Life insurance pricing with fuzzy random variables

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Abstract
In this paper, we develop life insurance pricing models with fuzzy quantification of the interest rates within the time horizon. We use the fuzzy random variables to assess the present value of these contracts. We develop an approach to price life insurance policy that combines the stochastic behavior of mortality of the insured and the quantification of interest rates with fuzzy numbers. This approach maintains stochastic and fuzzy sources of uncertainty throughout the valuation process. We provide numerical illustration of these models.

Keywords: life insurance, fuzzy random variable, stochastic behavior of mortality

JEL Classification: C1

1. Introduction

Pricing life insurance can be a difficult task, as an actuary has to consider different types of risks, so far modeled by random variables. These have to describe the uncertainty encountered in demography and in the financial parameters.

In this paper, we develop life insurance pricing models replacing the random variables with fuzzy random variables, in order to quantify better the uncertainty related to the interest rates within the time horizon, when modeling the present value of life insurance contracts.

In section 2 we describe the main concepts used: fuzzy number (FN), triangular fuzzy number (TFN), fuzzy random variable FRV), expectation. In section 3 we develop a model to calculate the pure net premium and the net mathematical reserve for an \textit{n}-year pure endowment life insurance, using fuzzy random variables. Section 4 contains the numerical illustration.

2. Basic concepts

\textbf{Definition.} (Zadeh, 1965). A fuzzy set \(X\) in the universal (classical) set \(U\) is defined (characterized) by a membership function \(\mu_X:U \rightarrow [0,1]\), where \(\mu_X(x)\) is the grade of membership of \(x\) in \(X\), \(\forall x \in U\).

Let the set \(U\) be the real line, and let \(X\) be a fuzzy set in \(U\), then:

a) \(X\) is a normal fuzzy set if and only if \(\sup_{x \in U} \mu_X(x) = 1\).

b) \(X\) is a convex fuzzy set if and only if for every \(x, y \in U\) and every \(\lambda \in [0,1]\) we have \(\mu_X(\lambda \cdot x + (1-\lambda) \cdot y) \geq \mu_X(x) \land \mu_X(y)\), where \(\alpha \land \beta = \min \{\alpha, \beta\}\).

\textbf{Definition.} A fuzzy number (FN) \(X\) is a normal, convex set in the universal set \(U\).

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Definition. A triangular fuzzy number (TFN) $X$ is defined by the triplet $(a,b,c)$ and the membership function
\[
\mu_X(x) = \begin{cases} 
\frac{x-a}{b-a}, & a \leq x \leq b \\
\frac{c-x}{c-b}, & b < x \leq c 
\end{cases}.
\]

\hspace{1cm} (1)

Let $(\Omega, K, \mathcal{B})$ be a probability space and $\mathcal{B}$ the borel sets on $\mathbb{R}$.

We will denote by $\mathcal{F}(\mathbb{R})$ the set of all fuzzy subsets $\mu: \mathbb{R} \to [0,1]$ with the following properties:

a) $\{x \in \mathbb{R} | \mu(x) \geq \alpha\}$ is a compact set for every $0 < \alpha \leq 1$.

b) $\{x \in \mathbb{R} | \mu(x) = 1\} \neq \emptyset$.

Definition. A fuzzy random variable (FRV) is a function $X: \Omega \to \mathcal{F}(\mathbb{R})$ such that $\{(\omega,x) | x \in \tilde{X}_\alpha(\omega)\} \in K \times \mathcal{B}$ for any $\alpha \in [0,1]$, where $\tilde{X}_\alpha : \Omega \to \mathcal{B}(\mathbb{R})$ is defined as $\tilde{X}_\alpha(\omega) = \{x \in \mathbb{R} | \tilde{X}(\omega)(x) \geq \alpha\}$, we have $\tilde{X}(\omega)(x) = \mu_{\tilde{X}(\omega)}(x)$. $\tilde{X}_\alpha$ is called the $\alpha$-level representation of the FRV $\tilde{X}$.

For any FRV $\tilde{X}$ and any $\alpha \in [0,1]$, it can be defined an infima RV, $\underline{X}_\alpha$, and a suprema RV, $\overline{X}_\alpha$. Their realizations are the lower and the upper extreme values of the $\alpha$-cuts of $\tilde{X}(\omega)$, $\forall \omega \in \Omega$. Therefore, the $\alpha$-representation is $\underline{X}_\alpha(\omega) = \{x \in \mathbb{R} | \mu_{\tilde{X}(\omega)}(x) \geq \alpha\} = \tilde{X}_\alpha(\omega), \overline{X}_\alpha(\omega)]$. (2)

Let us consider the Aumann integral $\int F = \int_{\Omega} f \, dP \bigg| f \in S(F)$, where $F: \Omega \to \mathcal{B}(\mathbb{R})$ is a set valued function such that $F(\omega) \neq \emptyset$ for any $\omega \in \Omega$ and $S(F) = \{f \in L^1(P) | f(\omega) \in F(\omega) \text{ a.s.}\}$, where $L^1(P)$ is the the space of the P-integrables functions $f : \Omega \to \mathbb{R}$.

Let $\tilde{X} : \Omega \to \mathcal{F}(\mathbb{R})$ be an integrable and bounded fuzzy random variable.

Definition. The expected value of a FRV $\tilde{X}$, denoted $E(\tilde{X})$, is the fuzzy set $\mu \in \mathcal{F}(\mathbb{R})$ such that $\{x \in \mathbb{R} | \mu(x) \geq \alpha\} = \int \tilde{X}_\alpha$ for any $\alpha \in [0,1]$.

Remark. We have $E(\tilde{X})(x) = \sup\{\alpha \in [0,1] | x \in \int \tilde{X}_\alpha\}$ and its level sets are given by $E(\tilde{X})(x) \geq \alpha = \int \tilde{X}_\alpha$, $\alpha \in [0,1]$. (3)

3. The model for an $n$-year pure endowment life insurance

Our purpose is to calculate the net single premium ($\tilde{P}_{\text{net}}$) paid by the insured aged $x$ for an $n$-year pure endowment, when the interest rate in a fuzzy number. This type of insurance provides that the insurer makes a payment of $S$ m.u. at the end of the $n$ years if and only if the insured survives at least $n$ years from the time of policy issue (is alive). The
actuarial present value of the insurer’s obligation is given (repreented) by a discrete fuzzy random variable \( \tilde{X} \).

Let us consider that \( \tilde{i} \), the annual interest rate, is a triangular fuzzy number. The representation of the \( \alpha \)-cuts for the net single premium are

\[
\tilde{P}_{\text{net-single}}(\alpha) = S \cdot (1 + \tilde{i})^n \cdot \Pi_x
\]

and

\[
\tilde{P}_{\text{net-single}}(\alpha) = S \cdot (1 + \tilde{i})^n \cdot \Pi_x
\]

where the survival probability \( \Pi_n = \frac{l_x + \alpha}{l_x} \) will be determined using the survival function \( l_x \), its particular values are read from mortality tables.

4. Numerical illustration

Let us consider the triangular fuzzy number \( \tilde{i} = (0.01; 0.03; 0.06) \), which has the membership function \( \mu_{\tilde{i}}(x) = \begin{cases} 
0.01 \leq x \leq 0.03 & \frac{x - 0.01}{0.02} \\
0.03 < x \leq 0.06 & \frac{0.06 - x}{0.03}
\end{cases} \).

Then, we obtain

\[
\tilde{P}_{\text{net-single}}(\alpha) = S \cdot (1.06 - 0.03 \cdot \alpha) \cdot \Pi_x
\]

and

\[
\tilde{P}_{\text{net-single}}(\alpha) = S \cdot (1.01 + 0.02 \cdot \alpha) \cdot \Pi_x
\]

Let us consider the particular case \( x = 50 \) years old, \( n = 10 \) years, and \( S = 30000 \) m.u.

Using (6) and (7), we obtain the values given in Table 1.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \tilde{P}_{\text{net-single}}(\alpha) )</th>
<th>( \tilde{P}_{\text{net-single}}(\alpha) )</th>
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<tr>
<td>0.0</td>
<td>14087.73</td>
<td>22839.47</td>
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<td>14492.72</td>
<td>22392.09</td>
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<td>21525.96</td>
</tr>
<tr>
<td>0.4</td>
<td>15786.54</td>
<td>21106.78</td>
</tr>
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<td>16245.64</td>
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<td>19902.16</td>
</tr>
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<td>17713.41</td>
<td>19517.59</td>
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<tr>
<td>0.9</td>
<td>18234.61</td>
<td>19141.17</td>
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<td>1.0</td>
<td>18772.73</td>
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Figure 1. The net single premium depending on $\alpha$-cuts.

Remarks:
1. As the membership degree increases, the difference between the maximum and the minimum values of the net single premium decreases, while for certain membership, they are equal.
2. The minimum net single premium increases with the degree of membership and the maximum net single premium decreases as the degree of membership becomes larger.

Similarly, we calculate the net mathematical reserve at the time moments $t = 2, 5, 7, 9$ years from the moment of policy issue. The results are presented in Table 2.

Table 2. The net mathematical reserve (m. u.)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$t = 2$</th>
<th>$t = 5$</th>
<th>$t = 7$</th>
<th>$t = 9$</th>
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<td></td>
<td>$\overline{R}<em>{net}(t)</em>{\alpha}$</td>
<td>$\overline{R}<em>{net}(t)</em>{\alpha}$</td>
<td>$\overline{R}<em>{net}(t)</em>{\alpha}$</td>
<td>$\overline{R}<em>{net}(t)</em>{\alpha}$</td>
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<td>23530.31</td>
<td>20563.37</td>
<td>25560.35</td>
</tr>
<tr>
<td>0.2</td>
<td>16996.00</td>
<td>23161.57</td>
<td>20857.69</td>
<td>25309.27</td>
</tr>
<tr>
<td>0.3</td>
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<tr>
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Figure 2. The minimum of the net mathematical reserve depending on $\alpha$-cuts.

Figure 3. The maximum of the net mathematical reserve depending on $\alpha$-cuts.
5. Conclusions

In this paper, using fuzzy random variables, we developed an approach to price life insurance policies that combines the stochastic behavior of mortality of the insured and the quantification of interest rates with fuzzy numbers. This approach maintains stochastic and fuzzy sources of uncertainty throughout the valuation process. The model was illustrated by a numerical example in which the net single premium and the net mathematical reserve were calculated.

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References


