Efficient open domination in graph products

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Abstract

A graph $G$ is an efficient open domination graph if there exists a subset $D$ of $V(G)$ for which the open neighborhoods centered in vertices of $D$ form a partition of $V(G)$. We completely describe efficient domination graphs among direct, lexicographic and strong products of graphs. For the Cartesian product we give a characterization when one factor is $K_2$ and some partial results for grids, cylinders and toruses. A connection with total domination number is also established.

Keywords: efficient open domination; graph products; total domination

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1 Introduction and preliminaries

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use the standard notations $N_G(v)$ for the open neighborhood $\{u : uv \in E(G)\}$ and $N_G[v]$ for the closed neighborhood $N_G(v) \cup \{v\}$ for a graph $G$. By $u \sim v$ we denote adjacency of $u$ and $v$ from $V(G)$. Throughout the article we consider only simple graphs.

The domination number $\gamma(G)$ of a graph $G$ is one of the classical invariants in graph theory. It is the minimum cardinality of a set $S$ for which the union of closed neighborhoods centered in vertices of $S$ cover the whole vertex set of $G$. Such a set $S$ is called a dominating set of $G$. Hence each vertex of $G$ is either in $S$ or adjacent to a vertex in $S$. In other words, we can say that vertices of $S$ control each vertex outside of $S$. A classical question in such a situation is: who controls the vertices of $S$? One possible solution to this dilemma is the total domination. A set $D \subseteq V(G)$ is a total dominating set of $G$ if every vertex of $G$ is adjacent to a vertex of $D$. (Hence, also vertices of $D$ are controlled by $D$.) The total domination number of a graph $G$ is the minimum cardinality of a total dominating set of $G$ and is denoted by $\gamma_t(G)$. A total dominating set $D$ of cardinality $\gamma_t(G)$ is called $\gamma_t(G)$-set.

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The natural question for a graph $G$ is whether we can find a total dominating set $D$ for which its open neighborhoods not only cover $V(G)$ but also form a partition of $V(G)$, which means that $N_G(u) \cap N_G(v) = \emptyset$ for every different $u, v \in D$. The problem is known as efficient open domination. Hence, a graph $G$ is an efficient open domination graph if there exists a set $D$, called an efficient open dominating set, for which $\bigcup_{v \in D} N_G(v) = V(G)$ and $N_G(u) \cap N_G(v) = \emptyset$ for every pair $u$ and $v$ of distinct vertices of $D$. It is easy to see that paths $P_n$ and cycles $C_n$ are efficient open domination graphs if and only if $n \equiv 0 \pmod{4}$.

The problem of establishing whether a graph $G$ is an efficient open domination graph is an NP-complete problem, see [4]. Efficient open domination trees have been characterized recursively in [4]. In [3] various properties of efficient open domination graphs were presented. The latest work on this topic [12] deals with the efficient open domination graphs among Cayley graphs. However we have not found in the literature the following basic connection.

**Observation 1.1** If $G$ is an efficient open domination graph with an efficient open dominating set $D$, then $\gamma_t(G) = |D|$.

**Proof.** If $D$ is an efficient open dominating set of $G$, then $D$ is also a total dominating set of $G$ and $\gamma_t(G) \leq |D|$ follows. On the other hand, every vertex of $D$ has at least one neighbor in every $\gamma_t(G)$-set $D'$, since $\bigcup_{v \in D'} N_G(v) = V(G)$. Moreover, these neighbors must be different, since $\bigcup_{v \in D} N_G(v)$ form a partition of $V(G)$. Hence $\gamma_t(G) \geq |D|$ and the equality follows. □

From this simple observation immediately follows that all efficient open dominating sets of a graph $G$ have the same cardinality.

Similar approach is also known for dominating sets under the name 1-perfect graphs. Accurately, $G$ is 1-perfect graph if there exists a set $P \subseteq V(G)$ for which $V(G) = \bigcup_{v \in P} N_G[v]$ and $N_G[u] \cap N_G[v] = \emptyset$ for every pair $u$ and $v$ of distinct vertices of $P$. Set $P$ is called 1-perfect set of $G$. The name arose from codes, since one can discover and repair one error in such graphs.

Several graph products have been investigated in the last few decades and a rich theory involving the structure and recognition of classes of these graphs has emerged, cf. [6]. The most studied graph products are the Cartesian product, the strong product, the direct product and the lexicographic product which are also called standard products. One standard approach to graph products is to deduce properties of a product with respect to (the same) properties of its factors. See a short collection of these types involving total domination and perfect codes in [2, 5, 7, 8, 9, 10, 11].

The domination related problems on the Cartesian product seem to be the most challenging among standard products. We just mention the famous Vizing’s conjecture: $\gamma(G \square H) \geq \gamma(G)\gamma(H)$, which is probably the most challenging problem in the area of domination (see the latest survey on Vizing’s conjecture [11]). The efficient open domination seems to be not an exception.

In the next section we completely describe the efficient open domination graphs among direct, strong and lexicographic products. Afterwards, it follows a section where
we completely describe graphs for which their Cartesian product with \(K_2\) is an efficient open domination graph. In the last section we deal with some special Cartesian product graphs and we derive many additionally efficient open domination graphs among them.

2 The direct, lexicographic and strong products

The direct product \(G \times H\) of graphs \(G\) and \(H\) is a graph with vertex set \(V(G \times H) = V(G) \times V(H)\). Two vertices \((g, h)\) and \((g', h')\) are adjacent in \(G \times H\) whenever \(gg' \in E(G)\) and \(hh' \in E(H)\). For a fix \(h \in V(H)\) we call \(G^h = \{(g, h) \in V(G \times H) : g \in V(G)\}\) a \(G\)-layer in \(G \times H\). Symmetrically, \(H\)-layers \(H^g\) for a fix \(g \in V(G)\) are defined. Notice that in direct product graphs, we have that the subgraphs induced by a \(G\)-layer or an \(H\)-layer represent a graph without edges on \(|V(G)|\) or \(|V(H)|\) vertices, respectively.

The map \(p_G : V(G \times H) \rightarrow V(G)\) defined by \(p_G((g, h)) = g\) is called a projection map onto \(G\). Similarly, we define \(p_H\) as the projection map onto \(H\). Projections are defined as maps between vertices, but frequently it is more comfortably to see them as maps between graphs. In this case we observe the subgraphs induced by \(A \subseteq V(G \times H)\) and \(p_X(A)\) for \(X \in \{G, H\}\).

The commutativity of the direct product follows directly from the symmetric definition of adjacency. It is also associative \([6]\), which means that in several cases it is enough to observe only the direct product of two factors. This is the case of the present work. The open neighborhoods of vertices in direct product graphs are nicely connected to open neighborhoods of projections to the factors. Namely, \(N_{G \times H}(g, h) = N_G(g) \times N_H(h)\) for every vertex \((g, h) \in V(G \times H)\). The direct product is not always a connected graph even if both factors are. Indeed, \(G \times H\) is a connected graph if and only if both \(G\) and \(H\) are connected and at least one of \(G\) or \(H\) is non bipartite. Moreover, if both \(G\) and \(H\) are bipartite, \(G \times H\) has exactly two components (see \([8, 14]\)) where vertices \((g, h)\) and \((g, h')\) with \(hh' \in E(H)\) are in different components. In particular, a graph \(G \times H\) has an isolated vertex if and only if there exists an isolated vertex in \(G\) or in \(H\).

**Theorem 2.1** Let \(G\) and \(H\) be two graphs. The direct product \(G \times H\) is an efficient open domination graph if and only if \(G\) and \(H\) are efficient open domination graphs.

**Proof.** Let \(G\) and \(H\) be efficient open domination graphs and let \(D_G\) and \(D_H\), be its efficient open dominating sets, respectively. We will show that \(D_G \times D_H\) is an efficient open dominating set of \(G \times H\). Let \((g, h)\) be an arbitrary vertex of \(G \times H\). Obviously, \(g\) is adjacent to a unique vertex \(g'\) of \(D_G\) in \(G\) and \(h\) is adjacent to a unique vertex \(h'\) of \(D_H\) in \(H\). Thus, \((g, h)\) is adjacent to \((g', h')\) in \(G \times H\) and \(\bigcup_{g' \in D_G, h' \in D_H} N_{G \times H}(g', h') = V(G \times H)\). Moreover, \((g', h')\) is the only neighbor of \((g, h)\) from \(D_G \times D_H\), since \(g'\) and \(h'\) are unique. Therefore, \(D_G \times D_H\) is an efficient open dominating set of \(G \times H\).

Conversely, let \(G \times H\) be an efficient open domination graph with an efficient open dominating set \(D\). By the commutativity of the direct product, it is enough to show that one factor, say \(G\), is an efficient open domination graph. Every vertex of a layer \(G^h\) is dominated by exactly one vertex of \(D\). Let \(D'\) be the subset of \(D\), where each vertex
of \(D'\) has a neighbor in \(G^h\). We claim that \(p_G(D')\) is an efficient open dominating set of \(G\). Indeed, every vertex of \(G\) has a neighbor in \(p_G(D')\), since edges project into edges in the direct product. Moreover, if \(g \in N_G(g') \cap N_G(g'')\) for some distinct \(g', g'' \in p_G(D')\), then \((g, h) \in N_{G \times H}(g', h') \cap N_{G \times H}(g'', h'')\), where \((g', h'), (g'', h'') \in D'\), which is a contradiction, since \(D'\) is a subset of an efficient open dominating set \(D\). Therefore, \(G\) is an efficient open domination graph which ends the proof. □

Recall that cycles and paths are efficient open domination graphs if and only if their orders are equivalent to zero modulo four. Hence the next corollary follows.

**Corollary 2.2** Let \(G\) be a graph isomorphic to \(P_k \times P_t\), to \(P_k \times C_t\) or to \(C_k \times C_t\). Then \(G\) is an efficient open domination graph if and only if \(k \equiv 0 \mod 4\) and \(t \equiv 0 \mod 4\).

By Observation 1.1 we also obtain the following corollary.

**Corollary 2.3** If \(G\) and \(H\) are efficient open domination graphs, then \(\gamma_t(G \times H) = \gamma_t(G) \gamma_t(H)\).

The lexicographic product \(G \circ H\) (sometimes also denoted by \(G[H]\)) of graphs \(G\) and \(H\) is a graph with \(V(G \circ H) = V(G) \times V(H)\). Two vertices \((g, h)\) and \((g', h')\) are adjacent in \(G \circ H\) whenever \(gg' \in E(G)\) or \((g = g'\) and \(hh' \in E(H)\)). Layers and projections are defined similarly like in the direct product. Notice that the subgraph induced by a layer \(G^h\) or \(gH\) is isomorphic to \(G\) or \(H\), respectively, in contrast to the direct product. The lexicographic product is clearly not commutative, nevertheless it is associative [6].

**Theorem 2.4** Let \(G\) and \(H\) be graphs. The lexicographic product \(G \circ H\) is an efficient open domination graph if and only if either

(i) \(G\) is a graph without edges and \(H\) is an efficient open domination graph, or

(ii) \(G\) is an efficient open domination graph and \(H\) contains an isolated vertex.

**Proof.** If \(G\) is a graph without edges on \(n\) vertices, then \(G \circ H\) is isomorphic to \(n\) copies of \(H\). If in addition \(H\) is an efficient open domination graph, then also \(n\) copies of \(H\) form an efficient open domination graph. So, let \(G\) be an efficient open domination graph, \(D_G\) be its efficient open dominating set and \(h_0\) be an isolated vertex of \(H\). We will show that \(D_G \times \{h_0\}\) is an efficient open dominating set of \(G \circ H\). For this, notice that \(N_{G \circ H}(g, h_0) = N_G(g) \times V(H)\) and \(\bigcup_{g \in D_G} N_{G \circ H}(g, h_0) = V(G \times H)\) holds. Also, \(N_{G \circ H}(g, h_0) \cap N_{G \circ H}(g', h_0) \neq \emptyset, g, g' \in D_G\), implies that \(N_G(g) \cap N_G(g') \neq \emptyset\), which is a contradiction. Therefore, \(G \circ H\) is an efficient open domination graph.

Conversely, let \(G \circ H\) be an efficient open domination graph with an efficient open dominating set \(D\). Let \((g, h), (g', h') \in D\) be adjacent vertices. If \(g \neq g'\), then \(h\) (and by symmetry also \(h'\)) is an isolated vertex of \(H\). Namely, if \(h'' \in N_H(h)\), then \((g, h'') \in N_{G \circ H}(g, h) \cap N_{G \circ H}(g', h')\), which is a contradiction. Since \(H\) contains an isolated
vertex, $G$ has no isolated vertices, otherwise $G \circ H$ would contain isolated vertices, which is impossible for an efficient open domination graph. Hence, $(u, x), (u, y) \in D$ implies $x = y$ and that $x$ is an isolated vertex of $H$. If $u \in N_G(g) \cap N_G(g')$ for some $g, g' \in p_G(D)$, then $uH \subseteq N_{G \circ H}(g, h) \cap N_{G \circ H}(g', h')$ for $(g, h), (g', h') \in D$, which is a contradiction. Also $\bigcup_{u \in p_G(D)} N_G(u) = V(G)$, since $\bigcup_{(u, x) \in D} N_{G \circ H}(u, x) = V(G \circ H)$ and $D$ is an efficient open dominating set. Thus, $G$ is an efficient open domination graph (with an efficient open dominating set $p_G(D)$).

Now we can assume that all edges between vertices of $D$ have the same first coordinate: $(g, h)(g, h')$. Thus, $g$ is an isolated vertex of $G$, otherwise $uH \subseteq N_{G \circ H}(g, h) \cap N_{G \circ H}(g', h')$ for every neighbor $u$ of $g$ in $G$, which is not possible. Since $\{N_{G \circ H}(g, h) : (g, h) \in D\}$ forms a partition of $G \circ H$, every vertex $(u, x)$ is in some $N_{G \circ H}(g, h)$. Again $(g, h)$ is in some $N_{G \circ H}(g', h')$ and we have $g = g' = u$. Hence every vertex of $G$ is an isolated vertex and suppose that there are $n$ vertices in $G$. Every $H$-layer is isomorphic to $H$ and $G \circ H$ is isomorphic to $n$ copies of $H$. Since $G \circ H$ is an efficient open domination graph, every component of $G \circ H$ is such. Therefore, also $H$ is an efficient open domination graph which ends the proof. \hfill $\square$

We can easily generalize the construction of Theorem 2.4 (ii). Namely, let $G$ be any efficient open domination graph with $V(G) = \{g_1, \ldots, g_n\}$. Choose $n$ arbitrary graphs $H_1, \ldots, H_n$. Let $N_G(g_i) = \{g_{j_1}, g_{j_2}, \ldots, g_{j_{k_i}}\}$ for every $i \in \{1, \ldots, n\}$. Connect $g_i$ by an edge with every vertex of graphs $H_{j_1}, H_{j_2}, \ldots, H_{j_{k_i}}$ to obtain new graph $G^*$. It is easy to see that an efficient open dominating set $D$ of $G$ is also an efficient open dominating set of $G^*$. Moreover, we can add to $G^*$ arbitrary many edges between vertices of $H_i$ and $H_j$ as long as $g_i g_j \in E(G)$ and the obtained graph is still an efficient open domination graph. If we add all possible edges between $H_i$ and $H_j$ whenever $g_i g_j \in E(G)$ and all graphs $H_1, \ldots, H_n$ are isomorphic to a graph $H$, then the new graph is isomorphic to $G \circ (H \cup \{h\})$ where $h$ is an isolated vertex of $H \cup \{h\}$.

The strong product $G \boxtimes H$ of graphs $G$ and $H$ is a graph with $V(G \boxtimes H) = V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent in $G \times H$ whenever $(g g' \in E(G)$ and $h = h'$) or $(g = g' \text{ and } h h' \in E(H))$ or $(g g' \in E(G)$ and $h h' \in E(H))$. Clearly $E(G \times H) \subseteq E(G \boxtimes H)$. The commutativity of the strong product follows again from the symmetry of the definition of adjacency and for associativity see [6].

Since $G \circ H \cong G \boxtimes H$ if $G$ is a graph without edges, Theorem 2.4 (i) already gives a hint for the strong product. Surprisingly, these are the only graphs (up to the commutativity of the factors) among strong products which are efficient open domination graphs. This follows immediately from the fact that $|N_{G \boxtimes H}(g, h) \cap N_{G \boxtimes H}(g', h')| \geq 2$ for any two adjacent vertices $(g, h)$ and $(g', h')$, where both $g$ and $h$ are not isolated vertices of $G$ and $H$, respectively.

**Proposition 2.5** Let $G$ and $H$ be two graphs. The strong product $G \boxtimes H$ is an efficient open domination graph if and only if one factor is a graph without edges and the other is an efficient open domination graph.
3 Efficient open domination graphs $G \Box K_2$

The Cartesian product $G \Box H$ of graphs $G$ and $H$ is a graph with $V(G \Box H) = V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent in $G \Box H$ whenever $(gg' \in E(G)$ and $h = h')$ or $(g = g'$ and $hh' \in E(H))$. Hence $E(G \Box H) = E(G \Box H) \cup E(G \times H)$. The Cartesian product is commutative and associative (see [6]). Layers and projections are defined identically as for the direct product. The subgraph of $G \Box H$ induced by $G^h$ or $9H$ is isomorphic to $G$ or $H$, respectively.

As usual in domination related problems, also seems for the efficient open domination graphs, that the Cartesian product is the most problematic of all four standard products. We will first demonstrate this for the case when one factor is $K_2$. The Cartesian product $G \Box K_2$ can be described as two copies of $G$ with a matching between the corresponding vertices of each copy of $G$. Let $G \Box K_2$ be an efficient open domination graph with an efficient open dominating set $D$. If we wish to have all edges induced by $D$ in the above mentioned matching, it is not hard to see which properties are needed for $G$. Namely, if $G$ is 1-perfect graph with 1-perfect code $P$, then $D$ contains exactly $P$ in both copies of $G$. In particular notice that a path graph $P_k$ is a 1-perfect graph for every integer $k \geq 2$, and hence $P_k \Box K_2$ is an efficient open domination graph. However this is not the only possibility for $G \Box K_2$ to be an efficient open domination graph.

Before we describe other possibilities we need some notation. We denote $V(K_2) = \{1, 2\}$ and for a vertex $v \in V(G)$ we denote by $v^i$, $i \in \{1, 2\}$, the copy of $v$ in the $G^i$-layer. By $d_G(u, v)$ we mean the geodesic or shortest path distance, which is the number of edges on a shortest $u, v$-path in $G$. Distance $d_G(e, v)$ between edge $e$ and a vertex $v$ in $G$ is the shortest distance between end vertices of $e$ and $v$, while the distance $d_G(e_1, e_2)$ between edges $e_1$ and $e_2$ is the shortest distance between end vertices of $e_1$ and end vertices of $e_2$. In general, for $P, Q \subseteq V(G)$, the distance $d_G(P, Q)$ between them is the shortest distance between a vertex from $P$ and a vertex from $Q$.

Let $G$ be a graph on at least three vertices and $E' = \{e_1, \ldots, e_k\}$ be a subset of $E(G)$, where $e_i = u_iv_i$, with the following properties:

(i) $N_G(u_i) \cap N_G(v_i) = \emptyset$;

(ii) $d_G(e_i, e_j) \geq 2$ for every different pair $i, j \in \{1, \ldots, k\}$;

(iii) for every $w \in V(G) - \{u_i, v_i : i \in \{1, \ldots, k\}\}$ there exist unique $j$ and $\ell$, $j \neq \ell$, such that $d(w, e_j) = d(w, e_\ell) = 1$;

(iv) for every sequence of distinct edges $e_{i_1}, e_{i_2}, \ldots, e_{i_j}, j > 2$, with $d_G(e_{i_\ell}, e_{i_{\ell + 1(\text{mod}j)}}) = 2$ for $\ell \in \{1, \ldots, j\}$, $j$ must be an even number.

We call $E'$ the zig-zag set of $G$ and, if there exists a zig-zag set in $G$, we call $G$ as a zig-zag graph. (The motivation for this name follows from the property (iv) and the proof of the following theorem.) It is easy to see that zig-zag graphs among cycles are exactly $C_{6k}$ for a positive integer $k$ (we need six to fulfill the property (iv) of the definition).
Theorem 3.1 If $G$ is a zig-zag graph, then $G \square K_2$ is an efficient open domination graph.

Proof. Let $G$ be a zig-zag graph and $E' = \{e_1, \ldots, e_k\}$, $e_i = u_iv_i$, its zig-zag set. In addition we may assume that $E'$ is ordered such that for every $e_i$, $i > 1$, there exists $e_j$, $j < i$, with $d_G(e_i, e_j) = 2$. We call such an edge $e_j$ an ancestor of $e_i$. We define a subset $D$ of $V(G \square K_2)$ inductively as follows. Let $u_1^1, v_1^1 \in D$. For an edge $e_i$, $i > 1$, with ancestor $e_j$ we have either $(u_1^1, v_1^1 \in D$ if $u_2^j, v_2^j \in D$) or $(u_2^j, v_2^j \in D$ if $u_1^j, v_1^j \in D$). Since an edge $e_i$, $i > 1$, can have many ancestors, we need to show that $D$ is well defined. Suppose not, and let $e_i$ be the first edge which has two ancestors $e_j$ and $e_\ell$ with $u_1^j, v_1^j \in D$ and $u_2^\ell, v_2^\ell \in D$ (without loss of generality). There must exist a sequence $e_j, e_i, \ldots, e_\ell$ with $d(e_j, e_i) = d(e_i, e_\ell) = d(e_{i+1}, e_{i+\ell}) = 2$ for $p \in \{1, \ldots, t-1\}$ of odd length since among every two neighboring edges of this sequence one ancestor is ancestor of the other one. But this contradicts the property (iv) of the definition of zig-zag sets. Hence, $D$ is well defined and for every pair of edges $e_i, e_j \in E'$ with $d_G(e_i, e_j) = 2$ it follows $u_1^i, v_1^i, u_2^i, v_2^i \in D$, where $\{p, q\} = \{1, 2\}$.

Next we show that $D$ is an efficient open dominating set of $G \square K_2$. Notice that if $u_1^j, v_1^j \in D$, $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$, then they are dominated by $u_1^2$ and $v_1^2$, respectively. On the contrary, if $u_1^j, v_1^j \notin D$, then $u_1^\ell$ and $v_1^\ell$ dominate $u_1^j$ and $v_1^j$, respectively, where $\ell \in \{1, 2\} \setminus \{j\}$. Let now $w \in V(G) - \{u_i, v_i : i \in \{1, \ldots, k\}\}$. By the property (iii) of the definition of zig-zag sets, there exist unique $j$ and $\ell$ with $d(w, e_j) = d(w, e_\ell) = 1$. Hence, either $e_j$ is an ancestor of $e_\ell$ or vice versa. In each case we may assume without loss of generality that $u_1^j, v_1^j \in D$ and $u_1^\ell, v_1^\ell \in D$. Now $w$ is dominated by $u_1^j$ or $v_1^j$ and $w^2$ is dominated by $u_2^j$ or $v_2^j$. Hence $\bigcup_{v \in D} N_{G \square K_2}(v) = V(G \square K_2)$. By the property (i) of the definition of zig-zag sets we have that $N_{G \square K_2}(u_1^j) \cap N_{G \square K_2}(v_1^j) = \emptyset$ for $u_1^j, v_1^j \in D$ and $j \in \{1, 2\}$. If $d_G(e_i, e_j) = 2$, then $N_{G \square K_2}(x_1^i) \cap N_{G \square K_2}(y_1^j) = \emptyset$ for $x, y \in \{u, v\}$ since $\{p, q\} = \{1, 2\}$. Finally, if $d_G(e_i, e_j) > 2$, then clearly $N_{G \square K_2}(x_1^i) \cap N_{G \square K_2}(y_1^j) = \emptyset$ for $x, y \in \{u, v\}$ and $p, q \in \{1, 2\}$. Therefore, the neighborhoods of $D$ form a partition of $G \square K_2$ and, as a consequence, $G \square K_2$ is an efficient open domination graph.

To describe all efficient open domination graphs among Cartesian products of graphs with $K_2$ we need a combination of both: 1-perfect graphs and zig-zag graphs. Let $H_1$ be 1-perfect graph with 1-perfect code $P$ and $H_2$ be a zig-zag graph with zig-zag set $E' = \{e_1, \ldots, e_k\}$, $e_i = u_iv_i$. (Notice that $H_1$ and $H_2$ do not need to be connected.) A graph $G$ is called a 1-perfect zig-zag graph if $V(G) = V(H_1) \cup V(H_2)$, $E(H_1), E(H_2) \subseteq E(G)$ and for every $x \in V(H_1) - P$ and $w \in V(H_2) - \{u_i, v_i : i \in \{1, \ldots, k\}\}$ we may insert an edge $xw$ to $E(G)$ or not. In particular $G$ is isomorphic to the disjoint union of $H_1$ and $H_2$ if no edges of type $xw$ are added.

Theorem 3.2 Let $G$ be a graph. The Cartesian product $G \square K_2$ is an efficient open domination graph if and only if $G$ is 1-perfect zig-zag graph.

Proof. Let $G$ be a 1-perfect zig-zag graph built from 1-perfect graph $H_1$ with 1-perfect code $P$ and zig-zag graph $H_2$ with zig-zag set $E' = \{e_1, \ldots, e_k\}$, $e_i = u_iv_i$. 

7
Let $P^i$, $i \in \{1, 2\}$, be a copy of $P$ in the $G^i$-layer of $G \Box K_2$ and let $D$ be a set of vertices obtained from $E'$ as in the proof of Theorem 3.1. The set $D' = P^1 \cup P^2 \cup D$ dominates $G \Box K_2$ since $P^1 \cup P^2$ dominates $H_1 \Box K_2$ and $D$ dominates $H_2 \Box K_2$. In addition $N_{G \Box K_2}(x) \cap N_{G \Box K_2}(y) = \emptyset$ for any $x, y \in D'$, since all additional edges in $G$ besides those from $H_1$ and $H_2$ are between vertices which neither belong to $P$ nor to $\{u_1, v_1 : i \in \{1, \ldots, k\}\}$. Hence $G \Box K_2$ is an efficient open domination graph.

For the other direction, let $G \Box K_2$ be an efficient open domination graph with an efficient open dominating set $D$. Clearly $D$ contains adjacent pairs of vertices and we split them into two subsets as follows. In $D_1$ we put pairs of adjacent vertices from $D$ for which its edge projects to the edge of $K_2$ and in $D_2$ are all the remaining vertices (those ones whose edge projects to an edge of $G$). First we define a graph $H_1$ as follows. Let $V(H_1 \Box K_2) = \bigcup_{v \in D_1} N_{G \Box K_2}(v)$ and $H_1 \Box K_2$ is an induced subgraph of $G \Box K_2$ on $V(H_1 \Box K_2)$. It is clear that $H_1$ which is isomorphic to the layer $H_1^i$ (and also to the layer $H_2^i$) is 1-perfect graph with 1-perfect code $p_{G}(D_1)$.

Next, let $H_2 \Box K_2 = G \Box K_2 - V(H_1 \Box K_2)$. The projection of all edges induced by $D_2$ projects to edges of the first factor $H_2$. We will show that these edges, namely $p_{H_2}(D_2)$, form a zig-zag set of $H_2$. Let $|D_2| = 2k$ and denote adjacent vertices of $D_2$ by $u_i^1$ and $v_i^1$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$. Since $D_2$ is a subset of the efficient open dominating set $D$, it follows $N_{H_2 \Box K_2}(u_i^1) \cap N_{H_2 \Box K_2}(v_i^2) = \emptyset$ for every $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$, consequently $N_{H_2}(u_i) \cap N_{H_2}(v_i) = \emptyset$ and the property (i) holds.

For the property (ii), let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be two different edges induced by $p_{H_2}(D_2)$ and let $e_1', e_2' \in E(H_2 \Box K_2)$ be edges which project to $e_1$ and $e_2$, respectively. If $d_{H_2}(e_1, e_2) = 0$, then the two end vertices coincide and the other two differ. Without loss of generality, let $u_1 = u_2$ and $v_1 \neq v_2$. If both $e_1'$ and $e_2'$ lie in the same $G$-layer, say $G^1$, then $u_i^1 \in N_{H_2 \Box K_2}(v_i^1) \cap N_{H_2 \Box K_2}(v_i^2)$ which is not possible, since $D$ is an efficient open dominating set. If $e_1'$ and $e_2'$ lie in different $G$-layers, say $e_1'$ in $G^1$ and $e_2'$ in $G^2$, then $u_i^1 \in N_{H_2 \Box K_2}(v_i^1) \cap N_{H_2 \Box K_2}(u_i^2)$, which again yields to the same contradiction.

If $d_{H_2}(e_1, e_2) = 1$, then there are two end vertices, say $u_1$ and $u_2$, adjacent in $H_2$. If both $e_1'$ and $e_2'$ lie in the same $G$-layer, say $G^1$, then $u_i^1 \in N_{H_2 \Box K_2}(v_i^1) \cap N_{H_2 \Box K_2}(u_i^2)$, which is not possible by the same reason. If $e_1'$ and $e_2'$ lie in different $G$-layers, say $e_1'$ in $G^1$ and $e_2'$ in $G^2$, then $u_i^1 \in N_{H_2 \Box K_2}(v_i^1) \cap N_{H_2 \Box K_2}(u_i^2)$, which is not possible. Hence, $d_{H_2}(e_1, e_2) \geq 2$ and the property (ii) holds for $p_{H_2}(D_2)$.

Let $w = V(H_2) - p_{H_2}(D_2)$. Hence both $w^1$ and $w^2$ are not in $D_2$. Suppose that they are dominated by $u^1$ and $u^2$, respectively. Both $u^1$ and $u^2$ have a neighbor in $D_2$ in the same layer: $v^1$ and $v^2$, respectively. Let $e = u^1v^1$ and $e' = u^2v^2$. If $p_{H_2}(e) = p_{H_2}(e')$, then $u^1 \in N_{H_2 \Box K_2}(v^1) \cap N_{H_2 \Box K_2}(u^2)$, which is not possible. Moreover, by the property (ii) we have $d_{H_2}(p_{H_2}(e), p_{H_2}(e')) = 2$. Clearly $d_{H_2}(w, p_{H_2}(e)) = d_{H_2}(w, p_{H_2}(e')) = 1$. If there exists a third edge $p_{H_2}(e'')$ induced by $p_{H_2}(D_2)$ with $d_{H_2}(w, p_{H_2}(e'')) = 1$, then either $w^1$ or $w^2$ is dominated by two vertices of $D_2 \subseteq D$, which is not possible, and the property (iii) is satisfied by $p_{H_2}(D_2)$.

If the property (iv) does not hold, then there exists a sequence $e_{i_1}, e_{i_2}, \ldots, e_{i_j}$, $j > 2$, with $d(e_{i_\ell}, e_{i_{\ell+1} (mod j)}) = 2$ for $\ell \in \{1, \ldots, j\}$ and $j$ is an odd number. Since $d(e_{i_\ell}, e_{i_{\ell+1} (mod j)}) = 2$ for $\ell \in \{1, \ldots, j\}$, there exists a vertex $w_\ell$ for which $d(w_\ell, e_{i_\ell}) = \Box K_2$. Hence $G \Box K_2$ is an efficient open domination graph.
1 = d(w_\ell,e_{i+1(\mod j)}). By e'_\ell we denote the edge which projects to e_i for every \ell \in \{1,\ldots,j\}. Two consecutive edges e'_{\ell} and e'_{p(\mod j)} must be in the same H_2-layer, say H_2, since j is odd. Without loss of generality, we may assume that w_{i,p} is a neighbor of u_{i,p} and v_{i+1(\mod j)}. Thus w_{i,p} \in N_{H_2}K_2(u_{i,p}) \cap N_{H_2}K_2(v_{i+1(\mod j)}), a final contradiction. Hence, the property (iv) also holds, pH_2(D_2) is a zig-zag set of H_2 and H_2 is a zig-zag graph. Therefore, G is 1-perfect zig-zag graph, which ends the proof. □

To generalize this results from K_2 to K_p it is easy to see that no edge induced by an efficient open dominated set D of G\square K_p can project to K_p. Hence 1-perfect graphs have no analogue for p > 2. However, it seems that zig-zag graphs could be generalized to higher dimensions, where property (iv) represents the greatest problem.

4 Efficient open domination in toruses, cylinders and grids

In this section we use the following notation: U = \{u_0,\ldots,u_{r-1}\} and V = \{v_0,\ldots,v_{t-1}\} are the vertex sets of G and H, where G and H are isomorphic to a path or a cycle of order r and t, respectively. Operations with the subindexes of vertices of U and V are done modulo r and t, respectively. With respect to the previous section we assume that r, t \geq 3. The adjacency in G and H is defined as u_0 \sim u_1 \sim \ldots \sim u_{r-1} (\sim u_0) and v_0 \sim v_1 \sim \ldots \sim v_{t-1} (\sim v_0), respectively.

4.1 The torus C_r\square C_t

We begin with a general observation for regular graphs among Cartesian products, which follows directly from the fact that each vertex from an efficient open dominating set contains the same number of neighbors in both layers and that they appear by pairs.

**Observation 4.1** Let G_i be an r_i-regular graph of order n_i, i \in \{1,2\}. If G_1\square G_2 is an efficient open domination graph, then n_1n_2 \equiv 0 (\mod (2r_1 + 2r_2)).

**Theorem 4.2** Torus C_{4r}\square C_{4t} is an efficient open domination graph for every r, t \geq 1.

**Proof.** The result follows immediately from the fact that the set of vertices of C_{4r}\square C_{4t} given by the union of the following sets: \{u_0, u_4, \ldots, u_{4(r-1)}\} \times \{v_0, v_1, v_4, v_5, \ldots, v_{4t-4}, v_{4t-3}\} and \{u_2, u_6, \ldots, u_{4r-2}\} \times \{v_2, v_3, v_6, v_7, \ldots, v_{4t-2}, v_{4t-1}\}, is an efficient open dominating set in C_{4r}\square C_{4t}. □

**Theorem 4.3** Torus C_{4}\square C_{t}, t \geq 4, is an efficient open domination graph if and only if t \equiv 0 (\mod 4).

**Proof.** If t \equiv 0 (\mod 4), then it follows by Theorem 4.2 that C_{4}\square C_{t} is an efficient open domination graph.

Now suppose that the torus graph C_{4}\square C_{t}, t \geq 4, is an efficient open domination graph and let F be an efficient open dominating set in C_{4}\square C_{t}. By Observation 4.1 we
have that $t$ is even. So either $t \equiv 0 \pmod{4}$ or $t \equiv 2 \pmod{4}$. Suppose $t \equiv 2 \pmod{4}$. According to the symmetry of $C_4 \Box C_t$, we can suppose, without loss of generality, that $(u_0, v_0) \in F$ and $(u_0, v_{t-1}) \notin F$.

If $(u_0, v_1) \in F$, then we have that $(u_2, v_2), (u_2, v_3) \in F$ and consequently, the vertices $(u_0, v_4), (u_0, v_5), (u_2, v_6), (u_2, v_7), \ldots, (u_0, v_{t-2}), (u_0, v_{t-1})$ also belong to $F$, which is a contradiction since $(u_0, v_{t-1})$ is dominated by $(u_0, v_0)$ and $(u_0, v_{t-2})$. Analogously, we obtain a contradiction if $(u_1, v_0) \in F$. Therefore $t \equiv 0 \pmod{4}$. \hfill $\Box$

**Theorem 4.4** Torus $C_r \Box C_t$, $r \in \{3, 5, 6, 7\}$ and $t \geq r$, is not an efficient open domination graph.

**Proof.** If $r = 3$, then it is straightforward to observe that $C_3 \Box C_t$ is not an efficient open domination graph for every $t \geq 3$. Now suppose $r = 5$ and let $F_5$ be an efficient open dominating set in $C_5 \Box C_t$. According to the symmetry of $C_5 \Box C_t$, we consider without loss of generality that $(u_0, v_0) \in F_5$ and $(u_0, v_{t-1}) \notin F_5$. If $(u_1, v_0) \in F_5$, then we have that $(u_3, v_1), (u_3, v_2) \in F_5$ and consequently, $(u_0, v_3), (u_1, v_3) \in F_5$. Thus, we have that at least one of the vertices of the set $\{(u_2, v_4), (u_3, v_4), (u_4, v_4)\}$ cannot be efficiently open dominated by $F_5$, a contradiction. On the other hand, if $(u_0, v_1) \in F_5$, then $(u_2, v_2), (u_3, v_2) \in F_5$ and at least one of the vertices of the set $\{(u_0, v_3), (u_1, v_3), (u_4, v_3)\}$ cannot be efficiently open dominated by $F_5$, a contradiction again.

Suppose $r = 6$ and let $F_6$ be an efficient open dominating set in $C_6 \Box C_t$. We proceed similarly to the above case. We may assume that $(u_0, v_0) \in F_6$ and $(u_0, v_{t-1}) \notin F_6$. If $(u_0, v_1) \in F_6$, then we have the following cases.

Case 1: $(u_3, v_0), (u_3, v_1) \in F_6$. As a consequence, we have that two vertices of the set $\{(u_1, v_2), (u_2, v_2), (u_4, v_2), (u_5, v_2)\}$ cannot be efficiently open dominated by $F_6$, a contradiction.

Case 2: $(u_3, v_1), (u_3, v_2) \in F_6$. Consequently, either $(u_2, v_0)$ or $(u_4, v_0)$ cannot be efficiently open dominated by $F_6$, a contradiction.

Case 3: Either $(u_2, v_2), (u_2, v_3) \in F_6$ or $(u_4, v_2), (u_4, v_3) \in F_6$. Consequently, either $(u_4, v_1)$ or $(u_2, v_1)$, respectively, cannot be efficiently open dominated by $F_6$, a contradiction.

Case 4: $(u_3, v_2), (u_3, v_3) \in F_6$. Consequently, $(u_2, v_1)$ and $(u_4, v_1)$ cannot be efficiently open dominated by $F_6$, a contradiction.

Case 5: Either $(u_2, v_2), (u_3, v_2) \in F_6$ or $(u_3, v_2), (u_4, v_2) \in F_6$. Consequently, either $(u_4, v_1)$ or $(u_2, v_1)$, respectively, cannot be efficiently open dominated by $F_6$, a contradiction.

On the other hand, if $(u_1, v_0) \in F_6$, then we have the following cases.

Case 6: $(u_3, v_1), (u_4, v_1) \in F_6$. As a consequence, we have that two vertices of the set $\{(u_0, v_2), (u_1, v_2), (u_2, v_2), (u_5, v_2)\}$ cannot be efficiently open dominated by $F_6$, a contradiction.

Case 7: Either $(u_3, v_1), (u_3, v_2) \in F_6$ or $(u_4, v_1), (u_4, v_2) \in F_6$. Consequently, either $(u_5, v_1)$ or $(u_2, v_1)$, respectively, cannot be efficiently open dominated by $F_6$, a contradiction.
Now, if $r = 7$, then by using a similar cases analysis like in the constructive procedure of the set $F_6$ for $r = 6$, we obtain contradictions which lead to that $C_7 \Box C_t$ is not an efficient open domination graph. We leave the details to the reader. \qed

As a consequence of the above results we conjecture the following.

**Conjecture 4.5** Torus $C_r \Box C_t$, $r, t \geq 3$, is an efficient open domination graph if and only if $r, t \equiv 0 \pmod 4$.

### 4.2 The cylinder $P_r \Box C_t$

**Proposition 4.6** Cylinder $P_{2r+1} \Box C_{4t}$ is an efficient open domination graph for every $r, t \geq 1$.

**Proof.** The result follows immediately like in Theorem 4.2, from the fact that the set $F$ of vertices of $P_{2r+1} \Box C_{4t}$ given in the following way is an efficient open dominating set for $P_{2r+1} \Box C_{4t}$.

If $2r + 1 \equiv 1 \pmod 4$, then the set $F$ is the union of sets \( \{u_0, u_4, \ldots, u_{2r}\} \times \{v_0, v_1, v_4, v_5, \ldots, v_{4t-4}, v_{4t-3}\} \) and \( \{u_2, u_6, \ldots, u_{2r-2}\} \times \{v_2, v_3, v_6, v_7, \ldots, v_{4t-2}, v_{4t-1}\} \).

If $2r + 1 \equiv 3 \pmod 4$, then the set $F$ is the union of sets \( \{u_0, u_4, \ldots, u_{2r-2}\} \times \{v_0, v_1, v_4, v_5, \ldots, v_{4t-4}, v_{4t-3}\} \) and \( \{u_2, u_6, \ldots, u_{2r}\} \times \{v_2, v_3, v_6, v_7, \ldots, v_{4t-2}, v_{4t-1}\} \). \qed

**Proposition 4.7** Cylinder $P_r \Box C_4$, $r \geq 3$, is an efficient open domination graph if and only if $r$ is odd.

**Proof.** If $r$ is odd, then it follows by Proposition 4.6 that $P_r \Box C_4$ is an efficient open domination graph.

Now, suppose that cylinder $P_r \Box C_4$, $r \geq 4$, is an efficient open domination graph and let $F$ be an efficient open dominating set in $P_r \Box C_4$.

Clearly, at least one vertex of $u_0 C_4$ must be in $F$ and according to the symmetry of $P_r \Box C_4$, we can suppose, without loss of generality, that $(u_0, v_0) \in F$. If $(u_1, v_0) \in F$, then we have that the vertex $(u_0, v_2)$ cannot be efficiently open dominated by $F$, a contradiction.

Now, if $(u_0, v_1) \in F$, then $(u_2, v_2), (u_2, v_3) \in F$, and consequently only vertices of the type $(u_{2k}, v_2), (u_{2k}, v_3), (u_{4k}, v_0), (u_{4k}, v_1)$ belong to $F$, where $k, \ell$ are integers and $k$ is odd. Thus, if $r$ is even, then there exist two vertices $(u_{r-1}, v_i), (u_{r-1}, v_j), i \neq j$ and $i, j \in \{0, 1, 2, 3\}$, which cannot be efficiently open dominated by $F$, a contradiction. Therefore, $r$ is odd and the proof is complete. \qed

**Proposition 4.8**

(i) $P_r \Box C_3$ is an efficient open domination graph if and only if $r = 2$. 

11
(ii) $P_r \Box C_5$ is an efficient open domination graph if and only if $r = 4$.

(iii) $P_r \Box C_6$ is an efficient open domination graph if and only if $r = 2$.

(iv) $P_r \Box C_7$ is an efficient open domination graph if and only if $r = 6$.

**Proof.** It is easy to observe that $P_3 \Box C_3$, $P_4 \Box C_5$, $P_2 \Box C_6$ and $P_3 \Box C_7$ are efficient open domination graphs. Also, it is straightforward to check that if $P_r \Box C_3$ is an efficient open domination graph, then $r = 2$. So (i) is proved.

Now, suppose $P_r \Box C_5$ is an efficient open domination graph and let $F_5$ be an efficient open dominating set in $P_r \Box C_5$. Clearly, at least one vertex of $u_0 C_5$ must be in $F_5$ and according to the symmetry of $P_r \Box C_5$, we consider without loss of generality that $(u_0, v_0) \in F_5$ and $(u_0, v_1) \notin F_5$. If $(u_1, v_0) \in F_5$, then we vertices $(u_0, v_2)$ and $(u_0, v_3)$ cannot be efficiently open dominated by $F_5$, a contradiction. On the other hand, let $(u_0, v_1) \in F_5$. If $r = 2$, then $(u_0, v_3)$ is not efficiently dominated. If $r > 2$, then $(u_1, v_3), (u_2, v_3) \in F_5$. If $r = 3$, then $(u_2, v_0)$ and $(u_2, v_1)$ are not efficiently open dominated by $F_5$. Hence $r > 3$ and also $(u_3, v_0), (u_3, v_1) \in F_5$. Now, if $r > 4$, then one vertex of the set $\{(u_4, v_2), (u_4, v_3), (u_4, v_4)\}$ cannot be efficiently open dominated by $F_5$, a contradiction again. Thus, the only possible choice is $r = 4$ and (ii) is proved.

Assume $P_r \Box C_6$ is an efficient open domination graph and let $F_6$ be an efficient open dominating set in $P_r \Box C_6$. We proceed similarly to the above case. Consider $(u_0, v_0) \in F_6$ and $(u_0, v_5) \notin F_6$. If $(u_1, v_0) \in F_6$, then $(u_0, v_3), (u_1, v_3) \in F_6$. If $r > 2$, then two vertices of the set $\{(u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4), (u_2, v_5)\}$ cannot be efficiently open dominated by $F_6$, a contradiction. On the other hand, if $(u_0, v_1) \in F_6$, then $(u_1, v_3), (u_1, v_4) \in F_6$. Analogously, if $r > 2$, then two vertices of the set $\{(u_2, v_0), (u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4)\}$ cannot be efficiently open dominated by $F_6$, a contradiction. So, the only possible choice is $r = 2$ and (iii) is proved.

Finally, suppose $P_r \Box C_7$ is an efficient open domination graph and let $F_7$ be an efficient open dominating set in $P_r \Box C_7$. As above we consider $(u_0, v_0) \in F_7$ and $(u_0, v_6) \notin F_7$. If $(u_1, v_0) \in F_7$, then $(u_0, v_2)$ and $(u_0, v_5)$ can be efficiently open dominated only by $(u_0, v_3)$ and $(u_0, v_4)$, respectively. Also $r > 2$, since $(u_1, v_2)$ and $(u_1, v_5)$ are not efficiently open dominated yet. Now $(u_1, v_2)$ and $(u_1, v_5)$ can be efficiently open dominated only by $(u_2, v_2)$ and $(u_2, v_5)$, respectively, and these two with $(u_3, v_2)$ and $(u_3, v_5)$, respectively. Hence $r > 3$ and $r \neq 4$ since $(u_3, v_0)$ is not efficiently dominated. The vertex $(u_3, v_0)$ yields that $(u_4, v_0), (u_5, v_0) \in F_7$ and consequently $r \geq 6$. To continue, $(u_5, v_3), (u_5, v_4) \in F_7$ to efficiently open dominate $(u_4, v_3)$ and $(u_4, v_4)$. If $r = 6$, then we are done. If $r > 6$, then at least one vertex out of $\{(u_6, v_1), (u_6, v_2), (u_6, v_3), (u_6, v_6)\}$ cannot be efficiently open dominated by $F_7$, a contradiction. On the other hand, if $(u_0, v_1) \in F_7$, then $(u_0, v_4), (u_1, v_4) \in F_7$. If $r = 2$, then $(u_1, v_2)$ and $(u_1, v_6)$ are not dominated by $F_7$, a contradiction. Hence $r > 2$ and to efficiently open dominate these two vertices, $(u_2, v_2)$ and $(u_2, v_6)$, respectively, must be in $F_7$. Furthermore, to efficiently open dominate $(u_2, v_2)$ and $(u_2, v_6), (u_3, v_2)$ and $(u_3, v_6)$, respectively, must be in $F_7$. Hence $r > 3$ and if $r = 4$, then $(u_3, v_4)$ is not dominated by $F_7$. Thus, $r > 4$ and to efficiently open dominate $(u_3, v_4)$ we need $(u_4, v_4), (u_5, v_4) \in F_7$. Moreover, the vertices $(u_5, v_0), (u_5, v_1) \in F_7$ to efficiently open
dominate \((u_4, v_0)\) and \((u_4, v_1)\). If \(r = 6\), then we are done and if \(r > 6\), then at least one vertex of the set \(\{(u_6, v_2), (u_6, v_3), (u_6, v_5), (u_6, v_6)\}\) cannot be efficiently open dominated by \(F_7\), a contradiction. Therefore, the only possible choice is \(r = 6\) and (iv) is proved. □

4.3 The grid \(P_t □ P_t\)

Since \(G □ P_2\) is analyzed in Section 3 we already know that \(P_k □ P_2\) is an efficient open domination graph for every positive integer \(k\). Next we present more efficient open domination graphs among grid graphs.

**Theorem 4.9** Let \(t\) and \(k\) be positive integers where \(k ≥ 2t\) and let \(k ≡ x \pmod{2t + 1}\). If \(x \in \{1, 2t − 2, 2t\}\), then \(P_t □ P_k\) is an efficient open domination graph.

**Proof.** First we show inductively that \(P_{2t} □ P_{2t}\) and \(P_{2t} □ P_{2t+2}\) are efficiently open domination graphs for every \(t ≥ 1\). Clearly \(P_2 □ P_2 ≅ C_4\) is an efficient open domination graph. Also, it is easy to see that \(\{(u_0, v_0), (u_1, v_0), (u_0, v_3), (u_1, v_3)\}\) is an efficient open dominating set of \(P_2 □ P_4\) and the basis of induction is settled. Notice that in the procedure we use implicitly that \(P_{2t+4} □ P_{2t+2} ≅ P_{2t+2} □ P_{2t+4}\).

Suppose that \(A\) is an efficient open dominating set of \(G = P_{2t} □ P_{2t+2}\) obtained by this induction procedure. We will, roughly speaking, extend \(A\) from \(G\) to \(P_{2t+2} □ P_{2t+2}\) or \(P_{2t+4} □ P_{2t+2}\) by adding two additional vertices to \(P_{2t}\) on one or on both sides, respectively. Clearly, \(G\) is isomorphic to the subgraph \(H\) of \(P_{2t+4} □ P_{2t+2}\) induced by all its vertices with the exception of vertices lying in layers \(u_0 P_{2t+2}, u_1 P_{2t+2}, u_{2t+2} P_{2t+2},\) and \(u_{2t+3} P_{2t+2}\). Let \(A'\) be a subset of \(V(H) − \{u_i P_{2t+2} : i ∈ \{0, 1, 2t + 2, 2t + 3\}\}\) where vertices of \(A'\) correspond to vertices of \(A\) in \(G\). If \(t\) is odd, we set

\[
A_0 = \{(u_0, v_1), (u_0, v_2), (u_0, v_5), (u_0, v_6), \ldots, (u_0, v_{2t}), (u_0, v_{2t+1})\}
\]

and

\[
A_{2t+3} = \{(u_{2t+3}, v_1), (u_{2t+3}, v_2), (u_{2t+3}, v_5), (u_{2t+3}, v_6), \ldots, (u_{2t+3}, v_{2t}), (u_{2t+3}, v_{2t+1})\}.
\]

If \(t\) is even, then let

\[
A_0 = \{(u_0, v_0), (u_0, v_1), (u_0, v_4), (u_0, v_5), \ldots, (u_0, v_{2t}), (u_0, v_{2t+1})\}
\]

and

\[
A_{2t+3} = \{(u_{2t+3}, v_0), (u_{2t+3}, v_1), (u_{2t+3}, v_4), (u_{2t+3}, v_5), \ldots, (u_{2t+3}, v_{2t}), (u_{2t+3}, v_{2t+1})\}.
\]

In both cases \(A' ∪ A_0\) and \(A' ∪ A_0 ∪ A_{2t+3}\) are efficient open dominating sets of \(P_{2t+2} □ P_{2t+2}\) and \(P_{2t+4} □ P_{2t+2}\), respectively, which ends the induction.

Let now \(k ≥ 2t\) and let \(k ≡ x \pmod{2t + 1}\) where \(x ∈ \{1, 2t − 2, 2t\}\). If \(k ∈ \{2t, 2t+2\}\) we are done with \(P_{2t} □ P_k\) according to this induction procedure. Let \(B\) be an efficient open dominating set of \(P_{2t} □ P_{2t+2}\) obtained by the above induction. Notice that \(B\) is
symmetric with respect to the first and last layer of \( P_{2t+2} \). Hence, if \( k \equiv 1 \pmod{2t+1} \), then we can continue the pattern of \( B \) on every part between the \( u_{i(2t+1)} P_k \)-layer and the \( u_{(i+1)(2t+1)} P_k \)-layer of \( P_2 \square P_k \) where \( i \in \{0, \ldots, \frac{k-1}{2t+1} - 1\} \). See Figures 1 b) and 2 b) for \( P_4 \square P_k \) and \( P_6 \square P_k \), respectively. Therefore, \( P_{2t} \square P_k \) is an efficient open domination graph for \( k \equiv 1 \pmod{2t+1} \).

To finish the proof just notice that in the above construction for \( P_{2t} \square P_k \) where \( k \equiv 1 \pmod{2t+1} \), we can delete first and/or last two layers of \( P_k \) and the remaining graph is still an efficient open domination graph. If we delete first two layers, then we obtain \( P_{2t} \square P_{k-2} \) where \( k - 2 \equiv 2t \pmod{2t+1} \) and if we delete the first two and the last two layers, then we obtain \( P_{2t} \square P_{k-4} \) where \( k - 4 \equiv 2t - 2 \pmod{2t+1} \). Again see Figures 1 b) and 2 b) for \( P_4 \square P_k \) and \( P_6 \square P_k \), respectively. Therefore the proof is completed.

The total domination number of grid graphs attract some attention in the past decade. First in [5] formulae for \( \gamma_t(P_r \square P_t) \), \( r \in \{1, 2, 3, 4\} \) have been derived. This work was continued in [9] for \( r \in \{5, 6\} \). Unfortunately there was a partial mistake in [9] for \( P_6 \square P_t \), which was corrected recently in [10]. In view of Observation 1.1 we have constructed \( \gamma_t(P_{2t} \square P_k) \)-sets for \( k \geq 2t \), \( k \equiv x \pmod{2t+1} \) and \( x \in \{1, 2t, 2t+1\} \) in the proof of Theorem 4.11.

With respect to Theorem 4.11 we pose the following conjecture:

**Conjecture 4.10** A grid graph \( P_t \square P_k \), \( k \geq t \geq 3 \), is an efficient open domination graph if and only if \( t \) is an even number and \( k \equiv x \pmod{2t+1} \) for \( x \in \{1, t-2, t\} \).

In the last results of this paper we prove this conjecture for small integers \( r \) and \( t \).

**Proposition 4.11** For every \( t \geq 4 \), \( P_4 \square P_t \) is an efficient open domination graph if and only if \( t \equiv x \pmod{5} \) and \( x \in \{1, 2, 4\} \).

**Proof.** We proceed in a constructive way assuming that \( P_4 \square P_t \) is an efficient open domination graph. Since at least one vertex of the \( u_0 P_t \)-layer belongs to an efficient open dominating set of \( P_4 \square P_t \) we have that only the patterns presented in Figure 1 and the corresponding symmetric ones, can be realized as a possible efficient open dominating set of \( P_4 \square P_t \) (white vertices represent the efficient open dominating set). Notice that the pattern on \( P_4 \square P_t \) is forced by the starting position in the first layer \( P_4^0 \). The possible ends of the patterns having a valid efficient open dominating set are represented in double dotted vertical lines and those ones are precisely when \( t \equiv x \pmod{5} \) and \( x \in \{1, 2, 4\} \), since all patterns have a repetition after five vertices of \( P_t \).

**Proposition 4.12** For every \( t \geq 6 \), \( P_6 \square P_t \) is an efficient open domination graph if and only if \( t \equiv x \pmod{7} \) and \( x \in \{1, 4, 6\} \).

**Proof.** We proceed analogously to the proof of Proposition 4.11. Notice that only the patterns presented in Figure 2 and the corresponding symmetric ones, can be realized
as a possible efficient open dominating set of $P_t \square P_t$. The possible ends of the patterns having a valid efficient open dominating set are represented in double dotted vertical lines and those ones are precisely when $t \equiv x \pmod{7}$ and $x \in \{1, 4, 6\}$, since all patterns have a repetition after seven vertices of $P_t$. □

Proposition 4.13 Let $r \in \{3, 5, 7\}$. For every $t \geq r$, $P_r \square P_t$ is not an efficient open domination graph.

Proof. It is straightforward to see that $P_3 \square P_t$ is not an efficient open domination graph for every $t \geq 3$.

Let now $r = 5$. Since at least one vertex of the $u_0 P_t$-layer belongs to an efficient open dominating set $F$ of $P_5 \square P_t$, we have the next cases (we avoid the symmetric cases since they are analogous). Suppose $(u_0, v_0) \in F$. We consider the following.
Case 1: If \((u_0, v_1) \in F\), then \((u_2, v_0)\) must be efficiently open dominated by \((u_3, v_0)\) and we have either \((u_4, v_0) \in F\) or \((u_3, v_1) \in F\). If \((u_4, v_0) \in F\), then \((u_2, v_2), (u_2, v_3) \in F\) to efficiently open dominate vertex \((u_2, v_1)\). But then \((u_4, v_2)\) is not efficiently open dominated by \(F\), a contradiction. If \((u_3, v_1) \in F\), then \((u_1, v_3), (u_2, v_3) \in F\) to efficiently open dominate vertices \((u_1, v_2)\) and \((u_2, v_2)\). Again \((u_4, v_2)\) is not efficiently open dominated by \(F\), a contradiction.

Case 2: If \((u_1, v_0) \in F\), then \((u_3, v_1), (u_4, v_1) \in F\) to efficiently open dominate vertices \((u_3, v_0)\) and \((u_4, v_0)\). At least one of the vertices of the set \{\((u_0, v_2), (u_1, v_2), (u_2, v_2)\)\} is not efficiently open dominated by \(F\), a contradiction.

Now, suppose that \((u_0, v_0) \notin F\) and \((u_0, v_1) \in F\). If \((u_1, v_1) \in F\), then it is equivalent to Case 2. If \((u_0, v_2) \in F\), then \((u_2, v_0), (u_3, v_0) \in F\) to efficiently open dominate \((u_1, v_0)\). Also \((u_4, v_2), (u_4, v_3) \in F\) to efficiently open dominate \((u_4, v_1)\). But then the vertex \((u_2, v_2)\) cannot be efficiently open dominated by \(F\), a contradiction.

Finally, suppose that \((u_0, v_0), (u_0, v_1) \notin F\). Hence \((u_1, v_0) \in F\) to dominate \((u_0, v_0)\). If either \((u_2, v_0)\) or \((u_1, v_1)\) is in \(F\) to dominate \((u_1, v_0)\), we get a symmetry to one of the above cases. Thus \(P_t \boxtimes P_t, t \geq 5\), cannot be efficiently open dominated by \(F\).

By a similar, but more tedious, case analysis like for \(r = 5\) we can prove the result for \(r = 7\). The details are left to the reader. \(\square\)

References


