and complete the proof.

7 Examples

In this section we apply the preceding design methodology to the problem of stabilizing the positions of the pendulum on a cart, and an arbitrary position with zero roll angle and zero speed of the vertical takeoff and landing aircraft. For other applications we refer the reader to [1].

7.1 Pendulum on a cart

Figure 1: Planar pendulum on a cart.

The dynamic equations of the pendulum on a cart depicted in Fig. 1 are given by

$$ml^2 \ddot{q}_1 + ml \cos q_1 \dot{q}_2 - mgl \sin q_1 = 0$$  \hspace{1cm} (31)

$$(M + m)\ddot{q}_2 + ml \cos q_1 \dot{q}_1 - m \sin q_1 \dot{q}_1^2 = v.$$  \hspace{1cm} (32)
where \( q_1, q_2 \) denote the cart position and the pendulum angle with the upright vertical, \( m \) and \( l \) are, respectively, the mass and the length of the pendulum, \( M \) is the mass of the cart and \( g \) is the gravity acceleration, as usual. As pointed out in [33], the relationship between the input \( v \) and the acceleration \( \dot{q}_2 \) can be inverted and we can rewrite the system (31)–(32) as

\[
\begin{align*}
\ddot{q}_1 &= a \sin q_1 - b \cos q_1 u \\
\ddot{q}_2 &= u,
\end{align*}
\]

where \( a = \frac{g}{l}, b = \frac{1}{l} \) and \( u \) the new control input. The system (33)–(34) can be put in the desired form

\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= a \sin q_1 e_1 + \left[ -b \cos q_1 \right] u
\end{align*}
\]

Notice that \( G^\perp(q_1) = \eta(q_1)[1, \ b \cos q_1] \), where \( \eta(q_1) \) is a free function that will be defined below.

The equilibrium to be stabilized is the upward position of the pendulum with the cart placed in any desired location, which corresponds to \( q_{1s} = 0 \) and an arbitrary \( q_{2s} \).

We will show now that independently of the way we construct \( M_d(q_1) \), to satisfy the conditions of Proposition 1, namely, (18), (23) and (24), \( \eta(q_1) \) cannot be a constant. Let us denote

\[
M_d(q_1) = \begin{bmatrix}
m_{11}^d(q_1) & m_{12}^d(q_1) \\
m_{12}^d(q_1) & m_{22}^d(q_1)
\end{bmatrix}.
\]

On one hand, (23) imposes

\[
G^\perp(q_1)M_d(q_1)e_1 = \eta_1[m_{11}^d(q_1) + m_{12}^d(q_1)b \cos q_1] = \rho.
\]

Now, for all \( M_d(q_1) \), (25) which in our case reduces to

\[
\frac{dm_{11}^d}{dq_1} + \frac{dm_{12}^d}{dq_1} b \cos q_1 = 0,
\]

is a necessary condition for (24) to hold. Differentiating (36) and replacing the identity above we conclude that \( m_{12}^d(q_1) = 0 \). Using again (36) we consequently have that

\[
m_{11}^d(q_1) = \frac{\rho}{\eta},
\]

hence \( m_{11}^d \) is a constant, that furthermore should be positive to ensure positivity of \( M_d(q_1) \). On the other hand, \( G^\perp(q_1)s(q_1) = a \eta \sin q_1 \), and Assumption A.1 (at \( q_{1s} = 0 \)) imposes \( \frac{\eta \eta_s}{\rho} < 0 \), where we recall that \( a > 0 \). This condition is clearly in contradiction with positivity of (37), hence \( \eta \) cannot be taken as a constant.

We come back to the construction of \( M_d(q_1) \) suggested in Proposition 1. For simplicity we take \( \Psi(q_1) \) to be a scalar function and compute from (20)

\[
M_d(q_1) = \int_{q_{1s}}^{q_1} \Psi(\mu) \begin{bmatrix}
b \mu^2 \cos^2 \mu & -b \cos \mu \\
b \mu \cos \mu & 1
\end{bmatrix} d\mu + M_d^0,
\]

and

\[
G^\perp(q_1)M_d(q_1)e_1 = \eta(q_1) \left[ b \mu^2 \left( \int_{q_{1s}}^{q_1} \Psi(\mu) \cos^2 \mu d\mu - \cos q_1 \int_{q_{1s}}^{q_1} \Psi(\mu) \cos \mu d\mu \right) + m_{11}^0 + m_{12}^0 b \cos q_1 \right].
\]

We have to select a function \( \Psi(q_1) \) so that the term in brackets (evaluated at zero) is bounded away from zero (Assumption A.1 and (26)) and can be explicitly integrated. The first condition allows to define \( \eta(q_1) \) such that
G^\perp(q_1) M_d(q_1) e_1 = \rho, while the second one is needed to compute the control law—see Remark 5. Positivity of \( M_d(q_1) \) imposes yet another requirement on \( \Psi(q_1) \). It is easy to see that the simplest choice of \( \Psi(q_1) = \text{const} \) is, unfortunately, not adequate to satisfy the latter. Indeed, in this case

\[
m_{11}(q_1) = \Psi b^2 \left( \frac{1}{2} \cos q_1 \sin q_1 + q_1 \right) - \Psi b^2 \left( \frac{1}{2} \cos q_1^* \sin q_1^* + q_1^* \right) + m_{11}^0,
\]

that is clearly not a bounded function of \( q_1 \).

We propose then \( \Psi(q_1) = -k \sin q_1 \), with \( k > 0 \) a parameter to be determined, and select \( m_{11}^0 = \frac{kb^2}{3} \), \( m_{12}^0 = -\frac{kb}{2} \). This leads to

\[
M_d(q_1) = \begin{bmatrix}
\frac{kb^2}{2} \cos^3 q_1 & -\frac{kb}{2} \cos^2 q_1 & \frac{6}{kb^2} \sin q_1 \\
-\frac{kb}{2} \cos^2 q_1 & k(\cos q_1 - 1) + m_{22}^0 \\
\frac{6}{kb^2} \sin q_1 & \frac{6m_{22}^0}{kb} \tan q_1
\end{bmatrix}
\]

with \( m_{22}^0 \geq 0 \) arbitrary. It is easy to prove that this matrix is positive definite and bounded for all \( q_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \)—a domain where we will restrict our system to operate—provided \( m_{22}^0 > k \). With this selection of \( \Psi(q_1) \) (23) is satisfied selecting the scaling factor of \( G^\perp(q_1) \) as

\[
\eta(q_1) = -\frac{6\rho}{kb^2 \cos^3 q_1}.
\]

We will now verify Assumption A.2. For, we evaluate

\[
\frac{1}{\rho} G^\perp(q_1) s(q_1) = \frac{6a}{kb^2 \cos^2 q_1} \sin q_1
\]

whose derivative satisfies the required inequality at zero.

We have the following result.

**Proposition 4** A set of energy functions of the form (10) assignable via IDA–PBC to the pendulum on a cart system (35) is characterized by the locally positive definite and bounded inertia matrix (38), for all arbitrary positive numbers \( m_{22}^0 > k \), and the potential energy function

\[
V_d(q) = \frac{3a}{kb^2 \cos^2 q_1} + \frac{P}{2} \left[ q_2 - q_2^* + 3b \ln(\sec q_1 + \tan q_1) + \frac{6m_{22}^0}{kb} \tan q_1 \right]^2.
\]

that satisfies (11) for all constants \( P > 0 \).

Moreover, the IDA–PBC

\[
u = A_1(q_1) P(q_2 - q_2^*) + p^\top A_2(q_1) p - K_v A_3(q_1) p + A_4(q_1)
\]

Make notation consistent with Proposition

where the matrices \( A_i(q_1) \) are given in the Appendix, ensures asymptotic stability of the desired equilibrium \( (0, q_2^*, 0, 0) \) with a domain of attraction equal to the set \( (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^3 \).
Proof. The expressions for the control and the potential energy function are obtained, as suggested in Proposition 2, with a quadratic $\Phi(z)$.

Taking into account Proposition 2 it only remains to prove the claim regarding the estimate of the domain of attraction. For, we note that $H_d(q,p)$ is a radially unbounded function on the set $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^3$, hence any trajectory that starts inside this set will remain in it—eventually converging to the desired equilibrium. This completes the proof.

Simulations were made with the normalized values $a = b = 1$, the constant for the damping injection was fixed to $K_v = 0.01$ and the other parameters given by $m_0 = k = 0.01$ and $P = 1$. We tested a set of “limiting” initial conditions with the pendulum starting near the horizontal $[q(0), p(0)] = [\pi/2 - 0.2, -0.1, 0.1, 0]$ and the desired position for the cart $q_2 = 20$, that is, very far away from the origin. The result for full state feedback is shown in Fig. 2 where an excellent performance is observed. We should underscore that, in contrast with the proposed scheme, most of the existing controllers for this problem can stabilize the upward position of the pendulum with zero cart velocity, but the cart position cannot be arbitrarily fixed. Also, we would like to bring to the readers attention the shape of the control action, which is a smooth low amplitude signal that moves the cart at the right time instants in the right direction. Again, this should be compared with other controllers, e.g., those

Figure 2: Trajectories with the pendulum starting near the horizontal $(q(0), p(0)) = (\pi/2 - 0.2, -0.1, 0.1, 0)$, (full state feedback.)
stabilizing the homoclinic orbit, where the control action is essentially bang–bang—even in the upper half plane of the pendulum.

We also have made simulations of the proposed position feedback controller. The result is shown in Fig. 3. As expected, a slower performance is observed, due to the time needed by the nonlinear speed estimator to converge. The gains of the nonlinear speed estimator were \( \lambda = [0.02, 0.01] \).

Remark 9 Independently of the choice of \( \Psi(q_1) \), Assumption A.2 is satisfied if and only if \( m_{11}^0 + bm_{12}^0 < 0 \). Indeed, with some simple calculations it is possible to show that

\[
\frac{d}{dt} \left( \frac{\eta}{\rho} \tilde{G}^{-1} s \right)(0) = \frac{ab}{m_{11}^0 + bm_{12}^0},
\]

proving the claim.

7.2 Vertical takeoff and landing aircraft

Our final example is the vertical takeoff and landing (VTOL) aircraft depicted in Fig. 4 whose dynamics are, following [16, 29], given by

\[
\ddot{x} = -\sin \theta v_1 + \epsilon \cos \theta v_2
\]
where $\theta$ is the roll angle, the VTOL moves in the $(x, y)$ plane, $g$ is the gravity acceleration, $v_1, v_2$ are the control actions and $\epsilon$ is a parameter that captures the effect of the “slopped" wings—and clearly induces a coupling between the vertical and the roll dynamics. To apply the theory developed in this paper we introduce the (globally defined) change of input

$$
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix} = \frac{g}{\epsilon} \begin{bmatrix}
  -\cos \theta & \sin \theta \\
  -\sin \theta & \cos \theta
\end{bmatrix} u
$$

where $u = [u_1, u_2]^\top$ is the new control vector. This transformation yields the VTOL dynamics

$$
\begin{align*}
\dot{q} &= p \\
\dot{p} &= \frac{g}{\epsilon} \sin q_3 e_3 + G(q_3) u
\end{align*}
$$

where we have introduced the notation $q = [x, y, \theta]^\top$, and defined the matrix

$$
G(q_3) = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  \frac{1}{\epsilon} \cos q_3 & \frac{1}{\epsilon} \sin q_3
\end{bmatrix}.
$$

Control requirements for the VTOL are typically expressed in terms of asymptotic regulation from any initial condition to an arbitrary position with zero roll angle and zero speed, that is, the asymptotic stabilization of all equilibria of the form $\text{col}(q_1^*, q_2^*, 0, 0, 0, 0)$.

In Proposition 1 we fixed $M_d(q_3)$—according to (20)—and then computed $\eta(q_3)$ for the construction of $G^\perp(q_3)$. Our interest with this example is to provide an alternative parametrization of the set of assignable energy functions—as suggested in Remarks 4 and 8. Towards this end, we define $M_d(q_3)$ directly from the key equations (23) and (24), which for ease of reference are repeated here for this particular example

$$
G^\perp(q_3) M_d(q_3) e_3 = \rho
$$

$$
\frac{dM_d}{dq_3} = -\frac{2}{\rho} A(q_3) J^\top(q_3).
$$
We recall that the $3 \times 3$ matrix $J^T(q_3)$ is free, hence, $(43)$ can be alternatively expressed as \( \frac{dM_d(q_3)}{dq_3} \in \text{range}\{\mathcal{A}(q_3)\} \), \( i = 1, 2, 3 \). Besides the latter, $(42)$ further restricts the third column of $M_d(q_3)$ to have a constant projection along $G^\perp(q_3)$. We will characterize below all 3–dimensional vectors satisfying these constraints. Then, define the two remaining columns of $M_d(q_3)$ to obtain a positive definite matrix, with derivatives also living in the range space of $\mathcal{A}(q_3)$.

Setting the scaling factor of $G^\perp(q_3)$ equal to 1, we compute then

\[
G^\perp(q_3) = [\cos q_3, \sin q_3, -\epsilon], \quad \mathcal{A}(q_3) = \begin{bmatrix}
-\sin q_3 & \epsilon & 0 \\
\cos q_3 & 0 & \epsilon \\
0 & \cos q_3 & \sin q_3
\end{bmatrix}
\]

The following lemma characterizes all vector functions, $f(q_3) \triangleq M_d(q_3)e_3$, verifying $(42)$ and $(43)$.$^{10}$

**Lemma 4** Fix a constant $\rho \neq 0$ and define the set

\[
\{ f \triangleq \text{col}(f_1, f_2, f_3): \mathbb{R} \to \mathbb{R}^3 \mid G^\perp(q_3)f(q_3) = \rho \text{ and } \frac{df}{dq_3} \in \text{range}\{\mathcal{A}(q_3)\} \}
\]

*All elements of the set are generated as*

\[
f_1(q_3) = [\rho + \epsilon f_3(q_3)] \cos q_3, \quad f_2(q_3) = [\rho + \epsilon f_3(q_3)] \sin q_3
\]

where $f_3(q_3)$ is an arbitrary differentiable function.

**Proof.** Express $f_1(q_3), f_2(q_3)$ in polar coordinates as

\[
f_1(q_3) = r(q_3) \cos q_3 \cos[\phi(q_3)], \quad f_2(q_3) = r(q_3) \sin[\phi(q_3)].
\]

Using the trigonometric identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$, and the definition of $G^\perp(q_3)$ we see that $G^\perp(q_3)f(q_3) = \rho$ if and only if

\[
\begin{aligned}
r(q_3) \cos[q_3 - \phi(q_3)] &= \rho + \epsilon f_3(q_3).
\end{aligned}
\]

This equation has a solution for some $r(q_3)$ and $\phi(q_3)$ if and only if

\[
r(q_3) \geq \rho + \epsilon f_3(q_3).
\]  

\[
(44)
\]

Extracting $\phi(q_3)$ from the expression above, replacing it in $f_1(q_3), f_2(q_3)$ and using the identity $\sin[\arccos A] = \sqrt{1 - A^2}$ we conclude that all functions which satisfy $G^\perp(q_3)f(q_3) = \rho$ are generated as

\[
\begin{aligned}
f_1(q_3) &= [\rho + \epsilon f_3(q_3)] \cos q_3 + \sqrt{r^2(q_3) - [\rho + \epsilon f_3(q_3)]^2} \sin q_3 \\
f_2(q_3) &= [\rho + \epsilon f_3(q_3)] \sin q_3 - \sqrt{r^2(q_3) - [\rho + \epsilon f_3(q_3)]^2} \cos q_3
\end{aligned}
\]

where $f_3(q_3)$ is arbitrary and $r(q_3)$ is any function verifying $(44)$.

Now, since $\text{rank}\{\mathcal{A}(q_3)\} = 2$ and $G^\perp(q_3) \in \ker\{\mathcal{A}(q_3)\}$ then $\frac{df}{dq_3} \in \text{range}\{\mathcal{A}(q_3)\}$ if and only if $G^\perp(q_3) \frac{df}{dq_3} = 0$. Some simple calculations show that this is true if and only if $(44)$ holds with the equality sign.

footnote{The second author thanks William Pasillas for help with the proof of this lemma.}
In the sequel we pick one element of the class characterized in the lemma above and—for the sake of simplicity—choose the function that parameterizes the set to be a constant, that is \( f_3 = k_2 \), this yields

\[
M_d(q_3)e_3 = \begin{bmatrix}
  k_1 \cos q_3 \\
  k_1 \sin q_3 \\
  k_2
\end{bmatrix}
\] (45)

where, for ease of notation, we have defined

\( k_1 \triangleq \rho + ek_2 \). (46)

We still have to decide the two remaining columns of the inertia matrix, that is (using the notation \( M_d(q_3) = \{ m_{ij}(q_3) \} \)) we look for functions \( m_{ij}(q_3) \), \( i, j = 1, 2 \) such that

\[
\begin{bmatrix}
  \frac{dm_{11}}{dq_3} \\
  \frac{dm_{12}}{dq_3} \\
  -k_1 \sin q_3
\end{bmatrix}, \begin{bmatrix}
  \frac{dm_{12}}{dq_3} \\
  \frac{dm_{22}}{dq_3} \\
  k_1 \cos q_3
\end{bmatrix} \in \text{range} \{ A(q_3) \}. \] (47)

Our first observation is that, since \( e_3 \) is not in the range space of \( A(q_3) \), we cannot take the \( m_{ij}(q_3) \) to be constant.

A more detailed analysis reveals that there is actually only one of the \( m_{ij}(q_3) \), \( i, j = 1, 2 \), that can be chosen freely, reducing our task to the selection of this free function in such a way that the resulting \( M_d(q_3) \) is positive definite. Indeed, multiplying the first two columns of \( M_d(q_3) \) by \( G_\perp(q_3) \), and recalling that \( G_\perp(q_3)A(q_3) = 0 \), we obtain the relations

\[
\begin{align*}
\frac{dm_{11}}{dq_3} &= -\sin q_3 \cos q_3 [k_1 \epsilon + \frac{dm_{12}}{dq_3}], \\
\frac{dm_{22}}{dq_3} &= \cos q_3 \sin q_3 [k_1 \epsilon - \frac{dm_{12}}{dq_3}].
\end{align*}
\] (48)

To avoid singularities, we impose the constraints

\[
\frac{dm_{12}}{dq_3}(0) = k_1 \epsilon, \quad \frac{dm_{12}}{dq_3}(\pi/2) = -k_1 \epsilon,
\]

that immediately suggests the choice

\[
\frac{dm_{12}}{dq_3} = k_1 \epsilon \cos(2q_3). \] (49)

This choice, besides enforcing (47), ensures that \( \frac{dm_{11}}{dq_3} \) and \( \frac{dm_{22}}{dq_3} \) are bounded functions and, consequently, that their integral can be rendered positive adding a suitable constant of integration.

We are in position to present the following proposition.

**Proposition 5** A set of energy functions of the form (10) assignable via IDA–PBC to the VTOL system (41) is characterized by the globally positive definite and bounded inertia matrix

\[
M_d(q_3) = \begin{bmatrix}
  k_1 \epsilon \cos^2 q_3 + k_3 & k_1 \epsilon \cos q_3 \sin q_3 & k_1 \cos q_3 \\
  k_1 \epsilon \cos q_3 \sin q_3 & -k_1 \epsilon \cos^2 q_3 + k_3 & k_1 \sin q_3 \\
  k_1 \cos q_3 & k_1 \sin q_3 & k_2
\end{bmatrix}
\] (50)

with \( k_1 \) an arbitrary positive number and \( k_2, k_3 \) verifying

\[
k_3 > 5k_1 \epsilon, \quad \frac{k_1}{\epsilon} > k_2 > \frac{k_1}{2\epsilon}.
\]
and the potential energy function

\[
V_d(q) = -\frac{g}{k_1 - k_2} \cos q_3 + \frac{1}{2} \begin{bmatrix}
q_1 - q_{1s} - \frac{k_1}{k_1 - k_2} \sin q_3 \\
q_2 - q_{2s} + \frac{k_1}{k_1 - k_2} (\cos q_3 - 1)
\end{bmatrix}^T \begin{bmatrix}
q_1 - q_{1s} - \frac{k_1}{k_1 - k_2} \sin q_3 \\
q_2 - q_{2s} + \frac{k_1}{k_1 - k_2} (\cos q_3 - 1)
\end{bmatrix}
\]

that satisfies (11) for all \( P = P^T > 0 \).

Moreover, the IDA–PBC law

\[
u = A_1(q_3) P \begin{bmatrix}
q_1 - q_{1s} \\
q_2 - q_{2s}
\end{bmatrix} + \begin{bmatrix}
p^T A_2(q_3) p \\
p^T A_3(q_3) p
\end{bmatrix} - K_e A_4(q_3) p + A_5(q_3)
\]

where the matrices \( A_i(q_3) \) are given in the Appendix, ensures almost global asymptotic stability of the desired equilibrium \((q_{1s}, q_{2s}, 0, 0, 0)\).\(^{11}\)

**Proof.** First, we need to verify that the proposed \( M_d(q_3) \) satisfies (42) and (43). The first condition is clearly fulfilled given the derivations above. The verification that (50) also satisfies (43) follows replacing (49) in (48), integrating and using the identities

\[
1 + \cos(2A) = 2 \cos^2 A, \quad 1 - \cos(2A) = 2 \sin^2 A.
\]

It remains to prove positive definiteness of \( M_d(q_3) \) and Assumption A.2. The latter is checked computing \( G^\perp(q_3) s(q_3) = -g \sin(q_3) \) which, upon differentiation yields the constraint \( \rho > 0 \), that in view of (46) translates into \( k_1 > \epsilon k_2 \).

Note that to comply with the requirement of positivity of the inertia matrix, we impose \( k_2 > 0 \). We will prove positivity decomposing it into the sum of two matrices. For, we write \( k_3 = a + b \) and start with the matrix

\[
\begin{bmatrix}
b & 0 & k_1 \cos q_3 \\
0 & b & k_1 \sin q_3 \\
k_1 \cos q_3 & k_1 \sin q_3 & k_2
\end{bmatrix}.
\]

It is easy to show that \( b > 4k_1 \epsilon, k_2 > \frac{k_1}{2} \) ensures positive definiteness, and is consistent with the additional requirement \( k_1 > \epsilon k_2 \), imposed by the potential energy shaping. On the other hand, the matrix

\[
\begin{bmatrix}
k_1 \epsilon \cos^2 q_3 + a & k_1 \epsilon \cos q_3 \sin q_3 & 0 \\
k_1 \epsilon \cos q_3 \sin q_3 & -k_1 \epsilon \cos^2 q_3 + a & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

is positive semi-definite if and only if \( a > k_1 \epsilon \). Adding the lower bounds on \( a \) and \( b \) completes the proof of positivity of the inertia matrix.

The expression for the potential energy and the control law are immediately obtained replacing (50) in (21), (22) and doing some simple calculations.

Finally, to prove the almost global claim we note that the system lives in the set \( \mathbb{R}^2 \times [-\pi, \pi] \times \mathbb{R}^3 \), and that the energy function \( H_d(q, p) \) is *positive definite and proper* throughout this set. Then, since \( H_d \leq 0 \), we have that *all solutions* are bounded. From the analysis above we know that the desired equilibrium is asymptotically stable. We will now show that the other equilibria are unstable. Indeed, the linearization of the closed-loop system at these equilibria has eigenvalues with strictly positive real part and at least one eigenvalue with strictly negative real part. Associated to the latter there is a stable manifold, and trajectories starting in this manifold will converge to these equilibria. However, it is well–known that an \( s \)-dimensional invariant manifold of an \( n \)-dimensional system has Lebesgue measure zero if \( s < n \). Consequently, the set of initial conditions that

\(^{11}\)That is, the domain of attraction is the whole state space minus a set of Lebesgue measure zero.
Simulations were carried out with a twofold objective, first to show how the energy shaping controller proposed in this paper ensures a satisfactory response for strong coupling coefficients $\epsilon > 0$, and second to illustrate the tuning flexibility provided by the design parameters. All simulations are made with a strong value of coupling $\epsilon = 1$. The damping injection matrix was fixed to $K_v = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}$.

The normal conditions of maneuvering for the VTOL aircraft is to keep an accurate lateral motion near the ground. This problem has been normally solved in two steps (see for instance [29]): decoupling the altitude output from the lateral motion and rolling moment by means of a pre-feedback control law and then, designing a control law for the new decoupled system; this procedure renders satisfactory results for small enough $\epsilon$. With the energy shaping controller, independently of the value of $\epsilon$, it is possible to “virtually decouple” the outputs using the weighting matrix $P$ in the potential energy (21). To illustrate this point two simulations were made, first with a “bad” potential energy taking $P$ diagonal and the weights equal to $1$ and $0.1$. This simulation for a lateral motion is shown in Fig. 5. The same simulation was made for a “good” potential energy taking the $P$ again diagonal but with the weights now $1/2$ and $1$, with the response shown in Fig. 6—notice the different scales in the graphs. The posture of the VTOL aircraft along the trajectory for both cases is shown (at the same scale) in Fig. 7. It can be seen that, for the first case, the altitude ($q_2 = y$) makes very large excursions to drive the VTOL
to rest, while in the second one a simple slow amplitude rocking motion achieves the objective.

The third simulation, depicted in Fig. 8, shows the behavior of the controlled system in an aggressive maneuver, from a limit position, upside down, for the roll angle \( q_3 \) and a great step on the lateral motion \( q_1 \) and altitude \( q_2 \). The high performance of the controlled system is clearly seen from the figure. The posture of the VTOL aircraft along the trajectory is shown in Fig.9.

We also have made the third simulation taking into account only position measurements. The result is shown in Fig. 10. A slower performance is observed again, as in the pendulum example. A matrix gain, \( \lambda = \text{diag}\{0.005, 0.005, 0.01\} \), was used for the nonlinear speed estimator.

## 8 Conclusions

In this paper we have identified a class of underactuated mechanical systems for which the IDA–PBC design methodology gives a complete solution to the stabilization problem—without the need to solve any PDE. The main assumptions made on the system are that it has underactuation degree one and that, roughly speaking, the dynamics is determined by only one generalized coordinate. See equation (6). Also, a position feedback implementation—with provable stability properties—is presented. This class contains several practically interesting benchmark examples, some of which are studied in the paper. In particular, we present an almost globally
Figure 7: Posture of the VTOL along the trajectory; $q = (x, y, \theta)$. To the left the “badly” tuned controller and to the right the well tuned one.

Figure 8: Upside down simulation; $q = (x, y, \theta)$. Initial conditions $(q(0), p(0)) = (5, -5, \pi, 0.1, -0.1, 0.1)$ and references $q_* = (-5, 5, 0)$. 
Figure 9: Upside down simulation. Posture of the VTOL along the trajectory for the simulation of Fig. 8.

Figure 10: Upside down simulation; $q = (x, y, \theta)$. Initials conditions $(q(0), p(0)) = (5, -5, \pi, 0.1, -0.1, 0.1)$ and references $q_\ast = (-5, 5, 0)$, (position feedback.)
stabilizing controller for the VTOL aircraft that ensures asymptotic regulation from any initial condition to an arbitrary position with zero roll angle and zero speed; and a controller for the pendulum on the cart that can swing–up the pendulum from any position in the (open) upper half plane and stop the cart at any desired location.

We present in [2] a characterization of all mechanical systems that are feedback–equivalent to this subclass, which is given in terms of solvability of a set of PDE’s with algebraic constraints.

Besides ensuring asymptotic stability the IDA–PBC methodology provides the designer with some degrees of freedom to improve the transient performance and the robustness. These degrees of freedom are given in terms of parameterized expressions for the assignable energy functions. More precisely, the total energy function can be effectively shaped via the selection of the scaling matrix $\Psi(q_r)$ and the constant matrix $M_d^0$ in the inertia matrix (20) and the choice of the function $\Phi(z)$ in the potential energy (21). An additional tuning parameter is the damping injection gain $K_v$ that may be any positive definite (possibly state–dependent) matrix.

For simplicity we have chosen in our simulations a quadratic function $\Phi(z)$ for the potential energy, but motivated by other considerations, e.g., input constraints or rate saturations, we could have also taken other (logarithmic or saturated) functions. An advantage of a quadratic function is that the control law takes a very nice expression (28), which consists of the sum of three types of terms that are modulated by functions of the distinguished coordinate $q_r$:

- (“proportional–like”) linear terms on the additional coordinate error $S(q - q_\star)$ that contribute to the potential energy shaping;\(^{12}\)
- (“derivative–like”) linear terms in $p$ due to the damping injection that enforce asymptotic stability;
- (“gyroscopic–like”) quadratic terms in $p$ that come from the interconnection matrix $J_2(q, p)$. These terms, which serve to propagate the damping through the well–known mechanism of feedback interconnection of passive and strictly passive systems [22], are essential for the solution of the present problem.

Current research is under way to extend the present work in the following directions:

1. The case when the underactuation degree is larger than one. The difficulty here is that $G^\perp(q_r)$ is now a matrix and the key step of “linearizing” the kinetic energy PDE is unavailable. Another case of interest is when the inertia matrix and the potential energy of the original system depend on more than one coordinate—in this case the proposed form of $M_d(q_r)$ still solves the kinetic energy PDE, but the potential energy PDE does not admit an immediate solution as in the paper.

2. Comparison of the class studied here with the one identified, via elegant geometric conditions, in [10]. See also [9]. Also, it would be interesting to explore the connections with the recent work of [15], where the authors consider underactuation degree one mechanical systems with a cyclic coordinate.

3. As pointed out in Remark 5 it is possible to implement numerically the controller without the need to compute analytically the integral that defines $\Psi(q_r)$. This opens up new possibilities to the method.

4. The proposed controllers should be tested experimentally and confronted with other existing schemes. The outcome of this research will be reported elsewhere.

5. A final interesting open question is to let the matrix $J_2(q, p)$ range, not in the set of skew–symmetric matrices, but in the larger set of matrices whose symmetric part is negative semi-definite, that is, $J_2(q, p) + J_2^\top(q, p) \preceq 0$. This is clearly enough to prove stability—and it amounts to assigning damping injection matrices other than $R_d(q_r) = \begin{bmatrix} 0 & 0 \\ 0 & G(q_r)K_vG^\top(q_r) \end{bmatrix}$. An immediate consequence of this relaxation is that the $A(q_r)$ matrix (16) takes a different form. Some preliminary results on the use of this additional degree of freedom have been reported in [23].

\(^{12}\)We have shown with examples the importance of a suitable selection of the relative weights of the configuration coordinates (the matrix $P$).
References

[1] J. Acosta, Ph D.


A Appendix

In this appendix the matrices $A_i(q_r)$, for the controllers (39) and (52) are given explicitly.
### A.1 The pendulum on a cart

Make notation consistent with Proposition

\[ A_1(q_1) = -(m_{12}^d f'(q_1) + m_{22}^d) \]
\[ A_2(q_1) = -\frac{1}{2} m_{22}^d M_d^{-1} \begin{bmatrix} \alpha_{11} & 0 \\ \alpha_{12} & 0 \end{bmatrix} M_d^{-1} \]

where the vector \( \alpha_1(q_1) \) from the Proposition 1 has been defined as

\[ \alpha_1 \triangleq \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix} = -\frac{1}{2\eta(q_1)} \begin{bmatrix} kb \cos q_1 \sin q_1 \\ -k \cos q_1 \end{bmatrix} \]

\[ A_3(q_1) = G^T M_d^{-1} = \begin{bmatrix} -b \cos q_1 & 1 \end{bmatrix} M_d^{-1} \]

\[ A_4(q_1) = - \begin{bmatrix} m_{12}^d \left( \frac{6a \sin q_1}{kb^2 \cos^3 q_1} + Pf(q_1)f'(q_1) \right) + Pm_{22}^d f(q_1) \end{bmatrix} M_d^{-1} \]

where, for convenience, we defined

\[ f(q_1) \triangleq \frac{3}{b} \ln(\sec q_1 + \tan q_1) + \frac{6m_{22}^d}{kb} \tan q_1. \]

### A.2 The strongly coupled VTOL aircraft

\[ A_1(q_3) = - \begin{bmatrix} P_{11}(m_{11} + m_{13}) + P_{12}m_{12} \\ P_{11}(m_{12} + m_{23}) + P_{12}m_{22} \end{bmatrix} \]
\[ A_2(q_3) = M_d^{-1} \begin{bmatrix} m_{13}m_{11}'/2 + \alpha_{11} \\ m_{13}m_{12}'/2 + \alpha_{12} \end{bmatrix} + \begin{bmatrix} m_{13}m_{13}'/2 + \alpha_{21} \\ m_{13}m_{23}'/2 + \alpha_{22} \end{bmatrix} + \begin{bmatrix} \alpha_{31} \\ \alpha_{32} \end{bmatrix} \]
\[ A_3(q_3) = M_d^{-1} \begin{bmatrix} m_{23}m_{11}'/2 - \alpha_{11} \\ m_{23}m_{12}'/2 - \alpha_{12} \end{bmatrix} + \begin{bmatrix} m_{23}m_{13}'/2 + \alpha_{31} \\ m_{23}m_{23}'/2 + \alpha_{32} \end{bmatrix} + \begin{bmatrix} \alpha_{31} \\ \alpha_{32} \end{bmatrix} \]

where the vectors \( \alpha_i(q_3), i = 1, 2, 3 \) from the Proposition 1 have been defined as

\[ \alpha_1 \triangleq \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \end{bmatrix} = -\frac{k_1 \rho}{2} \begin{bmatrix} -2c \cos q_3 \\ 2c \sin q_3 \\ 1 \end{bmatrix} \]
\[ \alpha_2 \triangleq \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \end{bmatrix} = -\frac{k_1 \rho}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \]
\[ \alpha_3 \triangleq \begin{bmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{bmatrix} = -\frac{k_1 \rho}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \]

\[ A_4(q_3) = G^T M_d^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cos q_3/\epsilon \]
\[ A_5(q_3) = - \begin{bmatrix} m_{11}(P_{11}F_1 + P_{12}F_2) + m_{12}(P_{12}F_1 + P_{22}F_2) + m_{13}(P_{11}F_1 + P_{22}F_2) + m_{22}(P_{12}F_1 + P_{22}F_2) + m_{23}(P_{11}F_1 + P_{22}F_2) + m_{23}(P_{12}F_1 + P_{22}F_2) \end{bmatrix} \]

where

\[ F_1(q_3) \triangleq \frac{k_3}{\rho} \sin q_3, \]
\[ F_2(q_3) \triangleq \frac{k_3 - k_1 \epsilon}{\rho} \cos q_3. \]