Explicit formulas for determinantal representations of the Drazin inverse solutions of some matrix and differential matrix equations.

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Abstract

The Drazin inverse solutions of the matrix equations $AX = B$, $XA = B$ and $AXB = D$ are considered in this paper. We use both the determinantal representations of the Drazin inverse obtained earlier by the author and in the paper. We get analogs of the Cramer rule for the Drazin inverse solutions of these matrix equations and using their for determinantal representations of solutions of some differential matrix equations, $X' + AX = B$ and $X' + XA = B$, where the matrix $A$ is singular.

Keywords: Drazin inverse, matrix equation, Drazin inverse solution, Cramer rule, differential matrix equation

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1. Introduction

In this paper we shall adopt the following notation. Let $\mathbb{C}^{m \times n}$ be the set of $m$ by $n$ matrices with complex entries and $I_m$ be the identity matrix of order $m$. Denote by $a_{j}$ and $a_{i}$, the $j$th column and the $i$th row of $A \in \mathbb{C}^{m \times n}$, respectively. Let $A_{j}(b)$ denote the matrix obtained from $A$ by replacing its $j$th column with the vector $b$, and by $A_{i}(b)$ denote the matrix obtained from $A$ by replacing its $i$th row with $b$.

Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. Then $|A_{\alpha\beta}|$ denotes the minor of $A$ determined by the rows indexed by $\alpha$ and the columns indexed by $\beta$. Clearly, $|A_{\alpha\alpha}|$ be a principal minor determined by the rows and columns indexed by $\alpha$. For $1 \leq k \leq n$, denote by

$$L_{k,n} := \{ \alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n \}$$

the collection of strictly increasing sequences of $k$ integers chosen from the set $\{1, \ldots, n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let

$$I_{k,m}\{i\} := \{ \alpha : \alpha \in L_{k,m}, i \in \alpha \}, \quad J_{k,n}\{j\} := \{ \beta : \beta \in L_{k,n}, j \in \beta \}.$$
Matrix equation is one of the important study fields of linear algebra. Linear matrix equations, such as
\[ AX = B, \]  
(1)
\[XA = B,\]  
(2)
and
\[AXB = D,\]  
(3)
play an important role in linear system theory therefore a large number of papers have presented several methods for investigating these matrix equations (for example, see [1]-[6]).

In some situations, however, people pay more attention to the Drazin inverse solutions of singular linear systems and matrix equations [7]-[13]. Moreover, Xu Zhao-liang and Wang Guo-rong in [14] proved that the Drazin inverse solutions of the matrix equations (1), (2) and (3) with some restricts are their unique solutions. The Cramer rule for the Drazin inverse solution of the restricted system of linear equations are used in [15]-[17]. The Cramer rules for solutions of the restricted matrix equations (1), (2) and (3), in particular for the Drazin inverse solution, are established in [18]-[21].

In this paper, we obtain explicit formulas for determinantal representations of the Drazin inverse solutions of the matrix equations (1), (2) and (3) and using their for determinantal representations of solutions of some differential matrix equations. The paper is based on principles used in [22], where we obtained analogs of the Cramer rule for the minimum norm least squares solutions of the matrix equations (1), (2) and (3). Liu et al. in [24] deduce the new determinantal representations of \( A^{(2)}_{T,S} \) and the Cramer rule for the restricted matrix equation (9) based on these principles as well. Since the Drazin inverse and the group inverse \(A\) are outer inverses \(A^{(2)}_{T,S}\) for some specific choice of \(T\) and \(S\), then the results obtained in [24] generalize in some ways some results of the paper. But we get the more detailed representation of the Drazin inverse solutions, and therefore we can used their for determinantal representations of solutions of some differential matrix equations.

The paper is organized as follows. We start with some basic concepts and results about the Drazin inverse in Section 2. We use the determinantal representation of the Drazin inverse obtained in [22] and also another determinantal representation is obtained in this section. In Section 3, we derive explicit formulas for determinantal representations of the Drazin inverse solutions for the matrix equations (1), (2) and (3). These formulas generalize the well-known Cramer rule. In Section 4, we demonstrate their using for determinantal representations of solutions of some differential matrix equations, \(X' + AX = B\) and \(X' +XA = B\), where the matrix \(A\) is singular. In Section 5, we show numerical examples to illustrate the main results.
2. Determinantal representations of the Drazin inverse

For any matrix $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$, where a positive integer $k = \min_{k \in \mathbb{N} \cup \{0\}} \{\text{rank} A^{k+1} = \text{rank} A^k\}$, the Drazin inverse is the unique matrix $X$ that satisfies the following three properties

1) $A^{k+1}X = A^k$;  
2) $XAX = X$;  
3) $AX =XA$. 

It is denoted by $X = A^D$.

In particular, when $\text{Ind} A = 1$, then the matrix $X$ in (4) is called the group inverse and is denoted by $X = A^g$.

If $\text{Ind} A = 0$, then $A$ is nonsingular, and $A^D \equiv A^{-1}$.

Remark 2.1. Since the equation 3) of (4), the equation 1) can be replaced by follows

1a) $X A^{k+1} = A^k$.

The Drazin inverse can be represented explicitly by the Jordan canonical form as follows.

Theorem 2.1. [25] If $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$ and

$$A = P \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} P^{-1}$$

where $C$ is nonsingular and $\text{rank} C = \text{rank} A^k$, and $N$ is nilpotent of order $k$, then

$$A^D = P \begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$ 

We use the following theorem about the limit representation of the Drazin inverse.

Theorem 2.2. [25] If $A \in \mathbb{C}^{n \times n}$, then

$$A^D = \lim_{\lambda \to 0} \left(\lambda I_n + A^{k+1}\right)^{-1} A^k,$$

where $k = \text{Ind} A$, $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.

The following theorem can be obtained by analogy to Theorem 2.2

Theorem 2.3. If $A \in \mathbb{C}^{n \times n}$, then

$$A^D = \lim_{\lambda \to 0} A^k \left(\lambda I_n + A^{k+1}\right)^{-1},$$

where $k = \text{Ind} A$, $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.
Denote by $a_{ij}^{(k)}$ and $a_{ij}^{(k)}$ the $j$th column and the $i$th row of $A^k$ respectively.

Lemma 2.1. ([22], Lemma 3.1) If $A \in \mathbb{C}^{n \times n}$ with $Ind A = k$, then for all $i, j = 1, \ldots, n$,

$$\text{rank } A^{k+1} (a_{ij}^{(k)}) \leq \text{rank } A^{k+1}.$$

Using Theorem 2.2 and Lemma 2.1 we obtained in [22] the following determinantal representations of the Drazin and group inverses and the identity $A^D A$ on $R(A^k)$.

Theorem 2.4. ([22], Theorem 3.3) If $Ind A = k$ and $\text{rank } A^{k+1} = \text{rank } A^{k} = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the Drazin inverse $A^D = (a_{ij}^D) \in \mathbb{C}^{n \times n}$ possess the following determinantal representations:

$$a_{ij}^D = \frac{\sum_{\beta \in J_{r,n}} \left| \left( A^{k+1} (a_{ij}^{(k)}) \right)^{\beta}_{\beta} \right|}{\sum_{\beta \in J_{r,n}} \left| (A^{k+1})^\beta_{\beta} \right|},$$

(5)

for all $i, j = 1, \ldots, n$.

Corollary 2.1. ([22], Corollary 3.1) If $Ind A = 1$ and $\text{rank } A^2 = \text{rank } A = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the group inverse $A^G = (a_{ij}^G) \in \mathbb{C}^{n \times n}$ possess the following determinantal representation:

$$a_{ij}^G = \frac{\sum_{\beta \in J_{r,n}} \left| (A^2 (a_{ij}))^\beta_{\beta} \right|}{\sum_{\beta \in J_{r,n}} \left| (A^2)^\beta_{\beta} \right|},$$

for all $i, j = 1, \ldots, n$.

Corollary 2.2. ([22], Corollary 3.2) If $Ind A = k$ and $\text{rank } A^{k+1} = \text{rank } A^{k} = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the matrix $A^D A = (p_{ij}) \in \mathbb{C}^{n \times n}$ possess the following determinantal representation

$$p_{ij} = \frac{\sum_{\beta \in J_{r,n}} \left| \left( A^{k+1} (a_{ij}^{(k+1)}) \right)^{\beta}_{\beta} \right|}{\sum_{\beta \in J_{r,n}} \left| (A^{k+1})^\beta_{\beta} \right|},$$

(6)

for all $i, j = 1, \ldots, n$.

Using Theorem 2.3 we can obtain another determinantal representation of the Drazin inverse. At first we consider the following auxiliary lemma similar to Lemma 2.1.
Lemma 2.2. If $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$, then for all $i, j = 1, \ldots, n$

$$\text{rank} A_i^{k+1} \left( a_{ij}^{(k)} \right) \leq \text{rank} A_i^{k+1}.$$

PROOF. The matrix $A_i^{k+1} \left( a_{ij}^{(k)} \right)$ may be represent as follows

$$\begin{pmatrix}
\sum_{s=1}^{n} a_{1s} a_{s1}^{(k)} & \cdots & \sum_{s=1}^{n} a_{1s} a_{sn}^{(k)} \\
\vdots & \ddots & \vdots \\
\sum_{s=1}^{n} a_{ns} a_{s1}^{(k)} & \cdots & \sum_{s=1}^{n} a_{ns} a_{sn}^{(k)}
\end{pmatrix}$$

Let $P_{li} (-a_{lj}) \in \mathbb{C}^{n \times n}$, $(l \neq i)$, be a matrix with $-a_{lj}$ in the $(l, i)$ entry, 1 in all diagonal entries, and 0 in others. It is a matrix of an elementary transformation. It follows that

$$A_i^{k+1} \left( a_{ij}^{(k)} \right) \cdot \prod_{l \neq i} P_{li} (-a_{lj}) = \begin{pmatrix}
\sum_{s=1}^{n} a_{1s} a_{s1}^{(k)} & \cdots & \sum_{s=1}^{n} a_{1s} a_{sn}^{(k)} \\
\vdots & \ddots & \vdots \\
\sum_{s=1}^{n} a_{ns} a_{s1}^{(k)} & \cdots & \sum_{s=1}^{n} a_{ns} a_{sn}^{(k)}
\end{pmatrix}$$

The obtained above matrix has the following factorization.

$$\begin{pmatrix}
\sum_{s \neq j} a_{1s} a_{s1}^{(k)} & \cdots & \sum_{s \neq j} a_{1s} a_{sn}^{(k)} \\
\vdots & \ddots & \vdots \\
\sum_{s \neq j} a_{ns} a_{s1}^{(k)} & \cdots & \sum_{s \neq j} a_{ns} a_{sn}^{(k)}
\end{pmatrix} = \begin{pmatrix}
a_{11} & \cdots & 0 & \cdots & a_{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & 0 \\
a_{n1} & \cdots & 0 & \cdots & a_{nn}
\end{pmatrix} \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}$$

Denote the first matrix by

$$\tilde{A} := \begin{pmatrix}
a_{11} & \cdots & 0 & \cdots & a_{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & 0 \\
a_{n1} & \cdots & 0 & \cdots & a_{nn}
\end{pmatrix}$$

ith.
The matrix \( \tilde{A} \) is obtained from \( A \) by replacing all entries of the \( i \)th row and the \( j \)th column with zeroes except for 1 in the \((i, j)\) entry. Elementary transformations of a matrix do not change its rank. It follows that \( \text{rank} \ A_{i} = \text{rank} \ (a_{i}) \), so that \( \text{rank} \ A_{i} = \text{rank} \ (a_{i}) \). Since \( \text{rank} \ A \geq \text{rank} \ A_{i} \) the proof is completed.

**Theorem 2.5.** If \( \text{Ind} A = k \) and \( \text{rank} A^{k+1} = \text{rank} A^k \leq n \) for \( A \in \mathbb{C}^{n \times n} \), then the Drazin inverse \( A^D = (a_{ij}^D) \in \mathbb{C}^{n \times n} \) possess the following determinantal representations:

\[
a_{ij}^D = \frac{\sum_{\alpha \in I_{r,n}(j)} \left| \left( A_{\alpha}^{k+1} \right)^{(k)} \right|}{\sum_{\alpha \in I_{r,n}} \left| \left( A_{\alpha}^{k+1} \right)^{(k)} \right|},
\]

for all \( i, j = 1, n \).

**Proof.** If \( \lambda \in \mathbb{R}_+ \), then \( \text{rank} (\lambda I + A^{k+1}) = n \). Hence, there exists the inverse matrix

\[
(\lambda I + A^{k+1})^{-1} = \frac{1}{\det (\lambda I + A^{k+1})} \begin{pmatrix} R_{11} & R_{21} & \ldots & R_{n1} \\ R_{12} & R_{22} & \ldots & R_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{1n} & R_{2n} & \ldots & R_{nn} \end{pmatrix},
\]

where \( R_{ij} \) is a cofactor in \( \lambda I + A^{k+1} \) for all \( i, j = 1, n \). By Theorem 2.3, \( A^D = \lim_{\lambda \to 0} A^{k} (\lambda I_{n} + A^{k+1})^{-1} \), so that

\[
A^D = \lim_{\lambda \to 0} \frac{1}{\det (\lambda I + A^{k+1})} \begin{pmatrix} \sum_{s=1}^{n} a_{1s}^k R_{1s} & \ldots & \sum_{s=1}^{n} a_{1s}^k R_{ns} \\ \vdots & \ddots & \vdots \\ \sum_{s=1}^{n} a_{ns}^k R_{1s} & \ldots & \sum_{s=1}^{n} a_{ns}^k R_{ns} \end{pmatrix} = \lim_{\lambda \to 0} \begin{pmatrix} \frac{\det (\lambda I + A^{k+1})}{\det (\lambda I + A^{k+1})} & \frac{\det (\lambda I + A^{k+1})}{\det (\lambda I + A^{k+1})} & \ldots & \frac{\det (\lambda I + A^{k+1})}{\det (\lambda I + A^{k+1})} \\ \frac{\det (\lambda I + A^{k+1})}{\det (\lambda I + A^{k+1})} & \frac{\det (\lambda I + A^{k+1})}{\det (\lambda I + A^{k+1})} & \ldots & \frac{\det (\lambda I + A^{k+1})}{\det (\lambda I + A^{k+1})} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\det (\lambda I + A^{k+1})}{\det (\lambda I + A^{k+1})} & \frac{\det (\lambda I + A^{k+1})}{\det (\lambda I + A^{k+1})} & \ldots & \frac{\det (\lambda I + A^{k+1})}{\det (\lambda I + A^{k+1})} \end{pmatrix}.
\]

Similar to the characteristic polynomial, we have

\[
\det (\lambda I + A^{k+1}) = \lambda^n + d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \ldots + d_n,
\]

where \( d_s = \sum_{\alpha \in I_{r,n}} \left| \left( A_{\alpha}^{k+1} \right)^{(s)} \right| \) is a sum of the principal minors of \( A^{k+1} \) of order \( s \), for all \( s = 1, n - 1 \), and \( d_n = \det A^{k+1} \). Since \( \text{rank} A^{k+1} = r \), then \( d_n = d_{n-1} = \ldots = d_{r+1} = 0 \) and

\[
\det (\lambda I + A^{k+1}) = \lambda^n + d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \ldots + d_r \lambda^{n-r}.
\]
Similarly we have for all \( i, j = 1, n \)
\[
\det (\lambda I + A^{k+1})_{ij}(a^{(k)}_{i,:}) = \ell^{(i)}(j)\lambda^{n-1} + \ell^{(2)}(j)\lambda^{n-2} + \ldots + \ell^{(n)}(j),
\]
where for all \( s = 1, n-1 \),
\[
\ell^{(i)}(j) = \sum_{\alpha \in I_{s,n}(j)} \left| \left( A^{k+1}_{ij}(a^{(k)}_{\alpha,:}) \right) \right|,
\]
and \( \ell^{(i)}(j) = \det A^{k+1}_{ij}(a^{(k)}_{i,:}) \).

By Lemma 22, \( \text{rank} A^{k+1}_{ij}(a^{(k)}_{i,:}) \leq r \), so that if \( s > r \), then for all \( \alpha \in I_{s,n}(i) \) and for all \( i, j = 1, n \),
\[
\left| \left( A^{k+1}_{ij}(a^{(k)}_{\alpha,:}) \right) \right| = 0.
\]

Therefore if \( r + 1 \leq s < n \), then for all \( i, j = 1, n \),
\[
\ell^{(i)}(j) = \sum_{\alpha \in I_{s,n}(j)} \left| \left( A^{k+1}_{ij}(a^{(k)}_{\alpha,:}) \right) \right| = 0,
\]
and \( \ell^{(i)}(j) = \det A^{k+1}_{ij}(a^{(k)}_{i,:}) = 0. \) Finally we obtain
\[
\det (\lambda I + A^{k+1})_{ij}(a^{(k)}_{i,:}) = \ell^{(i)}(j)\lambda^{n-1} + \ell^{(2)}(j)\lambda^{n-2} + \ldots + \ell^{(n)}(j)\lambda^{n-r}. \tag{10}
\]

By replacing the denominators and the nominators of the fractions in the entries of the matrix (8) with the expressions (9) and (11) respectively, finally we obtain
\[
A^D = \lim_{\lambda \to 0} \left( \begin{array}{cccc}
\frac{\ell^{(1)}(1)\lambda^{n-1} + \ldots + \ell^{(1)}(n)\lambda^{n-r}}{\lambda^0 + d_1\lambda^{n-1} + \ldots + d_r\lambda^{n-r}} & \ldots & \frac{\ell^{(1,n)}(1)\lambda^{n-1} + \ldots + \ell^{(1,n)}(n)\lambda^{n-r}}{\lambda^0 + d_1\lambda^{n-1} + \ldots + d_r\lambda^{n-r}} \\
\ldots & \ldots & \ldots \\
\frac{\ell^{(n)}(1)\lambda^{n-1} + \ldots + \ell^{(n)}(n)\lambda^{n-r}}{\lambda^0 + d_1\lambda^{n-1} + \ldots + d_r\lambda^{n-r}} & \ldots & \frac{\ell^{(n,1)}(1)\lambda^{n-1} + \ldots + \ell^{(n,1)}(n)\lambda^{n-r}}{\lambda^0 + d_1\lambda^{n-1} + \ldots + d_r\lambda^{n-r}} \\
\end{array} \right) = \\
\left( \begin{array}{cccc}
\ell^{(1)} & \ldots & \ell^{(1,n)} \\
\ldots & \ldots & \ldots \\
\ell^{(n)} & \ldots & \ell^{(n,1)} \\
\end{array} \right),
\]
where for all \( i, j = 1, n \),
\[
\ell^{(i)}(j) = \sum_{\alpha \in I_{r,n}(j)} \left| \left( A^{k+1}_{ij}(a^{(k)}_{\alpha,:}) \right) \right|.
\]

This completes the proof.
Using Theorem 2.5 we evidently can obtain another determinantal representation of the group inverse and the following determinantal representation of the identity $AA^D$ on $R(A^k)$

**Corollary 2.3.** If $\text{Ind} A = 1$ and rank $A^2 = \text{rank} A = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the group inverse $A^g = \left( a^g_{ij} \right) \in \mathbb{C}^{n \times n}$ possess the following determinantal representations:

$$a^g_{ij} = \frac{\sum_{\alpha \in I_{r,n}(j)} \left| (A^2_j \cdot (a_i^j))_\alpha \right| \sum_{\alpha \in I_{r,n}} \left| (A^2)^n_\alpha \right|}{\sum_{\alpha \in I_{r,n}} \left| (A^2)^n_\alpha \right|}, \quad (11)$$

for all $i, j = 1, n$.

**Corollary 2.4.** If $\text{Ind} A = k$ and rank $A^{k+1} = \text{rank} A^k = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the matrix $AA^D = (q_{ij}) \in \mathbb{C}^{n \times n}$ possess the following determinantal representation

$$q_{ij} = \frac{\sum_{\alpha \in I_{r,n}(j)} \left| (A^{k+1}_j \cdot (a_i^{(k+1)}))_\beta \right| \sum_{\alpha \in I_{r,n}} \left| (A^{k+1})^n_\beta \right|}{\sum_{\alpha \in I_{r,n}} \left| (A^{k+1})^n_\beta \right|}, \quad (12)$$

for all $i, j = 1, n$.

### 3. Cramer’s rule of the Drazin inverse solutions of some matrix equations

Consider a matrix equation

$$AX = B, \quad (13)$$

where $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$, $B \in \mathbb{C}^{n \times m}$ are given and $X \in \mathbb{C}^{n \times m}$ is unknown.

**Theorem 3.1.** ([14], Theorem 1) If the range space $R(B) \subset R(A^k)$, then the matrix equation (13) with constrain $R(X) \subset R(A^k)$ has a unique solution

$$X = A^D B.$$

We denote $A^kB = \hat{B} = \left( \hat{b}_{ij} \right) \in \mathbb{C}^{n \times m}$.

**Theorem 3.2.** If rank $A^{k+1} = \text{rank} A^k = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then for Drazin inverse solution $X = A^D B = \left( x_{ij} \right) \in \mathbb{C}^{n \times m}$ of (13) we have for all $i = 1, n$, $j = 1, m$,

$$x_{ij} = \frac{\sum_{\beta \in J_{r,n}(i)} \left| (A^{k+1} \cdot (\hat{b}_j))^\beta \right| \sum_{\beta \in J_{r,n}} \left| (A^{k+1})^\beta \right|}{\sum_{\beta \in J_{r,n}} \left| (A^{k+1})^\beta \right|}, \quad (14)$$
Proof. By Theorem 2.4 we can represent the matrix $A^D$ by (5). Therefore, we obtain for all $i = 1, n, j = 1, m$,
\[
x_{ij} = \sum_{s=1}^{n} a_{is} b_{sj} = \sum_{s=1}^{n} \frac{\sum_{\beta \in J_{r,n}(i)} \left| \begin{pmatrix} A^{k+1} \end{pmatrix}_{ij} \right|_{\beta} \cdot b_{sj}}{\sum_{\beta \in J_{r,n}} \left| \begin{pmatrix} A^{k+1} \end{pmatrix}_{\beta} \right|}.
\]

Since $\sum_{s} a_{is}^{(k)} b_{sj} = \hat{b}_{ij}$, then it follows (14).

Corollary 3.1. (22, Theorem 4.2.) If $\text{Ind} A = k$ and $\text{rank} A^{k+1} = \text{rank} A^k = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, and $y = (y_1, \ldots, y_n)^T \in \mathbb{C}^n$, then for Drazin inverse solution $x = A^D y =: (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ of the system of linear equations
\[
A \cdot x = y,
\]
we have for all $j = 1, n$,
\[
x_j = \frac{\sum_{\beta \in J_{r,n}(j)} \left| \begin{pmatrix} A^{k+1} \end{pmatrix}_{j} \right|_{\beta} \cdot b_{sj}}{\left| \begin{pmatrix} A^{k+1} \end{pmatrix}_{\beta} \right|},
\]
where $f = A^k y$.

Consider a matrix equation
\[
XA = B, \tag{15}
\]
where $A \in \mathbb{C}^{m \times m}$ with $\text{Ind} A = k$, $B \in \mathbb{C}^{n \times m}$ are given and $X \in \mathbb{C}^{n \times m}$ is unknown.

Theorem 3.3. (14, Theorem 2) If the null space $N(B) \supset N(A^k)$, then the matrix equation (15) with constrain $N(X) \supset N(A^k)$ has a unique solution
\[
X = BA^D.
\]
We denote $BA^k =: \tilde{B} = (\hat{b}_{ij}) \in \mathbb{C}^{n \times m}$. 

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Theorem 3.4. If rank $A^{k+1} = \text{rank } A^k = r \leq m$ for $A \in \mathbb{C}^{m \times m}$, then for Drazin inverse solution $X = BA^D = (x_{ij}) \in \mathbb{C}^{n \times m}$ of (17), we have for all $i = 1, n, j = 1, m$,

$$x_{ij} = \frac{\sum_{\alpha \in I_{r,m}} \left| \begin{pmatrix} A^{k+1} & B \end{pmatrix} \right|_\alpha}{\sum_{\alpha \in I_{r,m}} \left| (A^{k+1})^\alpha \right|_\alpha}.$$  \hfill (16)

PROOF. By Theorem 3.3 we can represent the matrix $A^D$ by (7). Therefore, for all $i = 1, n, j = 1, m$, we obtain

$$x_{ij} = \frac{\sum_{s=1}^m b_{is} a_{sj}^D = \sum_{s=1}^m b_{is} \sum_{\alpha \in I_{r,m}} \left| \begin{pmatrix} A^{k+1} & (a_s^{(k)}) \end{pmatrix} \right|_\alpha}{\sum_{\alpha \in I_{r,m}} \left| (A^{k+1})^\alpha \right|_\alpha} = \frac{\sum_{s=1}^m b_{ik} \sum_{\alpha \in I_{r,m}} \left| \begin{pmatrix} A^{k+1} & (a_s^{(k)}) \end{pmatrix} \right|_\alpha}{\sum_{\alpha \in I_{r,m}} \left| (A^{k+1})^\alpha \right|_\alpha}.$$  

Since for all $i = 1, n$

$$\sum_{s} b_{is} a_s^{(k)} = \left( \sum_{s} b_{is} a_s^{(k)} \sum_{s} b_{is} a_s^{(k)} \cdots \sum_{s} b_{is} a_s^{(k)} \right) = \tilde{b}_i,$$

then it follows (16).

Consider a matrix equation

$$AXB = D,$$  \hfill (17)

where $A \in \mathbb{C}^{n \times n}$ with $\text{Ind } A = k_1$, $B \in \mathbb{C}^{m \times m}$ with $\text{Ind } B = k_2$ and $D \in \mathbb{C}^{n \times m}$ are given, and $X \in \mathbb{C}^{n \times m}$ is unknown.

Theorem 3.5. (17), Theorem 3) If $R(D) \subset R(A^{k_1})$ and $N(D) \supset N(B^{k_2})$, $k = \max \{k_1, k_2\}$, then the matrix equation (17) with constrain $R(X) \subset R(A^k)$ and $N(X) \supset N(B^k)$ has a unique solution

$$X = A^{D}DB^D.$$  

We denote $A^{k_1}DB^{k_2} =: \tilde{D} = (\tilde{d}_{ij}) \in \mathbb{C}^{n \times m}$.

Theorem 3.6. If rank $A^{k_1+1} = \text{rank } A^{k_1} = r_1 \leq n$ for $A \in \mathbb{C}^{m \times m}$, and rank $B^{k_2+1} = \text{rank } B^{k_2} = r_2 \leq m$ for $B \in \mathbb{C}^{n \times m}$, then for the Drazin inverse solution $X = A^{D}DB^D =: (x_{ij}) \in \mathbb{C}^{n \times m}$ of (17) we have

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1, n}^{(1)}} \left| \begin{pmatrix} A^{k_1+1} & (d_B^{(1)}) \end{pmatrix} \right|_\beta}{\sum_{\beta \in J_{r_2, m}^{(1)}} \left| (A^{k_1+1})^\beta \right|_\beta \sum_{\alpha \in I_{r_2, m}} \left| (B^{k_2+1})^\alpha \right|_\alpha},$$  \hfill (18)
or

\[ x_{ij} = \frac{\sum_{\alpha \in I_{r_2,m}(j)} |B^{k_2+1}_{ij} (d^A)_{\alpha}|}{\sum_{\beta \in J_{r_1,n}} |(A^{k_1+1})_\beta| |B^{k_2+1}_{j\alpha}|}, \]  

(19)

where

\[
d^B = \left[ \sum_{\alpha \in I_{r_2,m}(j)} |B^{k_2+1}_{ij} (\tilde{d}_i)_{\alpha}|, \ldots, \sum_{\alpha \in I_{r_2,m}(j)} |B^{k_2+1}_{jn} (\tilde{d}_n)_{\alpha}| \right]^T, \tag{20}
\]

\[
d^A = \left[ \sum_{\beta \in J_{r_1,n}(i)} |A^{k_1+1}_{ij} (\tilde{d}_j)_{\beta}|, \ldots, \sum_{\beta \in J_{r_1,n}(i)} |A^{k_1+1}_{in} (\tilde{d}_n)_{\beta}| \right]
\]

are the column-vector and the row-vector, \(\tilde{d}_i\) and \(\tilde{d}_j\) respectively the \(i\)th row and the \(j\)th column of \(D\) for all \(i = 1, n, j = 1, m\).

**Proof.** By Theorems 2.4 and 2.5 the Drazin inverses \(A^D = (a^D_{ij}) \in \mathbb{C}^{n \times n}\) and \(B^D = (b^D_{ij}) \in \mathbb{C}^{m \times m}\) possess the following determinantal representations, respectively,

\[
a^D_{ij} = \frac{\sum_{\beta \in J_{r_1,n}(i)} |A^{k_1+1}_{ij} (a^D)_{\beta}|}{\sum_{\beta \in J_{r_1,n}} |(A^{k_1+1})_\beta|},
\]

(21)

\[
b^D_{ij} = \frac{\sum_{\alpha \in I_{r_2,m}(j)} |B^{k_2+1}_{ij} (b^D)_{\alpha}|}{\sum_{\alpha \in I_{r_2,m}(j)} |(B^{k_2+1})_\alpha|}.
\]

Then an entry of the Drazin inverse solution \(X = A^DDB^D =: (x_{ij}) \in \mathbb{C}^{n \times m}\) is

\[ x_{ij} = \sum_{s=1}^{m} \left( \sum_{t=1}^{n} a^D_{it} d_{ts} \right) b^D_{sj}. \tag{22} \]

Denote by \(\tilde{d}_s\) the \(s\)th column of \(A^sD =: \tilde{D} = (\tilde{d}_{ij}) \in \mathbb{C}^{n \times m}\) for all \(s = 1, m\).

It follows from \(\sum_t a^D_{it} d_{ts} = \tilde{d}_s\) that

\[
\sum_{t=1}^{n} a^D_{it} d_{ts} = \sum_{t=1}^{n} \sum_{\beta \in J_{r_1,n}(i)} |A^{k_1+1}_{ij} (a^D)_{\beta}| \cdot d_{ts} = \sum_{\beta \in J_{r_1,n}(i)} |(A^{k_1+1})_\beta| \cdot d_{ts} = \sum_{\beta \in J_{r_1,n}(i)} |(A^{k_1+1})_\beta| \cdot d_{ts} \tag{23}
\]
Substituting (22) and (21) in (22), we obtain

\[ x_{ij} = \sum_{s=1}^{m} \sum_{\beta \in J_{r_{1}, n} \setminus \{i\}} \left| A_{s,i}^{k_{1}+1} \left( d_{s}^{(k_{2})} \right) \right| \beta \ \sum_{\alpha \in I_{r_{2}, m} \setminus \{j\}} \left| B_{j}^{k_{2}+1} \left( b_{s}^{(k_{2})} \right) \right| \alpha \]

Suppose \( e_{x} \) and \( e_{s} \) are respectively the unit row-vector and the unit column-vector whose components are 0, except the \( s \)th components, which are 1. Since

\[ d_{s}^{(k_{2})} = \sum_{l=1}^{n} e_{l} d_{ls}^{(k_{2})}, \quad b_{s}^{(k_{2})} = \sum_{l=1}^{m} b_{st}^{(k_{2})} e_{t}, \quad \sum_{s=1}^{m} d_{st}^{(k_{2})} = \tilde{d}_{it}, \]

then we have

\[ x_{ij} = \]

\[ \sum_{s=1}^{m} \sum_{l=1}^{n} \sum_{\beta \in J_{r_{1}, n} \setminus \{i\}} \left| A_{s,i}^{k_{1}+1} \left( e_{l} \right) \right| \beta \ \sum_{\alpha \in I_{r_{2}, m} \setminus \{j\}} \left| B_{j}^{k_{2}+1} \left( e_{t} \right) \right| \alpha \]

\[ \sum_{\beta \in J_{r_{1}, n} \setminus \{i\}} \left| A_{s,i}^{k_{1}+1} \left( e_{l} \right) \right| \beta \ \sum_{\alpha \in I_{r_{2}, m} \setminus \{j\}} \left| B_{j}^{k_{2}+1} \left( e_{t} \right) \right| \alpha \]

\[ \sum_{\beta \in J_{r_{1}, n} \setminus \{i\}} \left| A_{s,i}^{k_{1}+1} \left( e_{l} \right) \right| \beta \ \sum_{\alpha \in I_{r_{2}, m} \setminus \{j\}} \left| B_{j}^{k_{2}+1} \left( e_{t} \right) \right| \alpha \]

Denote by

\[ d_{it}^{A} := \]

\[ \sum_{\beta \in J_{r_{1}, n} \setminus \{i\}} \left| A_{s,i}^{k_{1}+1} \left( \tilde{d}_{l} \right) \right| \beta \ \sum_{\alpha \in I_{r_{2}, m} \setminus \{j\}} \left| B_{j}^{k_{2}+1} \left( e_{t} \right) \right| \alpha \]

the \( t \)th component of a row-vector \( d_{i}^{A} = (d_{it}^{A}; \ldots; d_{im}^{A}) \) for all \( t = 1, m \). Substituting it in (23), we obtain

\[ x_{ij} = \frac{\sum_{l=1}^{m} d_{il}^{A} \sum_{\alpha \in I_{r_{2}, m} \setminus \{j\}} \left| B_{j}^{k_{2}+1} \left( e_{t} \right) \right| \alpha}{\sum_{\beta \in J_{r_{1}, n} \setminus \{i\}} \left| A_{s,i}^{k_{1}+1} \left( \tilde{d}_{l} \right) \right| \beta \ \sum_{\alpha \in I_{r_{2}, m} \setminus \{j\}} \left| B_{j}^{k_{2}+1} \left( e_{t} \right) \right| \alpha} \]

Since \( \sum_{l=1}^{m} d_{il}^{A} e_{t} = d_{i}^{A} \), then it follows (19).

If we denote by

\[ d_{ij}^{B} := \sum_{l=1}^{m} \tilde{d}_{lt} \sum_{\alpha \in I_{r_{2}, m} \setminus \{j\}} \left| B_{j}^{k_{2}+1} \left( e_{t} \right) \right| \alpha \]

\[ \sum_{\alpha \in I_{r_{2}, m} \setminus \{j\}} \left| B_{j}^{k_{2}+1} \left( e_{t} \right) \right| \alpha \]

Since \( \sum_{l=1}^{m} d_{il}^{A} e_{t} = d_{i}^{A} \), then it follows (19).
the $l$th component of a column-vector $d^B_j = (d^B_{1j}, ..., d^B_{nj})^T$ for all $l = 1, n$ and substitute it in (24), we obtain

$$x_{ij} = \frac{\sum_{l=1}^{n} \sum_{\beta \in J_{r_1,n} \setminus \{l\}} |A^{k_1+1}_{\cdot \cdot \cdot \cdot} (e_{\cdot \cdot \cdot \cdot})^{\beta}_{\beta} d^B_{lj} |}{\sum_{\beta \in J_{r_1,n}} |(A^{k_1+1})^{\beta}_{\beta} \sum_{\alpha \in J_{r_2,m}} |(B^{k_2+1})^{\alpha}_{\alpha}|}.$$ 

Since $\sum_{l=1}^{n} e_{l} d^B_{lj} = d^B_j$, then it follows (18).

4. Applications of the determinantal representations of the Drazin inverse to some differential matrix equations

Consider the matrix differential equation

$$X' + AX = B \quad (25)$$

where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times n}$ are given, $X \in \mathbb{C}^{n \times n}$ is unknown. It’s well-known that the general solution of (25) is found to be

$$X(t) = \exp^{-At} \left( \int \exp^{At} dt \right) B$$

If $A$ is invertible, then

$$\int \exp^{At} dt = A^{-1} \exp^{At} + G,$$

where $G$ is an arbitrary $n \times n$ matrix. If $A$ is singular, then the following theorem gives an answer.

**Theorem 4.1.** ([24], Theorem 1) If $A$ has index $k$, then

$$\int \exp^{At} dt = A^D \exp^{At} + (I - AA^D)t \left( I + \frac{A}{2t} + \frac{A^2}{3!t^2} + \ldots + \frac{A^{k-1}}{k!t^{k-1}} \right) + \frac{A^{k-1}}{k!} t^{k-1} + G.$$ 

Using Theorem 4.1 and the power series expansion of $\exp^{-At}$, we get an explicit form for a general solution of (25)

$$X(t) = \left\{ A^D + (I - AA^D)t \left( I - \frac{A}{2}t + \frac{A^2}{3!t^2} - \ldots + \frac{A^{k-1}}{k!t^{k-1}} \right) + G \right\} B.$$ 

If we put $G = 0$, then we obtain the following partial solution of (25),

$$X(t) = A^D B + \begin{pmatrix} B - A^D \end{pmatrix} t - \frac{1}{2}(AB - A^D A^2 B)t^2 + \ldots \frac{(-1)^{k-1}}{k!} (A^{k-1} B - A^D A^k B)t^k.$$ 

If we put $G = 0$, then we obtain the following partial solution of (25),

$$X(t) = A^D B + \begin{pmatrix} B - A^D \end{pmatrix} t - \frac{1}{2}(AB - A^D A^2 B)t^2 + \ldots \frac{(-1)^{k-1}}{k!} (A^{k-1} B - A^D A^k B)t^k.$$ 

Denote $A^l B = A^{l_1} B \in \mathbb{C}^{n \times n}$ for all $l = 1, 2, \ldots, k$. 

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Theorem 4.2. The partial solution \([26]\), \(X(t) = (x_{ij})\), possess the following determinantal representation,

\[
x_{ij} = \frac{\sum_{\beta \in J_r, n} \left| (A^{k+1} b_j^{(k+1)}) \right|_{\beta}}{\sum_{\beta \in J_r, n} \left| (A^{k+1}) \right|_{\beta}} \left( b_{ij} - \frac{\sum_{\beta \in J_r, n} \left| (A^{k+1} b_j^{(k+1)}) \right|_{\beta}}{\sum_{\beta \in J_r, n} \left| (A^{k+1}) \right|_{\beta}} \right) t + \cdots
\]

\[
-\frac{1}{2!} \left( \hat{b}_{ij}^{(1)} - \frac{\sum_{\beta \in J_r, n} \left| (A^{k+1} b_j^{(k+2)}) \right|_{\beta}}{\sum_{\beta \in J_r, n} \left| (A^{k+1}) \right|_{\beta}} \right) t^2 + \cdots
\]

\[
-\frac{(-1)^k}{k!} \left( \hat{b}_{ij}^{(k-1)} - \frac{\sum_{\beta \in J_r, n} \left| (A^{k+1} b_j^{(2k)}) \right|_{\beta}}{\sum_{\beta \in J_r, n} \left| (A^{k+1}) \right|_{\beta}} \right) t^k
\]

for all \(i, j = 1, n\).

PROOF. Using the determinantal representation of the identity \([6]\) we obtain the following determinantal representation of the matrix \(A^D A^m B := (y_{ij})\),

\[
y_{ij} = \sum_{s=1}^{n} p_s \sum_{t=1}^{n} a_{st}^{(m-1)} b_{ij} = \sum_{\beta \in J_r, n \{i\}} \sum_{s=1}^{n} \left| (A^{k+1} a_s^{(k+1)}) \right|_{\beta} \sum_{t=1}^{n} a_{st}^{(m-1)} b_{ij} = \sum_{\beta \in J_r, n \{i\}} \sum_{t=1}^{n} \left| (A^{k+1} b_j^{(k+m)}) \right|_{\beta} b_{ij} = \sum_{\beta \in J_r, n \{i\}} \sum_{\beta \in J_r, n} \left| (A^{k+1}) \right|_{\beta}
\]

for all \(i, j = 1, n\) and \(m = 1, k\). From this and the determinantal representation of the Drazin inverse solution \([14]\) and the identity \([6]\) it follows \([27]\).

Corollary 4.1. If \(\text{Ind} A = 1\), then the partial solution of \([26]\),

\[
X(t) = (x_{ij}) = A^g B + (B - A^g AB)t,
\]

possess the following determinantal representation

\[
x_{ij} = \frac{\sum_{\beta \in J_r, n} \left| (A^2 b_j^{(1)}) \right|_{\beta}}{\sum_{\beta \in J_r, n} \left| (A^2) \right|_{\beta}} \left( b_{ij} - \frac{\sum_{\beta \in J_r, n} \left| (A^2 b_j^{(2)}) \right|_{\beta}}{\sum_{\beta \in J_r, n} \left| (A^2) \right|_{\beta}} \right) t.
\]

for all \(i, j = 1, n\).

Consider the matrix differential equation

\[
X' + XA = B
\]
Corollary 4.2. If $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times n}$ are given, $X \in \mathbb{C}^{n \times n}$ is unknown. The general solution of (29) is found to be

$$X(t) = B \exp^{-A t} \left( \int \exp^{A t} \, dt \right)$$

If $A$ is singular, then an explicit form for a general solution of (29) is

$$X(t) = B \left\{ A^D + (I - AA^D)t \left( I - \frac{A}{2} t + \frac{A^2}{3!} t^2 + \ldots (-1)^{k-1} \frac{A^{k-1}}{k!} t^{k-1} \right) + G \right\}.$$ 

If we put $G = 0$, then we obtain the following partial solution of (29),

$$X(t) = BA^D + (B - BAA^D)t - \frac{1}{t} (BA - BA^2A^D)t^2 + \ldots \left( \frac{(-1)^{k-1}}{k!} (BA^{k-1} - BA^kA^D)t^k \right).$$

(30)

Denote $BA^l =: \tilde{B}^{(l)} (l) (i, j) \in \mathbb{C}^{n \times n}$ for all $l = 1, 2k$. Using the determinantal representation of the Drazin inverse solution (10), the group inverse (11) and the identity (12) we evidently obtain the following theorem.

Theorem 4.3. The partial solution (30), $X(t) = (x_{ij})$, possess the following determinantal representation,

$$x_{ij} = \sum_{\alpha \in \mathcal{I}_r, n(i)} \frac{\left( \sum_{\alpha \in \mathcal{I}_r, n(i)} \left( A^k \tilde{b}_{i,j}^{(k)} \right)_\alpha \right)}{\left| (A^k)_{ij} \right|} + \left( b_{ij} - \sum_{\alpha \in \mathcal{I}_r, n(i)} \frac{\left( \sum_{\alpha \in \mathcal{I}_r, n(i)} \left( A^k \tilde{b}_{i,j}^{(k+1)} \right)_\alpha \right)}{\left| (A^{k+1})_{ij} \right|} \right) t$$

$$- \frac{1}{2} \sum_{\alpha \in \mathcal{I}_r, n(i)} \frac{\left( \sum_{\alpha \in \mathcal{I}_r, n(i)} \left( A^k \tilde{b}_{i,j}^{(k+2)} \right)_\alpha \right)}{\left| (A^{k+2})_{ij} \right|} t^2 + \ldots$$

$$\frac{(-1)^{k}}{k!} \sum_{\alpha \in \mathcal{I}_r, n(i)} \frac{\left( \sum_{\alpha \in \mathcal{I}_r, n(i)} \left( A^k \tilde{b}_{i,j}^{(2k)} \right)_\alpha \right)}{\left| (A^{2k})_{ij} \right|} t^k$$

for all $i, j = 1, n$.

Corollary 4.2. If $\text{Ind} A = 1$, then the partial solution of (29),

$$X(t) = (x_{ij}) = BA^g + (B - BAA^g)t,$$

possess the following determinantal representation

$$x_{ij} = \sum_{\alpha \in \mathcal{I}_r, n(i)} \frac{\left( \sum_{\alpha \in \mathcal{I}_r, n(i)} \left( \tilde{A}^2 \tilde{b}_{i,j}^{(1)} \right)_\alpha \right)}{\left| \tilde{A}^2 \right|} + \left( b_{ij} - \sum_{\alpha \in \mathcal{I}_r, n(i)} \frac{\left( \sum_{\alpha \in \mathcal{I}_r, n(i)} \left( \tilde{A}^2 \tilde{b}_{i,j}^{(2)} \right)_\alpha \right)}{\left| \tilde{A}^2 \right|} \right) t.$$ 

for all $i, j = 1, n$. 

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5. Examples

In this section, we give examples to illustrate our results.

5.1. Example 1

Let us consider the matrix equation

\[ AXB = D, \quad (31) \]

where

\[
A = \begin{pmatrix}
2 & 0 & 0 \\
-i & i & i \\
-i & -i & -i \\
\end{pmatrix},
B = \begin{pmatrix}
1 & -1 & 1 \\
i & -i & i \\
-1 & 1 & 2 \\
\end{pmatrix},
D = \begin{pmatrix}
1 & i & 1 \\
i & 0 & 1 \\
1 & i & 0 \\
\end{pmatrix}.
\]

We shall find the Drazin inverse solution of (31) by (18). We obtain

\[
A^2 = \begin{pmatrix}
4 & 0 & 0 \\
2 - 2i & 0 & 0 \\
-2 - 2i & 0 & 0 \\
\end{pmatrix},
A^3 = \begin{pmatrix}
8 & 0 & 0 \\
4 - 4i & 0 & 0 \\
-4 - 4i & 0 & 0 \\
\end{pmatrix},
\]

\[
B^2 = \begin{pmatrix}
-i & i & 3 - i \\
1 & -1 & 1 + 3i \\
-3 + i & 3 - i & 3 + i \\
\end{pmatrix}.
\]

Since rank \( A = 2 \) and rank \( A^2 = \) rank \( A^2 = 1 \), then \( k_1 = \text{Ind} \ A = 2 \) and \( r_1 = 1 \).

Since rank \( B = \) rank \( B^2 = 2 \), then \( k_2 = \text{Ind} \ B = 1 \) and \( r_2 = 2 \). Then we have

\[
\tilde{D} = A^2DB = \begin{pmatrix}
-4 & 4 & 8 \\
-2 + 2i & 2 - 2i & 4 - 4i \\
2 + 2i & -2 - 2i & -4 - 4i \\
\end{pmatrix},
\]

and \( \sum_{\beta \in J_{1,3}} \left| (A^3)^{\beta} \right| = 8 + 0 + 0 = 8, \)

\[
\sum_{\alpha \in I_{2,3}} \left| (B^2)^{\alpha} \right| =
\]

\[
\det \begin{pmatrix}
-i & i \\
1 & -1 \\
\end{pmatrix} + \det \begin{pmatrix}
-1 & 1 + 3i \\
3 - i & 3 + i \\
\end{pmatrix} + \det \begin{pmatrix}
-i & 3 - i \\
-3 + i & 3 + i \\
\end{pmatrix} =
\]

\[
o + (-9 - 9i) + (9 - 9i) = -18i.
\]

By (20), we can get

\[
d_{1,1}^B = \begin{pmatrix}
12 - 12i \\
-12i \\
-12 \\
\end{pmatrix},
d_{1,2}^B = \begin{pmatrix}
-12 + 12i \\
12i \\
12 \\
\end{pmatrix},
d_{1,3}^B = \begin{pmatrix}
8 \\
-12 - 12i \\
-12 + 12i \\
\end{pmatrix}.
\]
Since $A_{31} \{d_{B1}\} = \begin{pmatrix} 12 - 12i & 0 & 0 \\ -12i & 0 & 0 \\ -12 & 0 & 0 \end{pmatrix}$, then finally we obtain

$$x_{11} = \frac{\sum_{\beta \in J_{1,3} \{1\}} A_{31} \{d_{B1}\}^{\beta}}{\sum_{\beta \in J_{1,3}} (A^{3})^{\beta} \sum_{\alpha \in J_{2,3}} |(B^{3})^{\alpha}|} = \frac{12 - 12i}{8 \cdot (-18i)} = \frac{1 + i}{12}. $$

Similarly,

$$x_{12} = \frac{-12 + 12i}{8 \cdot (-18i)} = \frac{-1 - i}{12}, x_{13} = \frac{8}{8 \cdot (-18i)} = \frac{i}{18},$$

$$x_{21} = \frac{-12i}{8 \cdot (-18i)} = \frac{1}{12}, x_{22} = \frac{12i}{8 \cdot (-18i)} = \frac{-1}{12}, x_{23} = \frac{-12 - 12i}{8 \cdot (-18i)} = \frac{-1 - i}{12},$$

$$x_{31} = \frac{12}{8 \cdot (-18i)} = \frac{-i}{12}, x_{32} = \frac{-12}{8 \cdot (-18i)} = \frac{i}{12}, x_{33} = \frac{-12 + 12i}{8 \cdot (-18i)} = \frac{-1 - i}{12}.$$ 

Then

$$X = \begin{pmatrix} \frac{1+i}{12} & \frac{-1-i}{12} & \frac{i}{12} \\ \frac{-1-i}{12} & \frac{1+i}{12} & \frac{i}{12} \\ \frac{-1-i}{12} & \frac{-1-i}{12} & \frac{-1+i}{12} \end{pmatrix}$$

is the Drazin inverse solution of (31).

### 5.2. Example 2

Let us consider the differential matrix equation

$$X' + AX = B, \quad (32)$$

where

$$A = \begin{pmatrix} 1 & -1 & 1 \\ i & -i & i \\ -1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & i & 1 \\ i & 0 & 1 \\ 1 & i & 0 \end{pmatrix}. $$

Since rank $A = rank A^2 = 2$, then $k = Ind A = 1$ and $r = 2$. The matrix $A$ is the group inverse. We shall find the partial solution of (32) by (28). We have

$$A^2 = \begin{pmatrix} -i & i & 3 - i \\ 1 & -1 & 1 + 3i \\ -3 + i & 3 - i & 3 + i \end{pmatrix}, \quad \hat{B}^{(1)} = AB = \begin{pmatrix} 2 - i & 2i & 0 \\ 1 + 2i & -2 & 0 \\ 1 + i & i & 0 \end{pmatrix},$$

$$\hat{B}^{(2)} = A^2 B = \begin{pmatrix} 2 - 2i & 2 + 3i & 0 \\ 2 + 2i & -3 + 2i & 0 \\ 1 + 5i & -2 & 0 \end{pmatrix}. $$
and
\[ \sum_{\alpha \in J_{2,3}} \left| (A^{2})^{\beta}_{\alpha} \right| = \det \begin{pmatrix} -i & i \\ 1 & -1 \end{pmatrix} + \det \begin{pmatrix} -1 & 1 + 3i \\ 3 - i & 3 + i \end{pmatrix} + \det \begin{pmatrix} -i & 3 - i \\ 3 + i & 3 + i \end{pmatrix} = 0 + (-9 - 9i) + (9 - 9i) = -18i. \]

Since \( (A^{2})^{1}_{1} (\hat{b}^{(1)})^{1}_{1} = \begin{pmatrix} 2 - i & i & 3 - i \\ 1 + 2i & -1 & 1 + 3i \\ 1 + i & 3 - i & 3 + i \end{pmatrix} \)
and
\[ (A^{2})^{1}_{1} (\hat{b}^{(2)})^{1}_{1} = \begin{pmatrix} 2 - 2i & i & 3 - i \\ 2 + 2i & -1 & 1 + 3i \\ 1 + 5i & 3 - i & 3 + i \end{pmatrix}, \]
then finally we obtain
\[ x_{11} = \frac{\sum_{\beta \in J_{2,3}^{(1)}} \left| (A^{2})^{\beta}_{\alpha} (\hat{b}^{(1)})^{\beta}_{\alpha} \right|}{\sum_{\beta \in J_{2,3}} \left| (A^{2})^{\beta}_{\alpha} \right|} + (b_{11} - \frac{\sum_{\beta \in J_{2,3}^{(1)}} \left| (A^{2})^{\beta}_{\alpha} (\hat{b}^{(2)})^{\beta}_{\alpha} \right|}{\sum_{\beta \in J_{2,3}} \left| (A^{2})^{\beta}_{\alpha} \right|}) t = \frac{3 - 3i}{18i} + (1 - \frac{-18i}{18i}) t = \frac{1 + i}{6}. \]

Similarly,
\[ x_{12} = \frac{-3 + 3i}{-18i} + \left( i - \frac{9 + 9i}{-18i} \right) t = -1 - i \frac{1}{6} + \frac{1 + i}{2} t, \quad x_{13} = 0 + (1 - 0) t = t, \]
\[ x_{21} = \frac{3 + 3i}{-18i} + \left( i - \frac{-18}{-18i} \right) t = -1 + i \frac{1}{6}, \]
\[ x_{22} = \frac{-3 - 3i}{-18i} + \left( 0 - \frac{-9 + 9i}{-18i} \right) t = 1 - i \frac{1}{6} + \frac{1 + i}{2} t, \quad x_{23} = 0 + (1 - 0) t = t, \]
\[ x_{31} = \frac{-12i}{-18i} + \left( 1 - \frac{-18i}{-18i} \right) t = \frac{2}{3} \]
\[ x_{32} = \frac{9 + 3i}{-18i} + \left( i - \frac{-18}{-18i} \right) t = -1 + 3i \frac{1}{6}, \quad x_{33} = 0 + (0 - 0) t = 0. \]

Then
\[ X = \frac{1}{6} \begin{pmatrix} 1 + i & -1 - i + (3 + 3i) t & t \\ -1 + i & 1 - i + (3 + 3i) t & t \\ 4 & -1 + 3i & 0 \end{pmatrix} \]
is the partial solution of \( (32) \).
References


