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Tensor Products of Complexes

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TENSOR PRODUCTS OF COMPLEXES

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1. Preliminaries. In this note we introduce a new tensor product functor in the category of complexes. We show that this tensor product is left adjoint to the *Hom* functor properly modified. With this tensor product we study flat complexes, pure exact sequences of complexes, pure injective complexes and give a complete description of flat pure injective complexes over a commutative noetherian ring. Also we define Gorenstein flat complexes using this new tensor product and we prove that over a commutative Gorenstein ring any complex has a Gorenstein flat cover.

In this article $\mathcal{C}$ will be the abelian category of complexes of left R-modules. This category has enough projectives and injectives. This can be seen from the fact that any complex of the form

$$\cdots \to 0 \to M \stackrel{id}{\to} M \to 0 \to \cdots$$

with M projective (injective) is projective (injective). For objects C and D of $\mathcal{C}$, $Hom(C, D)$ is the abelian group of morphisms from C to D in $\mathcal{C}$ and $Ext^i(C, D)$ for $i \geq 0$ will denote the groups we get from the right derived functor of Hom (the definition of $Hom(C, D)$ will be modified in the next section).

A complex

$$\cdots \to C^{-1} \stackrel{\delta^{-1}}{\to} C^0 \stackrel{\delta^0}{\to} C^1 \stackrel{\delta^1}{\to} \cdots$$

will be denoted C. We will use subscripts to distinguish complexes. So if $\{C_i\}_{i\in I}$ is a family of complexes, C_i will be

$$\cdots \to C_i^{-1} \stackrel{\delta^{-1}}{\to} C_i^0 \stackrel{\delta^0}{\to} C_i^1 \stackrel{\delta^1}{\to} \cdots .$$

Given M a left R-module, we will denote by $\overline{M}$ the complex

$$\cdots 0 \to 0 \to M \stackrel{id}{\to} M \to 0 \to 0 \cdots ,$$

with the M's in the -1 and 0-th position. Also we mean by $\underline{M}$ the complex with M in the 0-th place and 0 in the other places. Given a complex C

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and an integer m, $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ and whose boundary operators are $(-1)^m\delta^{m+n}$.

We note that $\underline{R}$ is a subcomplex of the projective complex $\overline{R}$ with quotient $\underline{R}[1]$. An element of $Hom(\underline{R}, C)$ corresponds to an element $x \in Z^0(C)$ which will be a boundary of C if and only if the corresponding map $\underline{R} \to C$ can be extended to $\overline{R} \to C$. Hence $Ext^1(\overline{R}[1], C) \cong H^0(C)$. More generally, $Ext^1(\overline{R}[n], C) \cong H^{-n+1}(C)$.

We will say that a complex C is finitely generated if, in case $C = \sum_{i\in I} D_i$, with $D_i \in \mathcal{C}$ subcomplexes of C, then there exists a finite subset $J \subseteq I$ such that $C = \sum_{i\in J} D_i$.

We will say that a complex C is finitely presented if C is finitely generated and for any exact sequence of complexes $0 \to K \to L \to M \to 0$ with L finitely generated, K is also finitely generated.

Let (F, δ) be a complex of left R-modules. We will say that F is a flat complex if F is a direct limit of projective complexes. We know that F is flat if and only if F is exact and $Ker(\delta^i)$ is flat for all $i \in \mathbb{Z}$ ([5, Theorem 2.4]). Note that a flat complex F with F^i finitely presented in R-Mod for all $i \in \mathbb{Z}$ (but F doesn't have to be a finitely presented complex) is projective

Let C be a complex of left R-modules (resp. of right R-modules) and let D be a complex of left R-modules. We will denote by $Hom^{\cdot}(C, D)$ (resp. $C \otimes^{\cdot} D$) the usual homomorphism complex (resp. tensor product) of the complexes C and D.

Remember, [1], that a complex F is called DG-flat if F^n is flat $\forall n \in \mathbb{Z}$ and for any exact complex E of right R-modules, the complex $E \otimes^{\cdot} F$ is exact. A complex D is called DG-injective (resp. DG-projective) if D^n is injective (resp. projective) $\forall n \in \mathbb{Z}$ and for any exact complex E, the complex $Hom^{\cdot}(E, D)$ is exact (resp. $Hom^{\cdot}(D, E)$ is exact).

We say that a complex (C, δ) is cotorsion if $Ext^1(F, C) = 0$ for any flat complex F, and (C, δ) is DG-cotorsion if C is exact and $Ker(\delta^i)$ is cotorsion in R-Mod for all $i \in \mathbb{Z}$.

2. Some canonical isomorphisms. Given two complexes C, D, $Hom(C, D) = Z^0(Hom^{\cdot}(C, D))$. We now modify the definition and let $\underline{Hom}(C, D) = Z(Hom^{\cdot}(C, D))$. We then see that $\underline{Hom}(C, D)$ can be made into a complex with $\underline{Hom}(C, D)^m$ the abelian group of morphisms from C to $D[m]$ and with boundary operator given by: $f \in \underline{Hom}(C, D)^m$, then $\delta^m(f) \colon C \to D[m+1]$ with $\delta^m(f)^n = (-1)^m\delta_D \circ f^n$, $\forall n \in \mathbb{Z}$.

If C is a complex of right R-modules and D is a complex of left R-

modules, let $C \otimes^{\cdot} D$ be the usual tensor product of the complexes. We define $C \otimes D$ to be $\dfrac{(C \otimes^{\cdot} D)}{B(C \otimes^{\cdot} D)}$. Then with the maps

$$\frac{(C \otimes^{\cdot} D)^n}{B^n(C \otimes^{\cdot} D)} \to \frac{(C \otimes^{\cdot} D)^{n+1}}{B^{n+1}(C \otimes^{\cdot} D)}, \quad x \otimes y \mapsto \delta_C(x) \otimes y$$

where $x \otimes y$ is used to denote the coset in $\dfrac{(C \otimes^{\cdot} D)^n}{B^n(C \otimes^{\cdot} D)}$, we get a complex.

We note that the new functor $\underline{Hom}(C, D)$ will have right derived functors whose values will be complexes. These values should certainly be denoted $\underline{Ext}^i(C, D)$. It is not hard to see that $\underline{Ext}^i(C, D)$ is the complex

$$\cdots \to Ext^i(C, D[n-1]) \to Ext^i(C, D[n]) \to Ext^i(C, D[n+1]) \to \cdots,$$

with boundary operator induced by the boundary operator of D.

For C a complex of left R-modules we have two functors $- \otimes C \colon \mathcal{C}_R \to \mathcal{C}_{\mathbb{Z}}$ and $\underline{Hom}(C, -) \colon {}_R\mathcal{C} \to \mathcal{C}_{\mathbb{Z}}$, where $\mathcal{C}_R$ (resp. ${}_R\mathcal{C}$) denotes the category of complexes of right R-modules (resp. left R-modules).

The following result shows that the above functors have similar properties to the functors defined for modules.

Proposition 2.1. *Given complexes E, C, D, we have the following isomorphisms of complexes:*

1) $\underline{Hom}(C \otimes D, E) \cong \underline{Hom}(C, \underline{Hom}(D, E))$.
2) *If R is commutative,* $C \otimes D \cong D \otimes C$.
3) $C \otimes (D \otimes E) \cong (C \otimes D) \otimes E$.
4) $\overline{M}[n] \otimes C \cong M \otimes_R C[n]$, $\forall M \in Mod - R$.
5) $(\varinjlim M_i) \otimes C \cong \varinjlim (M_i \otimes C)$ *where $\{M_i\}$ is a directed family of complexes of right R-modules.*
6) $\underline{Hom}(\overline{M}[n], D) \cong Hom_R(M, D)[-1-n]$ *and*

$$\underline{Hom}(D, \overline{M}[n]) = Hom_R(D, M)[-n]$$

for any $M \in R$-Mod.

Proof. 1) We define an isomorphism of complex of abelian groups in the following way. Given $m \in \mathbb{Z}$, we consider

$$\phi^m \colon \underline{Hom}(C \otimes D, E)^m \longrightarrow \underline{Hom}(C, \underline{Hom}(D, E))^m,$$

$$f \colon C \otimes D \to E[m] \ \mapsto \phi(f) \colon C \to \underline{Hom}(D, E)[m],$$

for $n \in \mathbb{Z}$,

$$\phi^m(f)^n \colon C^n \to \underline{Hom}(D,E)^{m+n} \quad x \mapsto \phi^m(f)^n(x) \colon D \to E[m+n]$$

and for $k \in \mathbb{Z}$

$$(\phi^m(f)^n(x))^k \colon D^k \to E^{m+n+k} \quad y \mapsto (-1)^{\rho(m,n,k)} f^{n+k}(x \otimes y)$$

where $\rho(m,n,k) = k + \binom{n+m}{2}$.

2) For $m \in \mathbb{Z}$ we define

$$\psi^m \colon (C \otimes D)^m \to (D \otimes C)^m$$

as $\psi^m(x \otimes y) = (-1)^{\alpha(d,t)} y \otimes x$ where $x \in C^d$, $y \in D^t$, $d+t=m$, and $\alpha(d,t) = \binom{d}{2} + \binom{t}{2} + \binom{m+1}{2}$.

3) Given $m \in \mathbb{Z}$, we take $(x \otimes y) \otimes z \in ((C \otimes D) \otimes E)^m$ with $x \in C^d$, $y \in D^t$, $z \in E^l$ and $d+t+l=m$. We define $h^m \colon ((C \otimes D) \otimes E)^m \longrightarrow (C \otimes (D \otimes E))^m$ as $h^m((x \otimes y) \otimes z) = (-1)^l x \otimes (y \otimes z)$.

Large calculations show that the maps defined in (1), (2) and (3) are isomorphisms of complexes.

(4) $\overline{M} \otimes^{\cdot} C$ is formed from the diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & M \otimes C^0 & \longrightarrow & M \otimes C^1 & \longrightarrow & \cdots \\
 & & \| & & \| & & \\
\cdots & \longrightarrow & M \otimes C^0 & \longrightarrow & M \otimes C^1 & \longrightarrow & \cdots
\end{array}$$

If we let D be the direct sum of all the bottom $M \otimes C^n$'s from the way the boundary of $\overline{M} \otimes C$ is defined we see that if $x \otimes y \in M \otimes C^n$ (with $M \otimes C^n$ from the top row), then for some $k \in \mathbb{Z}$,

$$x \otimes y + B(\overline{M} \otimes C) = (-1)^k (x \otimes \delta(y)) + B(\overline{M} \otimes C)$$

with $x \otimes \delta(y) \in M \otimes C^{n+1}$ on the bottom row. Hence $D \to \overline{M} \otimes C$ is surjective. But again from the definition of the boundary in $\overline{M} \otimes^{\cdot} C$ we see that $D \cap B(\overline{M} \otimes^{\cdot} C) = 0$. Hence $D \to \overline{M} \otimes C$ is bijective. But then the way that we make $\overline{M} \otimes C$ into a complex, we see that with D as a complex in the obvious way- $D \to \overline{M} \otimes C$ is an isomorphism of complexes. Hence $\overline{M} \otimes C \cong M \otimes_R C$

(5) follows by (1) and the Freyd's theorem about existence of adjoint functors.
(6) is easy to prove.

Lemma 2.2. *Any complex is the direct limit of finitely presented complexes*

Proof. Given C any complex, C is a direct union of bounded complexes. Hence we can suppose that C is bounded in order to prove the result. Let

$$C \equiv \cdots 0 \to C^0 \to \cdots \to C^n \to 0 \cdots .$$

We consider $F^i \to C^i \to 0$ a free presentation of C^i for $i = 0, ..., n$. Then $D = \oplus_{i=0}^{n} \overline{F^i}[-i] \to C \to 0$ is a projective presentation of C in $\mathcal{C}$ with D exact, D^j free for all j.
We consider the pairs (G, S) where $G \subseteq D$ is a finitely generated subcomplex with G^j free for all j and $S \subseteq G$ a finitely generated subcomplex of G. We order the family $\{(G, S)\}$ by $(G, S) \leq (G', S') \Leftrightarrow G \subseteq G'$, $S \subseteq S'$. Then G/S is finitely presented in $\mathcal{C}$ and $\varinjlim G/S = C$.

Lemma 2.3. *Let R and S be rings, L a complex of right S-modules, K a complex of (R, S)-bimodules and P a complex of left R-modules. Suppose that P is finitely presented and L is injective as complex of right S-modules. Then*

$$\underline{Hom}(K, L) \otimes P \cong \underline{Hom}(\underline{Hom}(P, K), L)$$

as complexes. This isomorphism is functorial in P, K and L.

Proof. We define

$$\lambda_P \colon \underline{Hom}(K, L) \otimes P \to \underline{Hom}(\underline{Hom}(P, K), L) \quad f \otimes p \mapsto \lambda_P(f \otimes p)$$

in the following way. For $m \in \mathbb{Z}$, we consider

$$\lambda_P^m \colon (\underline{Hom}(K, L) \otimes P)^m \to (\underline{Hom}(\underline{Hom}(P, K), L))^m$$

$$f \otimes p \mapsto \lambda_P^m(f \otimes p) \colon \underline{Hom}(P, K) \to L[m].$$

Suppose $f \in \underline{Hom}(K, L)^d$, $p \in P^t$ with $d + t = m$. Take $n \in \mathbb{Z}$, then

$$\lambda_P^m(f \otimes p)^n \colon \underline{Hom}(P, K)^n \to L^{m+n} \quad g \mapsto (-1)^{\beta(d,t,n)}(f^{n+t} \circ g^t)(p),$$

where $\beta(d,t,n) = dt + \binom{n+t+1}{2}$. Big calculations (but easy) show that λ_P is a map of complexes.
If we take $P = \overline{R}$, it is easy to see that $\lambda_{\overline{R}}$ is an isomorphism. Similarly if P is any finitely generated exact complex with all $Z^k(P)$ free. Since P is finitely presented we can find a sequence $H \to F \to P \to 0$ with H and F finitely generated exact complexes with all $Z^k(H)$, $Z^k(F)$ free. Since λ_H and λ_F are isomorphism, standard arguments show that λ_P is also an isomorphism.

Proposition 2.4. *Let (F,δ) be a complex of left R-modules. Then F is a flat complex if and only if $-\otimes F$ is an exact functor.*

Proof. Suppose F is flat. Then $F = \varinjlim P_i$ with P_i projective complexes. Hence $-\otimes F = \varinjlim(-\otimes P_i)$. Since P_i is a direct sum of complexes $\overline{Q}[n]$ with Q projective in R-Mod and $-\otimes \overline{Q}[n] \equiv (-\otimes_R Q)[n]$ it follows that $-\otimes F$ is an exact functor.
Conversely suppose $-\otimes F$ is exact. We only have to prove that $F^+ = \underline{Hom}(F, \overline{\mathbb{Q}/\mathbb{Z}})$ is an injective complex. Given $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ we have the commutative diagram

$$\begin{array}{ccccc} \underline{Hom}(B,F^+) & \longrightarrow & \underline{Hom}(A,F^+) & & \\ \downarrow & & \downarrow & & \\ (B\otimes F)^+ & \longrightarrow & (A\otimes F)^+ & \longrightarrow & 0 \end{array}$$

where the vertical arrows are isomorphisms. Therefore

$$\underline{Hom}(B,F^+) \to \underline{Hom}(A,F^+)$$

is an epimorphism.

Theorem 2.5. *The following conditions are equivalent for a short exact sequence $0 \to S \to C \to C/S \to 0$ in $\mathcal{C}$.*

(1) *$\underline{Hom}(P,C) \to \underline{Hom}(P,C/S) \to 0$ is exact for any finitely presented complex P.*

(2) $0 \to D \otimes S \to D \otimes C$ *is exact for any complex* D *or any finitely presented complex* D.

(3) $0 \to (C/S)^+ \otimes P \to C^+ \otimes P \to S^+ \otimes P \to 0$ *is exact for any finitely presented complex or the sequence splits.*

(4) $0 \to S \to C \to C/S \to 0$ *is a direct limit of splitting short exact sequences.*

Proof. Taking account of Proposition 2.1, Lemma 2.2 and Lemma 2.3, the proof follows by the same argument as in the case of modules (see for example [15, pg. 287]).

Definition 2.6. a) We will say that a short exact sequence in $\mathcal{C}$ is pure if it verifies one of the equivalent conditions in Theorem 2.5

b) We will say that a complex is pure injective if it is injective relative to any pure sequence.

Proposition 2.7.

1) *If* $N \to M$ *is pure monomorphism in* R*-Mod then* $\overline{N} \to \overline{M}$ *and* $\underline{N} \to \underline{M}$ *are pure monomorphisms in* $\mathcal{C}$.

2) *If a complex* (E, δ) *is pure injective then* E^n *and* $Ker(\delta^n)$ *are pure injective in* R*-Mod for all* $n \in \mathbb{Z}$.

3) C^+ *is pure injective for any complex* C.

4) *The evaluation map* $C \to C^{++}$ *is a pure monomorphism.*

5) *For a short exact exact sequence*

$$(*) \qquad 0 \to S \to C \to C/S \to 0$$

the following conditions are equivalent,

(i) $(*)$ *is pure;*

(ii) *every pure injective complex is injective with respect to* $(*)$.

(iii) F^+ *is injective with respect to* $(*)$ *for any complex of right* R*-modules* F.

(iv) S^{++} *is injective with respect to* $(*)$.

Proof. 1) $M^+ \to N^+$ splits in R-Mod, then $\overline{M}^+ \to \overline{N}^+$ and $\underline{M}^+ \to \underline{N}^+$ split in $\mathcal{C}$. Now the conclusion follows by Theorem 2.5.

2) follows by (1).

3), 4) and 5) are proved as in the case of modules.

Remark 2.8. It is easy to see that the inclusion $\underline{R} \subseteq \overline{R}$ is not a pure monomorphism. As a consequence of this and (5) in Proposition above we have that there are pure injective complexes which are not exact.

3. Pure injective and cotorsion envelopes. Let $\mathcal{A}$ be a class of objects in an abelian category $\mathcal{C}$. We recall the definition introduced in [3].

Definition 3.1. Let X be an object of $\mathcal{C}$. We say that E in $\mathcal{A}$ is a $\mathcal{A}$-preenvelope if there is exist an homomorphism $\phi\colon X \to E$ such that the triangle

$$\begin{array}{ccc} X & \xrightarrow{\phi} & E \\ \downarrow & \swarrow & \\ E' & & \end{array}$$

can be completed for each homomorphism $X \to E'$ with E' in $\mathcal{A}$.

If the triangle

$$\begin{array}{ccc} X & \xrightarrow{\phi} & E \\ \downarrow & \swarrow & \\ E & & \end{array}$$

can be completed only by automorphisms, we say that $\phi\colon X \to E$ is a $\mathcal{A}$-envelope.

An $\mathcal{A}$-preenvelope $\phi\colon X \to E$ is said to be special (or a special preenvelope) if $Ext^1(Coker(\phi), E') = 0$ for all $E' \in \mathcal{A}$.

Dually we have the concepts of $\mathcal{A}$-precover, $\mathcal{A}$-cover and special $\mathcal{A}$-precover.

Theorem 3.2. *Any complex has a pure injective envelope.*

Proof. Let C be a complex. By (3) and (4) in Proposition 2.7, we know that the evaluation $0 \to C \to C^{++} \to L \to 0$ is a pure injective preenvelope. Also we know that direct limit of pure sequences in $\mathcal{C}$ are pure (because, by Theorem 2.5, any pure sequence is a direct limit of splitting short exact sequences). Consider $\mathcal{M}$ the class of pure short exact sequences in $\mathcal{C}$ in the form $0 \to C \to K \to N \to 0$. Since this class has a generator, that means, the sequence $0 \to C \to C^{++} \to L \to 0$ is such that for any $0 \to C \to K \to T \to 0$ in $\mathcal{M}$ there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & K & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & C & \longrightarrow & C^{++} & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

But then by [8] or by an easy modification of the proof of [16, Section 2.2] we have guaranteed the existence of a minimal generator in $\mathcal{M}$, i.e., a generator $0 \to C \to V \to W \to 0$ such that for any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & C & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

g is an automorphism.
If $0 \to C \to V \to W \to 0$ is such minimal generator, it can be show that V is pure injective (in fact is a direct summand of C^{++}) and so $C \to V$ is a pure injective envelope.

Let $\mathcal{D}$ be a class of complexes. We let $\mathcal{D}^{\perp} = \{C \mid Ext^1(D, C) = 0 \ \forall D \in \mathcal{D}\}$ and ${}^{\perp}\mathcal{D} = \{C \mid Ext^1(C, D) = 0 \ \forall D \in \mathcal{D}\}$. We use analogous notation for any class of modules.

Lemma 3.3. *Let $\mathcal{F}$ be a class of left R-modules such that ${}^{\perp}(\mathcal{F})^{\perp} = \mathcal{F}$. Let (F, δ) be an exact complex with F^i, $Ker(\delta^i) \in \mathcal{F}$ and let (C, γ) be an exact complex with C^i, $Ker(\gamma^i) \in \mathcal{F}^{\perp}$. Then $Ext^1(F, C) = 0$ in $\mathcal{C}$.*

Proof. Given $(*)$ $0 \to C \to X \to F \to 0$ be an extension in $\mathcal{C}$, we have that $(*)$ splits at the module level, hence this sequence is isomorphic to $(**)$ $0 \to C \to M(f) \to F \to 0$ where $M(f)$ is the mapping cone associated to a map of complexes $f \colon F \to C[1]$. By the hypothesis it is easy to prove that f is homotopic to zero. Therefore we have that $(**)$ split and so $(*)$ also splits.

Proposition 3.4. *Let R be a right coherent ring. The following conditions are equivalent.*

(1) *C is a pure injective and flat complex.*

(2) *C is exact and $Z^k(C)$ are pure injective and flat in R-Mod for all $k \in \mathbb{Z}$.*

Proof. (1) $\Rightarrow$ (2) Since C is a flat complex then C is exact and $Z^k(C)$ are flat for all $k \in \mathbb{Z}$ ([5, Theorem 2.4]). By Proposition 2.7, the $Z^k(C)$ are pure injective for all $k \in \mathbb{Z}$.
(2) $\Rightarrow$ (1) Since R is right coherent, if $E \in R$-Mod is injective then E^+ is flat. So if $M \in R$-Mod is flat then M^{++} is also flat. Then $Z^k(C^{++}) = Z^k(C)^{++}$ and $C^{k\,++}$ are flat. On the other hand, the sequences $0 \to Z^k(C) \to C^k \to Z^{k+1}(C) \to 0$ split, hence the C^k are pure injective and flat.
Now the sequences

$$0 \to Z^k(C) \to Z^k(C^{++}) \to Z^k(C^{++}/C) \to 0$$

$$0 \to C^k \to C^{++\,k} \to (C^{++}/C)^k \to 0$$

split in R-Mod and therefore $Z^k(C^{++}/C)$ and $(C^{++}/C)^k$ are flat for all $k \in \mathbb{Z}$. Hence, by Lemma 3.3, the sequence

$$0 \to C \to C^{++} \to C^{++}/C \to 0$$

splits in $\mathcal{C}$ and so C is pure injective and flat.

Proposition 3.5. *Let R be a right coherent ring.*

1) *If F is a flat complex then the pure injective envelope of F, $PE(F)$, and $PE(F)/F$ are flat complexes.*

2) *A complex is flat and cotorsion if and only if it is flat and pure injective.*

Proof. 1) The sequence $0 \to F \to F^{++} \to L \to 0$ has F, F^{++} and L flat in $\mathcal{C}$. We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & PE(F) & \longrightarrow & D & \longrightarrow & 0 \\ & & \Vert & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & F & \longrightarrow & F^{++} & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Then f is a splitting monomorphism, hence $PE(F)$ is flat. Then it is clear that D is also flat.
2) If F is flat and cotorsion then the sequence $0 \to F \to PE(F) \to D \to 0$ has D flat, so the sequence split, therefore F is pure injective.

Theorem 3.6. *Let R be a commutative noetherian ring with finite Krull dimension. Then any complex C has a cotorsion envelope.*

Proof. The proof of this result follows that of [16, Theorem 3.4.6]. We will give it for completeness.
By [5, Theorem 4.6], we know that any complex over this ring has a flat cover. Given $C \in C$ we consider $0 \to C \to E(C) \to L \to 0$ and $F \to L \to 0$ a flat cover of L. We form the pull-back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & X & \to & F & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & E(C) & \to & L & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then K is cotorsion. Note also that $E(C)$ is cotorsion, hence X is cotorsion. Therefore $0 \to C \to X \to F \to 0$ gives a special cotorsion preenvelope. Since the class of short exact sequence

$$\{0 \to C \to V \to H \to 0 \mid C \text{ fixed and } H \text{ flat}\}$$

is closed under direct limits and it has a generator, we can find a minimal generator of this class and this gives a cotorsion envelope of C.

Corollary 3.7. *Let R be a commutative noetherian ring with finite Krull dimension.*

a) *If $\mathcal{D}$ is the class of cotorsion complexes and $\mathcal{F}$ is the class of flat complexes then ${}^{\perp}\mathcal{D} = \mathcal{F}$.*

b) *If F is a flat complex then the cotorsion envelope and the pure injective envelope of F coincide.*

Proof. a) Let $X \in {}^{\perp}\mathcal{D}$ and $0 \to L \to F \to X \to 0$ a short exact sequence with $F \to X$ a flat cover of X. The L is cotorsion. Therefore the sequence splits and so X is flat. The other inclusion ($\mathcal{F} \subseteq {}^{\perp}\mathcal{D}$) is clear.
b) Let $0 \to F \to C(F) \to D \to 0$ where $F \to C(F)$ is the cotorsion envelope of F. Then D is flat. Then diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & C(F) & \longrightarrow & D & \longrightarrow & 0 \\ & & \| & & \alpha\uparrow\downarrow\beta & & \uparrow\downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & PE(F) & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

gives $\alpha\beta$ and $\beta\alpha$ isomorphisms.

Theorem 3.8. *Let R be a commutative noetherian ring. Then any pure injective and flat complex is a direct product of complexes $\overline{T_p}[n]$ where p is any prime ideal, n is any integer and T_p is the completion of some free R_p-module.*

Proof. Given C a flat pure injective complex then $C = \oplus_{i\in\mathbb{Z}}\overline{D_i}[i]$ where D_i is a pure injective and flat module for all $i \in \mathbb{Z}$. By [16, Theorem 4.1.15], D_i is isomorphic in a unique way to a direct product of completions of free R_p-modules, where p is a prime ideal and R_p the localization of R in p, for different prime ideals p, i.e., $D_i \cong \prod_{p_i} \widehat{R_{p_i}^{(X_{p_i})}}$. Now the conclusion is clear.

4. Gorenstein flat complexes. Since $- \otimes C \colon \mathcal{C}_R \to {}_{\mathbb{Z}}\mathcal{C}$ is a right exact functor between abelian categories with enough projectives, we can construct right derived functors which we denote by $Tor_i(-, C)$.

Example 4.1. a) If M and N are R-modules then $\overline{Tor_1^R(M,N)} = Tor_1(\overline{M}, \overline{N})$.

b) Let M be a R-module and C a complex. Then $Tor_1(\overline{M}, C)$ is the complex

$$\cdots \to Tor_1^R(M, C^i) \to Tor_1^R(M, C^{i+1}) \to \cdots .$$

c) Let

$$L \equiv \ \ 0 \to A \to B \to C \to 0$$

$$M \equiv \ \ 0 \to P \to Q \to D \to 0$$

be two short exact sequences in R-Mod with A, B, P and Q projectives R-modules. If we consider L and M as complexes with B and Q the degree zero components of L and M respectively, then $Tor_1(M, L) = \underline{Tor_1^R(D, C)}[-1]$.

Lemma 4.2.
a) *A complex F is flat if and only if*

$$Tor_1(F, C) = 0 \ \ (Tor_i(F, C) = 0 \ \forall i > 0)$$

for any complex C.
b) *A complex D has finite flat dimension $\leq d$ if and only if D is exact and $Z^k(D)$ has finite flat dimension $\leq d$ for all $k \in \mathbb{Z}$.*
c) *A complex G is DG-flat if and only if $Tor_i(E, G) = 0$ for any exact complex E and all $i > 0$.*

Proof. a) Suppose F is flat. Take a projective presentation of F, $0 \to K \to P \to F \to 0$. By Theorem 2.5 this sequence is pure in $\mathcal{C}$. Also by properties of right derived functors $Tor_1(P, C) = 0$. Hence if we develop the long exact sequence of $- \otimes C$ we get the result.
Conversely, by standard arguments we prove that $F \otimes -$ is an exact functor. Therefore by Proposition 2.4 F is an flat complex.
b) Suppose that D has finite flat dimension $\leq d$. Let $0 \to F_d \to \cdots \to F_1 \to F_0 \to D \to 0$ be a flat resolution of D. Then, since F_i are flat, F_i are exact so D is also exact. On the other hand we have the exact sequences

$$0 \to Z^k(F_d) \to \cdots \to Z^k(F_1) \to Z^k(F_0) \to Z^k(D) \to 0$$

with $Z^k(F_i)$ flat modules. Hence $Z^k(D)$ has finite flat dimension $\leq d$.
Conversely suppose D has flat dimension $> d$ and let $\cdots \to F_1 \to F_0 \to D \to 0$ be a flat resolution. Since D is exact we have the resolutions of flat modules $\cdots \to Z^k(F_1) \to Z^k(F_0) \to Z^k(D) \to 0 \ \forall k \in \mathbb{Z}$. Then these sequences have, by hypothesis, length $\leq d$ a contradiction.
c) Let E be an exact complex. Since $(E \otimes^{\cdot} G)^+ \cong Hom^{\cdot}(E, G^+)$ we have that G is DG-flat if and only if G^+ is DG-injective. Then

$$Tor_1(E, G) = 0 \quad \text{if and only if} \quad 0 = (Tor_1(E, G))^+ = \underline{Ext}^1(E, G^+).$$

Now apply [7, Proposition 3.4]. If we take $0 \to K \to P \to G \to 0$ with G DG-flat and P projective then K is also DG-flat. Hence we get the sequence

$$\cdots 0 = Tor_2(E, P) \to Tor_2(E, G) \to Tor_1(E, K) = 0 \cdots,$$

therefore $Tor_2(E, G) = 0$. By induction the result follows.

In the next result we will suppose that R is a commutative Gorenstein ring. We will denote by $\mathcal{L}$ the class of complexes with finite injective dimension. By [4] and the Lemma above, it can be proved that this class coincides with the class of complexes of finite projective dimension and with the class of complexes of finite flat dimension. In [16] the existence of Gorenstein flat covers of R-modules over a Gorenstein ring is proved. We extend this result to the category of complexes.

Theorem 4.3. *Let R be a Gorenstein ring and C a complex of R-modules. The following conditions are equivalent.*

(1) *There is an exact sequence*

$$(*) \qquad \cdots \to F_1 \to F_0 \xrightarrow{\delta_0} F_{-1} \to F_{-2} \to \cdots$$

with F_i flat, $C = Ker(\delta_0)$ and $(*)$ remains exact when $E \otimes -$ is applied for any injective complex E.

(2) $Tor_1(L, C) = 0$ *for every* $L \in \mathcal{L}$.

(3) C^+ *is a Gorenstein injective complex.*

(4) C^n *is Gorenstein flat in R-Mod for all* $n \in \mathbb{Z}$.

(5) C *is a direct limit of Gorenstein projective complexes.*

(6) *For each finitely generated complex K and any map $K \to C$ there is a factorization $K \to P \to C$ where P is a finitely generated Gorenstein projective complex.*

Proof. (1) $\Rightarrow$ (2) We only need to prove that $Tor_i(E, M) = 0$ for all $i \leq 1$ and all E injective complex. Since the sequence

$$0 \to E \otimes C \to E \otimes F_0 \to E \otimes F_{-1} \to E \otimes F_{-2} \to \cdots$$

remains exact for any injective complex, we have the result.

(2) $\Rightarrow$ (1) Since R is coherent, any R-module has a pure injective flat envelope (see [6]). Let $C^i \to F^i$ be a pure injective flat envelope $\forall i \in \mathbb{Z}$. Then it is not hard to see that $C \to \oplus_{i \in \mathbb{Z}} \overline{F^i}[-i]$ is a pure injective flat preenvelope of C in $\mathcal{C}$. So we can construct a pure injective flat resolution

$$0 \to C \to F_0 \to F_{-1} \to \cdots$$

in $\mathcal{C}$. Let E be an injective complex. In order to show that

$$0 \to E \otimes C \to E \otimes F_0 \to E \otimes F_{-1} \to \cdots$$

is exact, it is enough to show that

$$0 \to (E \otimes F_{-1})^+ \to (E \otimes F_{-1})^+ \to \cdots$$

is exact. But $(E \otimes F_i)^+ \cong \underline{Hom}(F_i, E^+)$ for all i,

$$\underline{Hom}(F_{-1}, E^+) \cong \underline{Hom}(C, E^+)$$

and the sequence

$$\cdots \to \underline{Hom}(F_{-1}, E^+) \to \underline{Hom}(F_0, E^+) \to \underline{Hom}(C, E^+) \to 0$$

is exact because E^+ is pure injective flat. So if we take

$$\cdots \to F_2 \to F_1 \to C \to 0$$

a flat resolvent (see [5] for the existence of such a resolvent) we get the result.

(2) $\Leftrightarrow$ (3) This is clear since by [4] we know that $G \in \mathcal{C}$ is Gorenstein injective if and only if $\underline{Ext}^1(L, G) = 0$ for all $L \in \mathcal{L}$ and $Tor_1(L, C)^+ \cong \underline{Ext}^1(L, C^+)$.

(1) $\Rightarrow$ (4) For any $k \in \mathbb{Z}$ we have an exact sequence of flat modules

$$\cdots \to F_1^k \to F_0^k \overset{\delta_0^k}{\to} F_{-1}^k \to F_{-2}^k \to \cdots$$

with $C^k = Ker(\delta_0^k)$ and for any injective module E, the sequence remains exact when we apply $E \otimes_R - \equiv \overline{E} \otimes -$.

(4) $\Rightarrow$ (3) and (5) $\Rightarrow$ (3) are easy to prove.

(4) $\Rightarrow$ (6) and (6) $\Rightarrow$ (5) follows by the same arguments that of [5, Theorem 2.4] (by using [10, Theorem 2.1]).

Definition 4.4. We will say that a complex G is Gorenstein flat if G verifies one of the equivalent conditions of Theorem above.

Corollary 4.5. *If C is a Gorenstein projective complex then C is Gorenstein flat.*

Proof. Follow by (4) of above Theorem and [4, Theorem 2.7].

4.1. Existence of Gorenstein flat covers. Let R be a commutative Gorenstein ring. Let $D \in R$-Mod be an injective cogenerator. Then $\overline{D}$ is an injective cogenerator in $\mathcal{C}$.

Remark 4.6. Following [16, Lemma 5.7.1], let K be a pure injective flat complex (this is the same as DG-cotorsion flat) then K is a direct summand of $K^{++} = \underline{Hom}(\underline{Hom}(K, \overline{D}), \overline{D})$. Since K^+ and $\overline{D}$ are injective complexes, $\underline{Hom}(-, K)$ leaves a sequence exact if $\underline{Hom}(-, \underline{Hom}(E_1, E_2))$ leaves the sequence exact whenever E_1 and E_2 are injectives complexes. But $\underline{Hom}(-, \underline{Hom}(E_1, E_2))$ and $\underline{Hom}(E_1 \otimes -, E_2)$ are isomorphic functors.

Lemma 4.7. *If the sequence in $\mathcal{C}$*

$$\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

is an exact sequence of flat complexes such that $E \otimes -$ leaves the sequence exact for every E injective complex and K is DG-cotorsion and with finite flat dimension, then $\underline{Hom}(-, K)$ leaves the sequence exact.

Proof. By the Remark above if K is flat and DG-cotorsion then $\underline{Hom}(-, K)$ leaves the sequence exact. Now we consider K with finite flat dimension and DG-cotorsion. Let

$$0 \to H_n \to \cdots \to H_1 \xrightarrow{f_1} H_0 \xrightarrow{f_0} K \to 0$$

be exact in $\mathcal{C}$ with $H_i \to Ker(f_{i-1})$ flat covers. Then the H_i are DG-cotorsion for all $i = 0, ..., n$ since K is DG-cotorsion. Since $\underline{Hom}(-, H_i)$ leaves the sequence exact we see that $\underline{Hom}(-, K)$ leaves the sequence exact.

Corollary 4.8. *If $X \in \mathcal{C}$ is Gorenstein flat and K is DG-cotorsion with finite flat dimension then $\underline{Ext}^i(X, K) = 0$ for all $i \geq 1$.*

Theorem 4.9. *Every complex over a commutative Gorenstein ring has a Gorenstein flat cover.*

Proof. Since the functor $Tor_1(X, -)$ preserves direct limits for any $X \in \mathcal{C}$ it follows that the class of Gorenstein flat complexes is closed under direct limits. So we only have to find Gorenstein flat precovers. Let $C \in \mathcal{C}$. We have an exact sequence $0 \to K \to P \to C \to 0$ such that P is Gorenstein projective and $K \in \mathcal{L}$, (see [5, Lemma 4.7]). By [5, Theorem 4.10], we can find an exact sequence in $\mathcal{C}$ $0 \to V \to G \to K \to 0$ with $G \to K$ a flat cover. Then $V \in \mathcal{L}$ and V is DG-cotorsion (since K is exact). By Corollary above $Ext^i(H, V) = 0$ for every H Gorenstein flat complex and all $i > 0$. We consider the pushout diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & G & \longrightarrow & K & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & PE(G) & \longrightarrow & X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & = & F & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $G \to PE(G)$ is the pure injective envelope of G. Note that $PE(G)$ and F are flat because G is flat. Since $PE(G)$ is pure injective and flat it follows that $Ext^i(H, X) = 0$ for all Gorenstein flat complex H and all $i > 0$.

Now we take the pushout diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & F & = & F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where Y is Gorenstein flat since P and F are Gorenstein flat. Since X verifies $Ext^1(H, X) = 0$ for every H Gorenstein flat complexes we conclude that $Y \to C$ is a Gorenstein flat precover.

5. DG-pure sequences. Given a DG-flat complex G any sequence in $\mathcal{C}$, $0 \to C \to D \to G \to 0$, verifies that for every exact complex E,

$0 \to E \otimes C \to E \otimes D \to E \otimes G \to 0$ is exact. In this section we study some important properties of this kind of sequence. We will denote by $\mathcal{E}$ the class of exact complexes. Remember, [7], that any complex has an exact cover and a special exact preenvelope.

Definition 5.1. Let (*) $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{C}$.

a) We will say that (*) is $\mathcal{E}$-pure (resp. DG-pure) if $\underline{Hom}(E, B) \to \underline{Hom}(E, C) \to 0$ (resp. $0 \to E \otimes A \to E \otimes B$) is exact for all $E \in \mathcal{E}$.

b) We will say that $X \in \mathcal{C}$ is $\mathcal{E}$-pure injective (resp. DG-pure injective) if X is injective respect to any $\mathcal{E}$-pure sequence (resp. DG-pure sequence).

Proposition 5.2. *Let (*) $0 \to N \to L \to K \to 0$ be a short exact sequence in $\mathcal{C}$. The following conditions are equivalent.*

(1) (*) *is $\mathcal{E}$-pure.*

(2) a) (*) *is naturally isomorphic to a mapping cone short exact sequence* $0 \to D[-1] \to M(f)[-1] \to C \to 0$ *with* $f : C \to D$, *and*

b) *for every $E \in \mathcal{E}$ and every map $h : E \to C$, the composition* $E \xrightarrow{h} C \xrightarrow{f} D$ *is homotopic to* 0. (*or the exact cover of C, $E \to C$, verifies that the composition $E \to C \to D$ is homotopic to* 0).

(3) (2. a) *and if $p : E \to C$, $q : V \to D$ are the exact covers of C and D respectively then the induced map $g : E \to V$ given by the commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{g} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{f} & D \end{array}$$

is homotopic to 0.

(4) (2. a) *and in the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V[-1] & \longrightarrow & M(g)[-1] & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle (p,q)} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & D[-1] & \longrightarrow & M(f)[-1] & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

the top row splits.

Proof. (1) $\Rightarrow$ (2) Since for every $M \in R$-Mod the sequence

$$\underline{Hom}(\overline{M}, L) \to \underline{Hom}(\overline{M}, K) \to 0$$

is exact, the sequence $(*)$ splits at the module level. Then $(*)$ is isomorphic to a short exact sequence

$$0 \to D[-1] \to M(f)[-1] \xrightarrow{k} C \to 0$$

with $f\colon C \to D$. Let $E \in \mathcal{E}$ and $h\colon E \to C$. Then there is a map $v\colon E \to M(f)[-1]$ such that $k \circ v = h$. For each $n \in \mathbb{Z}$ consider the canonical projections $l^n\colon M(f)[-1]^n = C^n \oplus D^{n-1} \to D^{n-1}$ and the sequence

$$(**) \qquad 0 \to D[-1] \to M(fh)[-1] \xrightarrow{t} E \to 0.$$

Define $s\colon E \to M(fh)[-1]$ as

$$s^n\colon E^n \to E^n \oplus D^{n-1} \quad x \mapsto ((-1)^n x, l^n(v^n(x))).$$

Then it is not hard to see that s is a map of complexes and $t \circ s = 1$. Hence the sequence $(**)$ splits in $\mathcal{C}$ and so $f \circ h$ is homotopic to 0.
(2) $\Rightarrow$ (1) Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D[-1] & \longrightarrow & M(fh)[-1] & \longrightarrow & E & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle (h,1)} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & D[-1] & \longrightarrow & M(f)[-1] & \xrightarrow{k} & C & \longrightarrow & 0
\end{array}$$

Since $M(fh)[-1] \to E$ splits in $\mathcal{C}$, we find $v\colon E \to M(f)[-1]$ such that $k \circ v = h$.
(2) $\Rightarrow$ (4) Let $0 \to D[-1] \to M(f)[-1] \to C \to 0$ and $E \to C$, $V \to D$ exact covers. We form the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & E & \xrightarrow{p} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & U & \longrightarrow & V & \xrightarrow{q} & D & \longrightarrow & 0
\end{array}$$

which induces a mapping cone diagram

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$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U[-1] & \longrightarrow & M(\alpha)[-1] & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V[-1] & \longrightarrow & M(g)[-1] & \xrightarrow{\omega} & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle (p,q)} & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & D[-1] & \longrightarrow & M(f)[-1] & \xrightarrow{k} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Then $M(g)[-1] \to M(f)[-1]$ is a special exact precover. By hypothesis, there is a $v\colon E \to M(f)[-1]$ such that $k \circ v = p$. Then there is a $t\colon E \to M(g)[-1]$ such that $(p,q) \circ t = v$. Hence $p \circ \omega \circ t = k \circ (p,q) \circ t = k \circ v = p$. Therefore $\omega \circ t$ is an automorphism and so ω splits.
(4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are easy to prove.

Lemma 5.3.
a) *Given $M \in R$-Mod, the sequence $0 \to \underline{M} \to \overline{M} \to \underline{M}[1] \to 0$ is $\mathcal{E}$-pure if and only if M is injective in R-Mod. As a consequence, if ${}_RR$ is injective then any $\mathcal{E}$-pure injective complex is exact.*
b) *$E \in \mathcal{C}$ is exact if and only if $0 \to E \otimes \underline{R} \to E \otimes \overline{R}$ is exact. As a consequence, $0 \to \underline{R} \to \overline{R} \to \underline{R}[1] \to 0$ is always DG-pure and any DG-pure injective complex is exact.*

Proof. a) If M is injective then $\underline{M}$ is DG-injective. Therefore $\underline{Ext}^1(E, \underline{M}) = 0$ for any exact complex E.
Conversely, let $H \equiv\ 0 \to L \to T \to K \to 0$ be exact in R-Mod and $L \to M$ a map. If we consider H as a complex we can induce a map from H to

$\underline{M}[1]$. Since H is exact this map factorizes through $\overline{M}$. This factorization gives an extension $T \to M$ such that the triangle

$$\begin{array}{ccc} L & \longrightarrow & T \\ \downarrow & \swarrow & \\ M & & \end{array}$$

is commutative.
b) It is not hard to check that the kernel of $E \otimes \underline{R} \to E \otimes \overline{R}$ is precisely the homology complex of E.

Proposition 5.4. *Let* $(*)$ $0 \to A \to B \to C \to 0$ *be a short exact sequence in* $\mathcal{C}$. *If* $(*)$ *is DG-pure then* $0 \to C^+ \to B^+ \to A^+ \to 0$ *is* $\mathcal{E}$*-pure.*

Proof. Let E be an exact complex. We consider the diagram

$$\begin{array}{ccccc} \underline{Hom}(E, B^+) & \longrightarrow & \underline{Hom}(E, A^+) & & \\ \downarrow \cong & & \downarrow \cong & & \\ (E \otimes B)^+ & \longrightarrow & (E \otimes A)^+ & \longrightarrow & 0 \end{array}$$

where the bottom row is an epimorphism by hypothesis. Then $\underline{Hom}(E, B^+) \to \underline{Hom}(E, A^+)$ is an epimorphism.

Proposition 5.5. *The following conditions are equivalent for a complex D.*

(a) *D is DG-pure injective.*

(b) *D is pure injective and exact.*

Proof. (a)$\Rightarrow$(b) follows by (b) in Lemma 5.3.
(b)$\Rightarrow$(a) Let $0 \to A \to B \to C \to 0$ be a DG-pure sequence. We form the commutative diagram

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$$\begin{array}{ccccc} \underline{Hom}(B, D^{++}) & \longrightarrow & \underline{Hom}(A, D^{++}) & & \\ \downarrow \cong & & \downarrow \cong & & \\ (B \otimes D^{+})^{+} & \longrightarrow & (A \otimes D^{+})^{+} & \longrightarrow & 0 \end{array}$$

where the botton row is an epimorphism because D^{+} is exact. Hence

$$\underline{Hom}(B, D^{++}) \to \underline{Hom}(A, D^{++})$$

is an epimorphism. Since, by hypothesis, the evaluation map $D \to D^{++}$ is a splitting monomorphism, it is not hard to see that $\underline{Hom}(B, D) \to \underline{Hom}(A, D)$ is also an epimorphism.

Theorem 5.6. *Any complex C has a DG-pure injective envelope.*

Proof. Let $(*)$ $0 \to C \to D \to P \to 0$ be an exact sequence with $C \to D$ a special exact preenvelope, hence P is DG-projective and so DG-flat. Then $(*)$ is DG-pure. If we consider the evaluation $D \to D^{++}$, it is easy to see that the composition $C \to D \to D^{++}$ is a DG-pure injective preenvelope and it is a DG-pure monomorphism. Now, since the class of DG-pure sequence is closed under direct limits, we can apply the same argument as in the proof of Theorem 3.2 to get the result.

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