Uplink-Downlink Duality For Integer-Forcing in Cloud Radio Access Networks

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***Abstract*—Consider a cloud radio access network where neighboring basestations are able to jointly encode and decode their signals via rate-limited links to a central processor. One promising approach to such systems is for the basestations to simply quantize their observations and send them to the central processor (during the uplink phase) and to emit the quantized signals generated by the central processor (during the downlink phase). Several recent works have proposed compression-based architectures based on sequential source and channel coding. In prior work, we proposed an integer-forcing architecture for the uplink phase, and, in this paper, we propose an integer-forcing architecture for the downlink phase. As part of the achievability argument, we introduce a novel "reverse" integer-forcing source coding strategy that can be used to quantize sources so that their quantization noises are correlated, i.e., multivariate compression. We also establish uplink-downlink duality between our uplink and downlink integer-forcing architectures, and use this as the basis for optimizing the beamforming, equalization, and integer matrices.**

I. Introduction

Cloud-Radio Access Networks (C-RAN) aim to better utilize the limited resources available in cellular networks by employing joint signal processing and coding techniques at a central processor (CP) rather than relying on local encoding and decoding at each basestation (BS) [1]. Although joint signal processing and coding techniques are by now well understood, C-RANs face the additional obstacle that the links between the BSs and the CP are rate-limited, and thus some form of relaying is necessary. During the uplink phase, the users emit codewords, the BSs observe the resulting channel outputs and communicate with the CP, which decodes the users' messages. During the downlink phase, the CP has messages for the users and communicates with the BSs, which then emit signals that are observed and decoded by the users. In prior work, we proposed an end-to-end integer-forcing (IF) architecture for the uplink phase [2] and, in this paper, we propose an IF architecture for the downlink phase as well as establish uplink-downlink duality.

Several recent works have proposed downlink C-RAN architectures, including data-sharing of users' messages with some of the BSs prior to encoding [3], precoding the data digitally before sharing it with the BSs [4] (e.g., reverse compute-and-forward) or compressing the data after encoding and forwarding it to the BSs [5]. In this paper, we focus on compression-based schemes, where the CP generates codes, applies a beamforming matrix, and then quantizes the resulting analog signals so they can be relayed to the BSs through the backhaul links. The intuitive way to do this is to employ single-user encoders to compress each codeword and forward it to the desired BS. However, one can employ more sophisticated quantization strategies (such as joint typicality or sequential encoding) to introduce correlation between the quantization noises across BSs [5]. The correlation can then be shaped so that, after passing through the channel, the effective quantization noise seen by each user is minimized. It has been shown that compression-based strategies based on simultaneous joint typicality encoding and decoding can operate within a constant gap of the capacity region [6].

Here, we propose an end-to-end IF architecture for downlink C-RANs that combines integer-forcing beamforming for broadcast channel coding and "reverse" integer-forcing source coding for multivariate compression. Integer-forcing beamforming, as introduced in [7], steers the channel matrix towards an integer matrix, and precodes the messages by applying the inverse integer matrix, so that by decoding a linear combination, each user obtains its desired codeword. A similar idea underpins reverse integer-forcing source coding, which we propose in this paper: the encoder first applies the inverse integer matrix to the sources, then applies a lattice quantizer to each effective source, and finally takes linear combinations according to the integer matrix. This yields quantized versions of the original sources that have correlated quantization noises, without the need for joint typicality or sequential encoding. Overall, our architecture generates the signals to be transmitted via integer-forcing beamforming and quantizes them for the BSs via reverse integer-forcing source coding. We also establish that our downlink IF scheme satisfies uplink-downlink duality with the uplink IF scheme we introduced in [2]. That is, by transposing and exchanging the roles of the beamforming and equalization matrices and transposing the integer matrix, we can attain the same sum rate on the downlink as on the uplink (and vice versa). One immediate application for this duality relationship is to optimize the parameters of the downlink C-RAN using algorithms originally developed to optimize uplink C-RAN parameters.

The paper is organized as follows. Section II includes

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some necessary lattice preliminaries. Section III introduces the uplink channel and overviews our prior working on IF for uplink C-RAN. Section IV introduces the downlink channel and our IF scheme for downlink C-RAN, including the reverse integer-forcing source coding scheme. Section V establishes uplink-downlink duality and Section VI utilizes this duality relationship as part of an optimization algorithm. Finally, Section VII compares the performance of our scheme to competing strategies via simulations.

We denote column vectors by boldface lowercase (e.g., $\mathbf{x}$) and matrices by boldface uppercase (e.g., $\mathbf{X}$). Let $\mathbf{X}^\dagger$ denote the transpose of a matrix $\mathbf{X}$ and let $\mathbf{X}_{\mathcal{A},\mathcal{B}}$ be the matrix composed of the rows and columns of $\mathbf{X}$ with indices in the sets $\mathcal{A}$ and $\mathcal{B}$, respectively. When $\mathcal{A}=\mathcal{B}$, we write $\mathbf{X}_{\mathcal{A},\mathcal{B}}$ as $\mathbf{X}_{\mathcal{A}}$. We denote by $\mathbb{D}^L$, the set of square diagonal matrices of size L. Define $\log^+(x) \triangleq \max(0, \log(x))$. For simplicity, we focus on real-valued channels.[1] We use the superscripts 'ul' and 'dl' to denote symbols defined for the uplink and downlink channels, respectively.

II. Lattice Preliminaries

A lattice is a discrete additive subgroup of $\mathbb{R}^T$ that is closed under addition and reflection. The lattice quantizer maps any point in $\mathbb{R}^T$ to the nearest point in Λ, i.e.,

$$\mathcal{Q}_\Lambda(\mathbf{x}) \triangleq \underset{\boldsymbol{\lambda}\in\Lambda}{\arg\min} \|\mathbf{x}-\boldsymbol{\lambda}\|^2,$$

which in turns defines the fundamental Voronoi region $\mathcal{V}(\Lambda)$ as the set of points in $\mathbb{R}^T$ that quantize to the zero vector with ties broken in a systematic way. The $\bmod\ \Lambda$ operator returns the lattice quantization error

$$[\mathbf{x}] \bmod \Lambda \triangleq \mathbf{x} - \mathcal{Q}_\Lambda(\mathbf{x}).$$

The second moment of a lattice is

$$\sigma^2(\Lambda) \triangleq \frac{1}{T}\mathbb{E}\|\mathbf{x}\|^2$$

for $\mathbf{x} \sim \text{Unif}\left(\mathcal{V}\left(\Lambda\right)\right)$.

Lemma 1 (Crypto Lemma): For any real vector $\mathbf{y} \in \mathbb{R}^T$ and a dither $\mathbf{u} \sim \text{Unif}(\mathcal{V}(\Lambda))$ independent of $\mathbf{y}$, we have that $\mathbf{q} = [\mathbf{y}+\mathbf{u}] \bmod \Lambda$ is independent of $\mathbf{y}$ and $\mathbf{q} \sim \text{Unif}(\mathcal{V}(\Lambda))$. See [9] for a proof.

For nested lattices $\Lambda_C \subset \Lambda_F$, the associated nested lattice codebook $\Lambda_F \cap \mathcal{V}(\Lambda_C)$ consists of all of the fine lattice points that fall inside the fundamental Voronoi region of the coarse lattice. Recall that nested lattices satisfy a distributive law: for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^T$,

$$\begin{aligned}&[a\,[\mathbf{x}] \bmod \Lambda_C + b\,[\mathbf{y}] \bmod \Lambda_C] \bmod \Lambda_F \\ &= [a\mathbf{x} + b\mathbf{y}] \bmod \Lambda_F,\ \forall a, b \in \mathbb{Z}\end{aligned}$$

The following theorem restates nested lattice existence results from [10] in a form suitable for establishing our integer-forcing achievability results.

Lemma 2 ([10, Theorem 2]): For $\theta_1, \ldots, \theta_K, \epsilon > 0$, T large enough, there exists a nested lattice chain $\Lambda_K \subseteq \ldots \subseteq \Lambda_1$ (generated using Construction A from a p-ary linear code for a prime p) such that

1. $\theta_k \leq \sigma^2(\Lambda_k) < \theta_k + \epsilon$,
2. For any $\mathbf{z}_{\text{eff}} = \beta_0 \mathbf{z}_0 + \sum_{k=1}^{K} \beta_n \mathbf{z}_k$ where $\beta_0, \ldots, \beta_K \in \mathbb{R}$, $\mathbf{z}_0 \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, $\mathbf{z}_k \sim \text{Unif}(\mathcal{V}(\Lambda_k))$ and if $\beta_0^2 + \sum_{k=1}^{K} \beta_k^2 \theta_k < \theta_m$ then $\Pr\left([\mathbf{z}_{\text{eff}}] \bmod \Lambda_m \neq \mathbf{z}_{\text{eff}}\right) \leq \epsilon$,
3. For $\Lambda_\ell \subseteq \Lambda_m$ we have
$$\frac{1}{2}\log\left(\frac{\theta_\ell}{\theta_m}\right) \leq \frac{1}{T}\log\left|\mathcal{V}(\Lambda_\ell)\cap\Lambda_m\right| < \frac{1}{2}\log\left(\frac{\theta_\ell}{\theta_m}\right) + \epsilon.$$

III. Uplink C-RAN

A. Uplink Channel Model

Consider an uplink C-RAN, where we have a set $\mathcal{K} \triangleq \{1, \ldots, K\}$ of single-antenna users that want to communicate to a set $\mathcal{L} \triangleq \{1, \ldots, L\}$ of single-antenna BSs. The k^{th} user has a message $w_k^{ul} \in \left\{1, \ldots, 2^{TR_k^{ul}}\right\}$ with rate R_k^{ul}. The BSs are connected to a CP through noiseless backhaul links with finite sum-rate C_{tot}. The received signal at the BSs is given by

$$\mathbf{Y}^{ul} = \mathbf{H}^{ul}\mathbf{X}^{ul} + \mathbf{Z}^{ul}, \tag{1}$$

where $\mathbf{Y}^{ul} \triangleq [\mathbf{y}_1^{ul} \ \cdots \ \mathbf{y}_L^{ul}]^\dagger$, $\mathbf{y}_\ell^{ul} \in \mathbb{R}^T$ is the received signal at the ℓ^{th} BS, T is the blocklength, $\mathbf{H}^{ul} \in \mathbb{R}^{L\times K}$ is the channel from all users to all BSs, $\mathbf{X}^{ul} \triangleq [\mathbf{x}_1^{ul} \ \cdots \ \mathbf{x}_K^{ul}]^\dagger$, $\mathbf{x}_k^{ul} \in \mathbb{R}^T$ is the signal transmitted from the k^{th} user and $\mathbf{Z}^{ul} \in \mathbb{R}^{L\times T}$ is i.i.d. $\mathcal{N}(0,1)$ noise. The matrix $\mathbf{X}^{ul}$ satisfies a total power constraint[2] $\frac{1}{T}\mathbb{E}\text{Tr}\left(\mathbf{X}^{ul}\mathbf{X}^{ul\dagger}\right) \leq P_{\text{total}}$.

The k^{th} user encodes its message w_k^{ul} into a codeword $\mathbf{s}_k^{ul}$, then transmits $\mathbf{x}_k^{ul} = v_k^{ul}\mathbf{s}_k^{ul}$. For convenience, let us define $\mathbf{X}^{ul} = \mathbf{V}^{ul}\mathbf{S}^{ul}$ where $\mathbf{V}^{ul} = \text{diag}(v_1^{ul}, \ldots, v_K^{ul})$ is the beamforming matrix and $\mathbf{S}^{ul} = [\mathbf{s}_1^{ul} \ \cdots \ \mathbf{s}_K^{ul}]^\dagger$ is the codeword matrix. The total power constraint becomes

$$\text{Tr}\left(\mathbf{V}^{ul}\mathbf{P}^{ul}\mathbf{V}^{ul\dagger}\right) \leq P_{\text{total}} \tag{2}$$

where $\mathbf{P}^{ul} \triangleq \frac{1}{T}\mathbb{E}(\mathbf{S}^{ul}\mathbf{S}^{ul\dagger})$ is a diagonal coding power matrix.

B. Integer-Forcing for Uplink C-RAN

We begin with an overview of the integer-forcing scheme proposed for uplink C-RAN in [2]. Without loss of generality, we assume for both the source and channel coding stages that the identity permutation is admissible [11, Definition 2], which will impose constraints on the effective noises and integer matrices.

Uplink Source Coding. The ℓ^{th} BS uses a lattice codebook $\mathcal{C}_\ell \triangleq \Lambda_{F,\ell}\cap\mathcal{V}(\Lambda_{C,\ell})$ with rate C_ℓ^{ul} to quantize its observation $\mathbf{y}_\ell^{ul}$,

$$\boldsymbol{\lambda}_\ell^{ul} = \left[\mathcal{Q}_{F,\ell}(\mathbf{y}_\ell^{ul} + \mathbf{u}_\ell^{ul})\right] \bmod \Lambda_{C,\ell} \tag{3}$$

[1]Note that complex-valued channels can be handled via their real-valued decompositions [8].

[2]Although standard uplink channel models employ individual power constraints on the users, a total power constraint is necessary here to enable us to establish uplink-downlink duality.

where $\mathbf{u}_\ell^{ul}$ is a random dither independent of $\mathbf{y}_\ell^{ul}$ and uniformly distributed over $\mathcal{V}(\Lambda_{F,\ell})$, $\Lambda_{C,L} \subseteq \cdots \subseteq \Lambda_{C,1}$ are L nested coarse lattices nested in L fine lattices $\Lambda_{F,1} \subseteq \cdots \subseteq \Lambda_{F,L}$ and all lattices are chosen according to Lemma 2. The ℓ^{th} BS then forwards the index $i_\ell^{ul} \in \left\{1, \ldots, 2^{TC_\ell^{ul}}\right\}$ of $\boldsymbol{\lambda}_\ell^{ul}$ to the CP through the backhaul link.

Upon receiving the indices $i_1^{ul}, \ldots, i_L^{ul}$, the CP first recovers $\boldsymbol{\lambda}_1^{ul}, \ldots, \boldsymbol{\lambda}_L^{ul}$, removes the dithers $\mathbf{u}_1^{ul}, \ldots, \mathbf{u}_L^{ul}$ to get

$$\begin{aligned} \widetilde{\mathbf{y}}_\ell^{ul} &= \left[\boldsymbol{\lambda}_\ell^{ul} - \mathbf{u}_\ell^{ul}\right] \bmod \Lambda_{C,\ell} \\ &\overset{(a)}{=} \left[\mathbf{y}_\ell^{ul} + \mathbf{q}_\ell^{ul}\right] \bmod \Lambda_{C,\ell} \\ &= \left[\widehat{\mathbf{y}}_\ell^{ul}\right] \bmod \Lambda_{C,\ell} \end{aligned} \tag{4}$$

where $\mathbf{q}_\ell^{ul} = -[\mathbf{y}_\ell^{ul} + \mathbf{u}_\ell^{ul}] \bmod \Lambda_{F,\ell} \sim \text{Unif}(\mathcal{V}(\Lambda_{F,\ell}))$, (a) holds from the distributive law and Lemma 1 and $\widehat{\mathbf{y}}_\ell^{ul} \triangleq \mathbf{y}_\ell^{ul} + \mathbf{q}_\ell^{ul}$, then the CP proceeds to decode integer-linear combinations $\mathbf{v}_{s,1}^{ul}, \ldots, \mathbf{v}_{s,L}^{ul}$ where

$$\mathbf{v}_{s,m}^{ul} \triangleq \sum_{\ell=1}^{L} a_{s,m,\ell}^{ul} \widehat{\mathbf{y}}_\ell^{ul}, \ \forall m \in \mathcal{L}, \ a_{s,m,\ell}^{ul} \in \mathbb{Z}.$$

At the m^{th} decoding step (i.e., while recovering $\mathbf{v}_{s,m}^{ul}$) and assuming correct recovery of $\mathbf{v}_{s,1}^{ul}, \ldots, \mathbf{v}_{s,m-1}^{ul}$, the CP first recovers $\mathbf{t}_{m,\ell}^{ul} \triangleq \left[\mathbf{y}_\ell^{ul} + \mathbf{q}_\ell^{ul}\right] \bmod \Lambda_{C,m}, \ \forall \ell = 1, \ldots, L$ using [2, Lemma 12] as an intermediate decoding point.

Upon recovering $\mathbf{t}_{m,1}^{ul}, \ldots, \mathbf{t}_{m,L}^{ul}$, the CP makes the estimate

$$\begin{aligned} \widehat{\mathbf{v}}_{s,m}^{ul} &= \left[\sum_{\ell=1}^{L} a_{s,m,\ell}^{ul} \mathbf{t}_{m,\ell}^{ul}\right] \bmod \Lambda_{C,m} \\ &\overset{(a)}{=} \left[\sum_{\ell=1}^{L} a_{s,m,\ell}^{ul} \widehat{\mathbf{y}}_\ell^{ul}\right] \bmod \Lambda_{C,m} \overset{\text{w.h.p.}}{=} \sum_{\ell=1}^{L} a_{s,m,\ell}^{ul} \widehat{\mathbf{y}}_\ell^{ul} \end{aligned} \tag{5}$$

where (a) holds from the distributive law and (b) holds w.h.p. from Lemma 2 if $\frac{1}{T}\mathbb{E}\|\mathbf{v}_{s,m}^{ul}\|^2 < \sigma^2(\Lambda_{C,m})$.

Lemma 3 ([2, Lemma 9]): For a fixed channel $\mathbf{H}^{ul}$, beamforming matrix $\mathbf{V}^{ul}$, coding power matrix $\mathbf{P}^{ul}$, effective covariance matrix $\mathbf{K}_{YY}^{ul} \triangleq \mathbf{H}^{ul}\mathbf{V}^{ul}\mathbf{P}^{ul}\mathbf{V}^{ul\dagger}\mathbf{H}^{ul\dagger} + \mathbf{I}$ and by choosing a full-rank integer matrix $\mathbf{A}_s^{ul}$ with full-rank submatrices[3] $\mathbf{A}_{s,[1:m]}^{ul}$ for $m = 1, \ldots, L$ and target distortion levels $d_1^{ul}, \ldots, d_L^{ul}$ such that

$$R_{s,\ell}^{ul} \triangleq \frac{1}{2} \log \left(\frac{\mathbf{a}_{s,\ell}^{ul\dagger} \left(\mathbf{K}_{YY}^{ul} + \mathbf{D}^{ul}\right) \mathbf{a}_{s\ell}^{ul}}{d_\ell^{ul}} \right)$$

$$\sum_{\ell=1}^{L} R_{s,\ell}^{ul} \leq C_{\text{tot}}$$

$$\mathbf{a}_{s,1}^{ul\dagger}(\mathbf{K}_{YY}^{ul} + \mathbf{D}^{ul})\mathbf{a}_{s,1}^{ul} < \cdots < \mathbf{a}_{s,L}^{ul\dagger}(\mathbf{K}_{YY}^{ul} + \mathbf{D}^{ul})\mathbf{a}_{s,L}^{ul}, \tag{6}$$

the integer-forcing source coding scheme allows the CP to recover

$$\widehat{\mathbf{Y}}^{ul} = \mathbf{H}^{ul}\mathbf{V}^{ul}\mathbf{S}^{ul} + \mathbf{Z}^{ul} + \mathbf{Q}^{ul} \tag{7}$$

[3]Without loss of generality, we may assume that the BSs have been re-indexed so that this assumption holds.

with high probability where $\widehat{\mathbf{Y}}^{ul} \triangleq [\widehat{\mathbf{y}}_1^{ul} \ \cdots \ \widehat{\mathbf{y}}_L^{ul}]^\dagger$, $\mathbf{Q}^{ul} \triangleq [\mathbf{q}_1^{ul} \ \cdots \ \mathbf{q}_L^{ul}]^\dagger$ has effective covariance matrix $\mathbf{D}^{ul} \triangleq \frac{1}{T}\mathbb{E}(\mathbf{Q}^{ul}\mathbf{Q}^{ul\dagger}) = \text{diag}(d_1^{ul}, \ldots, d_L^{ul})$ and $\mathbf{a}_{s,\ell}^{ul\dagger}$ is the ℓ^{th} row of $\mathbf{A}_s^{ul}$.

Uplink Channel Coding. The users draw their codewords $\mathbf{s}_1^{ul}, \ldots, \mathbf{s}_K^{ul}$ from nested lattice codebooks, which are selected via Lemma 2. After reconstructing the quantized BS observations $\widehat{\mathbf{Y}}^{ul}$, the CP proceeds to successively decode integer-linear combinations $\mathbf{v}_{c,1}^{ul}, \ldots, \mathbf{v}_{c,K}^{ul}$ of channel codewords $\mathbf{s}_1^{ul}, \ldots, \mathbf{s}_K^{ul}$ where

$$\mathbf{v}_{c,m}^{ul} \triangleq \sum_{k=1}^{K} a_{c,m,k}^{ul} \mathbf{s}_k^{ul}, \ \forall m \in \mathcal{K}, \ a_{c,m,k} \in \mathbb{Z}.$$

At the m^{th} channel decoding step (i.e., when decoding $\mathbf{v}_{c,m}^{ul}$) and assuming correct decoding of $\mathbf{v}_{c,1}^{ul}, \ldots, \mathbf{v}_{c,m-1}^{ul}$, the CP first employs a linear equalizer $\mathbf{b}_m^{ul}$ to obtain

$$\begin{aligned} &\mathbf{b}_m^{ul\dagger}\widehat{\mathbf{Y}}^{ul} \\ &= \mathbf{v}_{c,m}^{ul\dagger} + \underbrace{(\mathbf{b}_m^{ul\dagger}\mathbf{H}^{ul}\mathbf{V}^{ul} - \mathbf{a}_{c,m}^{ul\dagger})\mathbf{S}^{ul} + \mathbf{b}_m^{ul\dagger}\mathbf{Z}^{ul} + \mathbf{b}_m^{ul\dagger}\mathbf{Q}^{ul}}_{\mathbf{z}_{\text{eff},m}^{ul\dagger}} \end{aligned}$$

where the effective noise $\mathbf{z}_{\text{eff},m}^{ul}$ has an effective variance

$$\begin{aligned} (\sigma_m^{ul})^2 &\triangleq \frac{1}{T}\mathbb{E}\|\mathbf{z}_{\text{eff},m}^{ul}\|^2 \qquad (8) \\ &= \|(\mathbf{b}_m^{ul\dagger}\mathbf{H}^{ul}\mathbf{V}^{ul} - \mathbf{a}_{c,m}^{ul\dagger})\mathbf{P}^{ul\frac{1}{2}}\|^2 + \|\mathbf{b}_m^{ul}\|^2 + \mathbf{b}_m^{ul\dagger}\mathbf{D}^{ul}\mathbf{b}_m^{ul}. \end{aligned}$$

It can be shown [12, Lemma 13] that the achievable end-to-end sum-rate for this IF strategy is

$$R_{\text{IF-CRAN}}^{ul}(\mathbf{H}^{ul}) = \max_{\substack{\mathbf{B}^{ul} \in \mathbb{R}^{K \times L}, \mathbf{A}_c^{ul} \in \mathbb{Z}^{K \times K} \\ \text{rank}(\mathbf{A}_c^{ul}) = K}} \sum_{k=1}^{K} \frac{1}{2} \log^+ \left(\beta_k^{ul}\right) \tag{9}$$

where $\beta_k^{ul} = P_k^{ul}/(\sigma_k^{ul})^2$ denotes the k^{th} effective SINR and the k^{th} effective variance $(\sigma_k^{ul})^2$ is given by (8).

Finally, it is worth noting that the MMSE equalizer that minimizes (8) and the corresponding variance are given by

$$\mathbf{b}_m^{ul\dagger} = \mathbf{a}_{c,m}^{ul\dagger}\mathbf{P}^{ul\dagger}\mathbf{V}^{ul\dagger}\mathbf{H}^{ul\dagger}\left(\mathbf{H}^{ul}\mathbf{V}^{ul}\mathbf{P}^{ul}\mathbf{V}^{ul\dagger}\mathbf{H}^{ul\dagger} + \mathbf{I} + \mathbf{D}^{ul}\right)^{-1} \tag{10}$$

$$(\sigma_m^{ul})^2 = \|\mathbf{F}_c^{ul}\mathbf{a}_{c,m}^{ul}\|^2 \tag{11}$$

where $\mathbf{F}_c^{ul}$ is any matrix that satisfies

$$\mathbf{F}_c^{ul\dagger}\mathbf{F}_c^{ul} = \left((\mathbf{P}^{ul})^{-1} + \mathbf{V}^{ul\dagger}\mathbf{H}^{ul\dagger}\left(\mathbf{I} + \mathbf{D}^{ul}\right)^{-1}\mathbf{H}^{ul}\mathbf{V}^{ul}\right)^{-1}. \tag{12}$$

Lemma 4: For an uplink channel $\mathbf{H}^{ul}$, beamforming matrix $\mathbf{V}^{ul}$, coding power matrix $\mathbf{P}^{ul}$, coding power vector $\boldsymbol{\rho}^{ul} = \text{diag}(\mathbf{P}^{ul})$ and equalization matrix $\mathbf{B}^{ul}$, we can write (8) as

$$\left(\mathbf{I} - \text{diag}(\boldsymbol{\beta}^{ul})\mathbf{M}^{ul}\right)\boldsymbol{\rho}^{ul} = \mathbf{J}^{ul}\boldsymbol{\beta}^{ul} \tag{13}$$

where $\boldsymbol{\beta}^{ul} = [\beta_1^{ul} \cdots \beta_K^{ul}]^\dagger$, $\beta_k^{ul} = P_k^{ul}/(\sigma_k^{ul})^2$ is the k^{th} effective SINR, $\mathbf{J}^{ul} = \text{diag}(J_1^{ul}, \ldots, J_K^{ul})$, $J_k^{ul} = \|\mathbf{b}_k^{ul}\|^2 + \sum_i \sum_j (b_{k,i}^{ul})^2 C_{i,j}^{ul} \|\mathbf{a}_{s,j}^{ul}\|^2$, $M_{k,\ell}^{ul} = (\mathbf{b}_k^{ul\dagger}\mathbf{h}_\ell^{ul}v_\ell^{ul} - a_{c,k,\ell}^{ul})^2 + \sum_i \sum_j (b_{k,i}^{ul})^2 C_{i,j}^{ul} (\mathbf{a}_{s,j}^{ul\dagger}\mathbf{h}_\ell^{ul})^2 (v_\ell^{ul})^2$ is the $(k,\ell)^{\text{th}}$ element of $\mathbf{M}^{ul}$ and $\mathbf{h}_\ell^{ul}$ is the ℓ^{th} column of $\mathbf{H}^{ul}$.

The proof of Lemma 4 is given in Appendix A.

IV. Downlink C-RAN

In the downlink channel, we employ the basic idea from reverse compute-and-forward [4]: the encoder first applies the inverse integer matrix so that, when an individual user (or BS) obtains a linear combination of codewords, it can directly obtain its desired message (or quantized source). For the source coding phase, we introduce an integer-forcing multivariate compression scheme that allows us to create correlated quantization noises, which ultimately help lower the effective noise variances seen at the users.

A. Downlink Channel

Consider the downlink channel where there is a CP connected to a set $\mathcal{L}$ of BSs through backhaul links with sum-rate C_{tot}. The CP wants to communicate K messages $w_k^{dl} \in \{1, \ldots, 2^{TR_k^{dl}}\}$ with rate R_k^{dl}, for $k \in \mathcal{K}$, to a set of users $\mathcal{K}$ where the k^{th} user is interested in w_k^{dl}. The received signal across all users is

$$\mathbf{Y}^{dl} = \mathbf{H}^{dl}\mathbf{X}^{dl} + \mathbf{Z}^{dl} \tag{14}$$

where $\mathbf{Y}^{dl} \triangleq [\mathbf{y}_1^{dl} \ \cdots \ \mathbf{y}_K^{dl}]^\dagger$, $\mathbf{y}_k^{dl} \in \mathbb{R}^T$ is the received signal at the k^{th} user, $\mathbf{H}^{dl} \in \mathbb{R}^{K\times L}$ is the channel matrix from the L BSs to the K users, $\mathbf{X}^{dl} \triangleq [\mathbf{x}_1^{dl} \ \cdots \ \mathbf{x}_L^{dl}]^\dagger$, $\mathbf{x}_\ell^{dl} \in \mathbb{R}^T$ is the transmitted signal from the ℓ^{th} BS and $\mathbf{Z}^{dl} \in \mathbb{R}^{K\times T}$ is AWGN. Similar to the uplink, we have a total power constraint $\frac{1}{T}\mathbb{E}\text{Tr}\left(\mathbf{X}^{dl}\mathbf{X}^{dl\dagger}\right) \le P_{\text{total}}$.

B. Integer-Forcing for Downlink C-RAN

We begin with an overview of the integer-forcing beamforming strategy for downlink channel coding, which was originally proposed in [4]. Afterwards, we introduce our integer-forcing multivariate compression strategy.

Downlink Channel Encoding. Due to space limitations, we only present integer-forcing beamforming for the special case of symmetric rates, and point to [11, Section VI] for the asymmetric case. The CP first forms the precoded messages $\widetilde{\mathbf{w}}_1^{dl}, \ldots, \widetilde{\mathbf{w}}_K^{dl}$ as suggested by [4]:

$$\begin{bmatrix} \widetilde{\mathbf{w}}_1^{dl\dagger} \\ \vdots \\ \widetilde{\mathbf{w}}_K^{dl\dagger} \end{bmatrix} \triangleq \mathbf{A}_{c,\text{inv}}^{dl} \begin{bmatrix} \mathbf{w}_1^{dl\dagger} \\ \vdots \\ \mathbf{w}_K^{dl\dagger} \end{bmatrix} \tag{15}$$

where p is a prime, $\mathbf{w}_k^{dl}$ is the p-ary expansion of w_k^{dl}, $\mathbf{A}_{c,\text{inv}}^{dl} \in \mathbb{Z}_p^{K\times K}$ is the inverse of $[\mathbf{A}_c^{dl}] \bmod p$ over $\mathbb{Z}_p$, and each row of $\mathbf{A}_c^{dl} \in \mathbb{Z}^{K\times K}$ contains the coefficients of the integer-linear combinations to be decoded at one of the users. The precoded messages are then mapped to lattice codewords $\mathbf{s}_k^{dl} \in \mathbb{R}^T$ for $k \in \mathcal{K}$. The pre-inversion step in (15) allows the m^{th} user, after decoding

$$\mathbf{v}_{c,m}^{dl} \triangleq \sum_{k=1}^{K} a_{c,m,k}^{dl}\mathbf{s}_k^{dl}$$

where $a_{c,m,k}^{dl}$ is the $(m,k)^{\text{th}}$ element of $\mathbf{A}_c^{dl}$, to map $\mathbf{v}_{c,m}^{dl}$ back to the desired message $\mathbf{w}_m^{dl}$.

After forming the channel codewords $\mathbf{S}^{dl} \triangleq [\mathbf{s}_1^{dl} \cdots \mathbf{s}_K^{dl}]^\dagger$, the CP uses a beamforming matrix $\mathbf{B}^{dl} \in \mathbb{R}^{L\times K}$ to form

$$\widetilde{\mathbf{S}}^{dl} = \mathbf{B}^{dl}\mathbf{S}^{dl} \tag{16}$$

where $\widetilde{\mathbf{S}}^{dl} \triangleq [\widetilde{\mathbf{s}}_1^{dl} \ \cdots \ \widetilde{\mathbf{s}}_L^{dl}]^\dagger$.

Downlink Source Coding. The CP then compresses $\widetilde{\mathbf{s}}_1^{dl}, \ldots, \widetilde{\mathbf{s}}_L^{dl}$ using a *reverse* integer-forcing source coding. Due to space limitations, we only provide a succinct description of our coding scheme. Detailed proofs will be provided in an extended version of this work. The goal is to quantize $\widetilde{\mathbf{s}}_1^{dl}, \ldots, \widetilde{\mathbf{s}}_L^{dl}$ so that the reconstructed versions at the BSs have correlated quantization noises. This idea was first introduced in [5] using Gaussian codebooks. Toward this end, the CP first pre-inverts $\widetilde{\mathbf{S}}^{dl}$ to get

$$\mathbf{V}_s^{dl} = \left(\mathbf{A}_{s,\text{inv}}^{dl}\right)^{-1}\widetilde{\mathbf{S}}^{dl} \tag{17}$$

where $\mathbf{V}_s^{dl} \triangleq [\mathbf{v}_{s,1}^{dl} \ \cdots \ \mathbf{v}_{s,L}^{dl}]^\dagger$ and $\mathbf{A}_s^{dl}$ is a full-rank integer matrix with full-rank submatrices $\mathbf{A}_{s,[1:m]}^{dl}, \forall m \in \mathcal{L}$, and $\mathbf{A}_{s,\text{inv}}$ is the inverse of $[\mathbf{A}_s^{dl}] \bmod p$ over $\mathbb{Z}_p$. Next, using lattice codebooks $\mathcal{C}_\ell \triangleq \Lambda_{F,\ell} \cap \mathcal{V}(\Lambda_{C,\ell})$, where $\Lambda_{C,1}, \ldots, \Lambda_{C,L}$ are L coarse lattices nested in L fine lattices $\Lambda_{F,1} \subseteq \cdots \subseteq \Lambda_{F,L}$, all are chosen according to Lemma 2, the CP quantizes and forms integer-linear combinations

$$\boldsymbol{\lambda}_m^{dl} = \left[\sum_{k=1}^{m} a_{m,k}^{dl}\mathcal{Q}_{\Lambda_{F,k}}(\mathbf{v}_{s,k}^{dl} + \mathbf{u}_k^{dl} + \mathbf{g}_k^{dl}) - \mathbf{t}_{F,m}^{dl}\right] \bmod \Lambda_{C,m} \tag{18}$$

where $\mathbf{u}_1^{dl}, \ldots, \mathbf{u}_L^{dl}$ are independent dithers with $\mathbf{u}_k^{dl} \sim \text{Unif}\left(\mathcal{V}\left(\Lambda_{F,k}\right)\right), \forall k \in \mathcal{L}$ and $\mathbf{g}_1^{dl}, \ldots, \mathbf{g}_m^{dl}, \mathbf{t}_{F,m}^{dl}$ are *DPC* parameters chosen as in Appendix D.

The index of $\boldsymbol{\lambda}_m^{dl}$ is then forwarded to BS m to recover

$$\begin{aligned} \widehat{\mathbf{s}}_m^{dl} &= \left[\boldsymbol{\lambda}_m^{dl} - \sum_{k=1}^{L} a_{m,k}^{dl}\mathbf{u}_k^{dl}\right] \bmod \Lambda_{C,m} \\ &\overset{(a)}{=} \left[\sum_{k=1}^{L} a_{m,k}^{dl}\left(\mathbf{v}_{s,k}^{dl} + \mathbf{q}_k^{dl}\right)\right] \bmod \Lambda_{C,m} \\ &= \left[\widetilde{\mathbf{s}}_m^{dl} + \sum_{k=1}^{L} a_{m,k}^{dl}\mathbf{q}_k^{dl}\right] \bmod \Lambda_{C,m} \\ &\overset{(b)}{=} \widetilde{\mathbf{s}}_m^{dl} + \sum_{k=1}^{L} a_{m,k}^{dl}\mathbf{q}_k^{dl} \end{aligned} \tag{19}$$

where $\mathbf{q}_k^{dl} = -[\mathbf{v}_{s,k}^{dl} + \mathbf{u}_k^{dl} + \mathbf{g}_k^{dl}] \bmod \Lambda_{F,k}$ is uniformly distributed over $\mathcal{V}(\Lambda_{F,k})$, (a) follows from Appendix D and (b) follows w.h.p. if

$$\text{var}\left(\widetilde{\mathbf{s}}_m^{dl}\right) + \mathbf{a}_{s,m}^{dl\dagger}\mathbf{D}^{dl}\mathbf{a}_{s,m}^{dl} < \sigma^2(\Lambda_{C,m})$$

where $\mathbf{D}^{dl}$ is the covariance matrix of $\mathbf{Q}^{dl} \triangleq [\mathbf{q}_1^{dl} \ \cdots \ \mathbf{q}_L^{dl}]^\dagger$ and $\text{var}\left(\widetilde{\mathbf{s}}_m^{dl}\right) \triangleq \frac{1}{T}\mathbb{E}\|\widetilde{\mathbf{s}}_m^{dl}\|^2 = \mathbf{b}_m^{dl\dagger}\mathbf{P}^{dl}\mathbf{b}_m^{dl}$.

Theorem 1: For the distributed decompression problem shown in Fig. 1, where the source wants to convey $\widetilde{\mathbf{s}}_1^{dl}, \ldots, \widetilde{\mathbf{s}}_L^{dl}$ to L independent decoders and the ℓ^{th} decoder wants to decode $\widehat{\mathbf{s}}_\ell^{dl} \triangleq \widetilde{\mathbf{s}}_\ell^{dl} + \widetilde{\mathbf{q}}_\ell^{dl}$ with a quantization noise $\widetilde{\mathbf{Q}}^{dl} =$

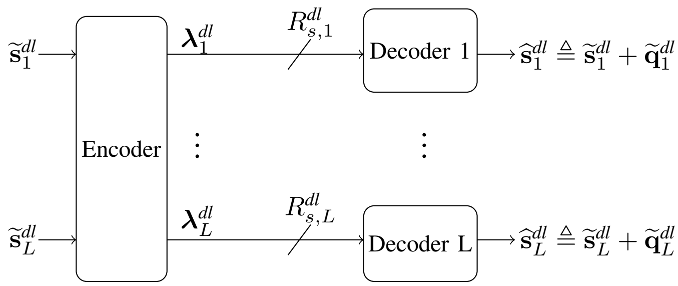

Fig. 1: The distributed decompression problem with a single source and distributed L decoders.

$[\widetilde{\mathbf{q}}_1^{dl} \; \cdots \; \widetilde{\mathbf{q}}_L^{dl}]^\dagger$ that has a target covariance matrix $\mathbf{\Omega} \triangleq \mathbf{A}_s^{dl}\mathbf{D}^{dl}\mathbf{A}_s^{dl\dagger}$ for $\mathbf{A}_s^{dl} \in \mathbb{Z}^{L\times L}$ and $\mathbf{D}^{dl} \in \mathbb{D}^L$, the following rates are achievable

$$R_{s,\ell}^{dl} \triangleq \frac{1}{2}\log\left(\frac{\text{var}\left(\widetilde{\mathbf{s}}_\ell^{dl}\right) + \mathbf{a}_{s,\ell}^{dl\dagger}\mathbf{D}^{dl}\mathbf{a}_{s,\ell}^{dl}}{d_\ell^{dl}}\right). \tag{20}$$

The proof of Theorem 1 is omitted due to space limitations.

Using Theorem 1 and by choosing $\mathbf{A}_s^{dl}$ and $\mathbf{D}^{dl}$ such that $\sum_{\ell=1}^{L} R_{s,\ell}^{dl} \le C_{\text{tot}}$, the BSs can recover, w.h.p., and re-transmit

$$\mathbf{X}^{dl} = \widetilde{\mathbf{S}}^{dl} + \widetilde{\mathbf{Q}}^{dl}$$

where $\widetilde{\mathbf{Q}}^{dl} \triangleq \mathbf{A}_s^{dl}\mathbf{Q}^{dl}$ is correlated quantization noise.

Downlink Channel Decoding. As discussed earlier, the k^{th} user wants to decode $\mathbf{v}_{c,k}^{\dagger dl} = \mathbf{a}_{c,k}^{dl\dagger}\mathbf{S}^{dl}$, which can be mapped back to its desired message $\mathbf{w}_k^{dl}$. To do so, it equalizes its received signal to get

$$\begin{aligned}\widetilde{\mathbf{y}}_k^{dl\dagger} &\triangleq v_k^{dl}\mathbf{y}_k^{dl\dagger} \\ &= \mathbf{v}_{c,k}^{dl\dagger} + \mathbf{z}_{\text{eff}}^{dl\dagger}\end{aligned} \tag{21}$$

where $\mathbf{z}_{\text{eff}}^{dl\dagger} \triangleq (v_k^{dl}\mathbf{h}_k^{dl\dagger}\mathbf{B}^{dl} - \mathbf{a}_{c,k}^{dl\dagger})\mathbf{S}^{dl} + v_k^{dl}\mathbf{z}_k^{dl\dagger} + v_k^{dl}\mathbf{h}_k^{dl\dagger}\mathbf{A}_s^{dl}\mathbf{Q}^{dl}$ is an effective noise with effective variance

$$\begin{aligned}(\sigma_k^{dl})^2 \triangleq \frac{1}{T}\mathbb{E}(\|\mathbf{z}_{\text{eff}}^{dl}\|^2) = \| \left(v_k^{dl}\mathbf{h}_k^{dl\dagger}\mathbf{B}^{dl} - \mathbf{a}_{c,k}^{dl\dagger}\right)\mathbf{P}^{dl\frac{1}{2}}\|^2 \\ + (v_k^{dl})^2 + (v_k^{dl})^2\mathbf{h}_k^{dl\dagger}\mathbf{A}_s^{dl}\mathbf{D}^{dl}\mathbf{A}_s^{dl\dagger}\mathbf{h}_k^{dl}.\end{aligned} \tag{22}$$

It can be shown that the achievable sum-rate for the IF strategy with algebraic successive decompression and parallel channel decoding is

$$R_{\text{IF-CRAN}}^{dl}(\mathbf{H}^{dl}) = \max_{\substack{\mathbf{B}^{dl}\in\mathbb{R}^{L\times K}, \mathbf{A}_c^{ul}\in\mathbb{Z}^{K\times K} \\ \text{rank}(\mathbf{A}_c^{dl})=K}} \sum_{k=1}^{K}\frac{1}{2}\log^+\left(\beta_k^{dl}\right). \tag{23}$$

where $\beta_k^{dl} \triangleq P_k^{dl}/(\sigma_k^{dl})^2$ is the k^{th} effective SINR for the k^{th} decoded combination (user).

Finally, the MMSE equalizer that minimizes the variance in (22) and the corresponding variance are given by

$$v_m^{dl} = \frac{\mathbf{a}_{c,m}^{dl\dagger}\mathbf{P}^{dl\dagger}\mathbf{B}^{dl\dagger}\mathbf{h}_m^{dl}}{1 + \mathbf{h}_m^{dl\dagger}\left(\mathbf{A}_s^{dl}\mathbf{D}^{dl}\mathbf{A}_s^{dl\dagger} + \mathbf{B}^{dl}\mathbf{P}^{dl}\mathbf{B}^{dl\dagger}\right)\mathbf{h}_m^{dl}} \tag{24}$$

$$(\sigma_m^{dl})^2 = \|\mathbf{F}_{c,m}^{dl}\mathbf{a}_{c,m}^{dl}\|^2 \tag{25}$$

where $\mathbf{F}_{c,m}^{dl}$ is the Cholesky decomposition $\mathbf{F}_{c,m}^{dl\dagger}\mathbf{F}_{c,m}^{dl} = \left((\mathbf{P}^{dl})^{-1} + \mathbf{B}^{dl\dagger}\mathbf{h}_m^{dl}\left(1 + \mathbf{h}_m^{dl\dagger}\mathbf{A}_s^{dl}\mathbf{D}^{dl}\mathbf{A}_s^{dl\dagger}\mathbf{h}_m^{dl}\right)^{-1}\mathbf{h}_m^{dl\dagger}\mathbf{B}^{dl}\right)^{-1}$.

Lemma 5: Let

$$\boldsymbol{\beta}^{dl} \triangleq \begin{bmatrix} P_1^{dl}/(\sigma_1^{dl})^2 \\ \vdots \\ P_K^{dl}/(\sigma_K^{dl})^2 \end{bmatrix} \text{ and } \boldsymbol{\rho}^{dl} \triangleq \begin{bmatrix} P_1^{dl} \\ \vdots \\ P_K^{dl} \end{bmatrix} \tag{26}$$

denote the effective SINR and coding power vectors, respectively, then (8) can be written as

$$\left(\mathbf{I} - \text{diag}(\boldsymbol{\beta}^{dl})\mathbf{M}^{dl}\right)\boldsymbol{\rho}^{dl} = \mathbf{J}^{dl}\boldsymbol{\beta}^{dl} \tag{27}$$

where $M_{\ell,k}^{dl} = (\mathbf{h}_\ell^{dl\dagger}\mathbf{b}_k^{dl}v_\ell^{dl} - a_{c,\ell,k}^{dl})^2 + \sum_i\sum_j b_{i,k}^{dl2}C_{j,i}^{dl}(\widetilde{\mathbf{a}}_{s,j}^{dl\dagger}\mathbf{h}_\ell^{dl})^2(v_\ell^{dl})^2$ is the $(\ell,k)^{\text{th}}$ element of $\mathbf{M}^{dl}$, $\mathbf{h}_\ell^{dl}$ is the ℓ^{th} column of $\mathbf{H}^{dl}$, $\widetilde{\mathbf{a}}_{s,j}^{dl}$ is the j^{th} column of $\mathbf{A}_s^{dl}$ and $\mathbf{J}^{dl} = \text{diag}\left((v_1^{dl})^2, \ldots, (v_K^{dl})^2\right)$.

The proof of Lemma 5 is given in Appendix B.

V. Uplink Downlink Duality for IF C-RAN

We now turn to establishing uplink-downlink duality. The next two lemmas allow us to express the distortion levels achieved by both the IF downlink scheme and the IF uplink scheme in terms of the rate allocation of the backhaul network for the uplink channel.

Lemma 6: For the uplink C-RAN with sum-rate backhaul constraint C_{tot} and an integer matrix $\mathbf{A}_s^{ul}$ that satisfies $\text{rank}(\mathbf{A}_{s,[1:m]}^{ul}) = m, \forall m \in \mathcal{L}$, the achievable distortion levels $d_1^{ul}, \ldots, d_L^{ul}$ can be written in terms of the achievable compression rates $C_1^{ul}, \ldots, C_L^{ul}$ as

$$\mathbf{d}^{ul} = \mathbf{C}^{ul}\mathbf{e}^{ul} \tag{28}$$

where d_ℓ^{ul} and $e_\ell^{ul} \triangleq \mathbf{a}_{s,\ell}^{ul\dagger}(\mathbf{H}^{ul}\mathbf{V}^{ul}\mathbf{P}^{ul}\mathbf{V}^{ul\dagger}\mathbf{H}^{ul\dagger} + \mathbf{I})\mathbf{a}_{s,\ell}^{ul}$ are the ℓ^{th} elements of $\mathbf{d}^{ul}$ and $\mathbf{e}^{ul}$, respectively, while

$$\mathbf{C}^{ul} \triangleq \begin{bmatrix} 2^{2C_1^{ul}} - (a_{s,1,1}^{ul})^2 & \cdots & -(a_{s,1,L}^{ul})^2 \\ \vdots & \ddots & \vdots \\ -(a_{s,L,1}^{ul})^2 & \cdots & 2^{2C_L^{ul}} - (a_{s,L,L}^{ul})^2 \end{bmatrix}^{-1} \tag{29}$$

for some rate allocation that satisfies $\sum_{\ell=1}^{L} C_\ell^{ul} \le C_{\text{tot}}$.

The proof of Lemma 6 follows from Lemma 3.

Lemma 7: For the downlink C-RAN with sum-rate capacity constraint C_{tot} and integer matrix $\mathbf{A}_s^{dl} = \mathbf{A}_s^{ul\dagger}$, the following distortion levels $d_1^{dl}, \ldots, d_L^{dl}$ are achievable using the previously discussed compression scheme

$$\mathbf{d}^{dl} = \mathbf{C}^{dl}\mathbf{e}^{dl} \tag{30}$$

where d_ℓ^{dl} and $e_\ell^{dl} \triangleq \mathbf{b}_\ell^{dl\dagger}\mathbf{P}^{dl}\mathbf{b}_\ell^{dl}$ are the ℓ^{th} elements of $\mathbf{d}^{dl}$ and $\mathbf{e}^{dl}$, respectively, while

$$\mathbf{C}^{dl} \triangleq \begin{bmatrix} 2^{2C_1^{ul}} - (a_{s,1,1}^{dl})^2 & \cdots & -(a_{s,1,L}^{dl})^2 \\ \vdots & \ddots & \vdots \\ -(a_{s,L,1}^{dl})^2 & \cdots & 2^{2C_L^{ul}} - (a_{s,L,L}^{dl})^2 \end{bmatrix}^{-1} = \mathbf{C}^{ul\dagger}$$

where $C_1^{ul}, \cdots, C_L^{ul}$ are the achievable rates for the corresponding uplink channel. The proof of Lemma 7 is given in Appendix C.

Remark 1: For the dual channel $\mathbf{H}^{dl} = \mathbf{H}^{ul\dagger}$ and by choosing $\mathbf{B}^{dl} = \mathbf{B}^{ul\dagger}, \mathbf{V}^{dl} = \mathbf{V}^{ul}, \mathbf{A}_c^{dl} = \mathbf{A}_c^{ul\dagger}$ and $\mathbf{A}_s^{dl} = \mathbf{A}_s^{ul\dagger}$, it can be shown that

$$\mathbf{C}^{dl} = \mathbf{C}^{ul\dagger} \quad (31)$$

$$\mathbf{M}^{dl} = \mathbf{M}^{ul\dagger}. \quad (32)$$

Theorem 2: Let $R_{\text{IF-CRAN}}^{ul}$ be the achievable sum-rate using integer-forcing equalization and compression for a given uplink channel $\mathbf{H}^{ul}$, integer matrices $\mathbf{A}_c^{ul}$ and $\mathbf{A}_s^{ul}$, coding power matrix $\mathbf{P}^{ul}$, equalization matrix $\mathbf{B}^{ul}$ and beamforming matrix $\mathbf{V}^{ul}$ that satisfies the total power constraint P_{total}. Then, for the dual downlink channel $\mathbf{H}^{dl} = \mathbf{H}^{ul\dagger}$, we can achieve a sum-rate $R_{\text{IF-CRAN}}^{dl} \geq R_{\text{IF-CRAN}}^{ul}$.

Proof : By setting $\mathbf{B}^{dl}, \mathbf{V}^{dl}, \mathbf{A}_c^{dl}$ and $\mathbf{A}_s^{dl}$ as in Remark 1, we have $\mathbf{M}^{dl} = \mathbf{M}^{ul\dagger}$. Let $\mathbf{d}^{ul} \geq 0$ be a solution for $(\mathbf{C}^{ul})^{-1}\mathbf{x} = \mathbf{e}^{ul}$ where $\left(\mathbf{C}^{ul}\right)^{-1}$ is a Z-matrix and $\mathbf{e}^{ul} > 0$. Then, it follows that $(\mathbf{C}^{ul})^{-1}$ is an M-matrix (and the same argument holds for $(\mathbf{C}^{dl})^{-1}$). Using [13, Theorem 1], it follows that $\mathbf{C}^{dl}$ (as well as $\mathbf{C}^{ul}$) is a non-negative matrix and so is $\mathbf{M}^{dl}$ (as well as $\mathbf{M}^{ul}$), respectively. This implies that $(\mathbf{I} - \text{diag } (\boldsymbol{\beta}^{ul})\mathbf{M}^{ul})$ is a Z-matrix.

Similarly, since $\boldsymbol{\rho}^{ul} \geq 0$ is a solution for (13) where $\mathbf{J}^{ul}\boldsymbol{\beta}^{ul} > 0$, it follows directly that $(\mathbf{I} - \text{diag } (\boldsymbol{\beta}^{ul})\mathbf{M}^{ul})$ is an M-matrix. Furthermore, it can be shown that diag $(\boldsymbol{\beta}^{ul})\mathbf{M}^{dl}$ and diag $(\boldsymbol{\beta}^{ul})\mathbf{M}^{ul}$ have the same eigenvalues. Thus, by setting $\boldsymbol{\beta}^{dl} = \boldsymbol{\beta}^{ul}$, we deduce from [13, Theorem 1] that there exist a unique non-negative downlink coding power vector

$$\boldsymbol{\rho}^{dl} = (\mathbf{I} - \text{diag } (\boldsymbol{\beta}^{dl})\mathbf{M}^{dl})^{-1}\mathbf{J}^{dl}\boldsymbol{\beta}^{dl}. \quad (33)$$

Now, it remains to check that this coding power vector satisfies the total power constraints. To this end, define

$$\boldsymbol{\rho}_{tot}^{ul} \triangleq \mathbf{G}^{ul}\boldsymbol{\rho}^{ul} \in \mathbb{R}^K$$
$$\boldsymbol{\rho}_{tot}^{dl} \triangleq \mathbf{G}^{dl}\boldsymbol{\rho}^{dl} \in \mathbb{R}^L$$

as the power allocated across transmitters for the uplink and downlink, where $\mathbf{G}^{ul} = \text{diag } ((v_1^{ul})^2, \ldots, (v_K^{ul})^2)$ and $G_{\ell,k}^{dl} = (b_{\ell,k}^{dl})^2 + \sum_{j=1}^{L}\sum_{i=1}^{L}(a_{s,\ell,j}^{dl})^2 C_{j,i}^{dl}(b_{i,k}^{dl})^2$, $\forall \ell \in \mathcal{L}$ and $\forall k \in \mathcal{K}$.

Since $\boldsymbol{\rho}^{ul}$ satisfies the total power constraint, we have

$$\begin{aligned}
P_{\text{total}} &= \mathbf{1}^\dagger \boldsymbol{\rho}_{tot}^{ul} \\
&= \mathbf{1}^\dagger \mathbf{G}^{ul}(\mathbf{I} - \text{diag}(\boldsymbol{\beta}^{ul})\mathbf{M}^{ul})^{-1}\mathbf{J}^{ul}\boldsymbol{\beta}^{ul} \\
&= \mathbf{1}^\dagger \mathbf{G}^{ul}(\mathbf{I} - \text{diag}(\boldsymbol{\beta}^{ul})\mathbf{M}^{ul})^{-1}\text{diag}(\boldsymbol{\beta}^{ul})\mathbf{J}^{ul}\mathbf{1} \\
&= \mathbf{1}^\dagger \mathbf{G}^{ul}(\text{diag}(\boldsymbol{\beta}^{ul})^{-1} - \mathbf{M}^{ul})^{-1}\mathbf{J}^{ul}\mathbf{1} \\
&= \mathbf{1}^\dagger \mathbf{J}^{dl\dagger}(\text{diag}(\boldsymbol{\beta}^{dl})^{-1} - \mathbf{M}^{dl\dagger})^{-1}\mathbf{J}^{ul}\mathbf{1} \\
&= \mathbf{1}^\dagger \text{diag}(\boldsymbol{\beta}^{dl})\mathbf{J}^{dl\dagger}(\mathbf{I} - \mathbf{M}^{dl\dagger}\text{diag}(\boldsymbol{\beta}^{dl}))^{-1}\mathbf{J}^{ul}\mathbf{1} \\
&= \boldsymbol{\beta}^{dl\dagger}\mathbf{J}^{dl\dagger}(\mathbf{I} - \mathbf{M}^{dl\dagger}\text{diag}(\boldsymbol{\beta}^{dl}))^{-1}\mathbf{J}^{ul}\mathbf{1} \\
&= \boldsymbol{\beta}^{dl\dagger}\mathbf{J}^{dl\dagger}(\mathbf{I} - \mathbf{M}^{dl\dagger}\text{diag}(\boldsymbol{\beta}^{dl}))^{-1}\mathbf{G}^{dl\dagger}\mathbf{1} \\
&= \boldsymbol{\rho}_{tot}^{dl\dagger}\mathbf{1}.
\end{aligned}$$

Finally using (9) and (23), similar to [11], and since the achievable SINRs for the uplink and downlink are equal, we have our result.

Theorem 3: Let $R_{\text{IF-CRAN}}^{dl}$ be the achievable sum-rate using integer-forcing equalization and compression for a given downlink channel $\mathbf{H}_d$, integer matrices $\mathbf{A}_{d,c}$ and $\mathbf{A}_{d,s}$, coding power matrix $\mathbf{P}_d$, equalization matrix $\mathbf{V}_d$ and beamforming matrix $\mathbf{B}_d$ that satisfies the total power constraint P_{total}. Then, for the dual uplink channel $\mathbf{H}_u = \mathbf{H}_d^\dagger$, we can achieve a sum-rate $R_{\text{IF-CRAN}}^{ul} \geq R_{\text{IF-CRAN}}^{dl}$.

Proof : The proof is similar to the proof of Theorem 2 and omitted due to space limitations.

VI. Downlink IF-CRAN Optimization

The problem of choosing the integer matrix $\mathbf{A}_c^{dl}$ and beamforming matrix $\mathbf{B}^{dl}$ to maximize the SINRs was considered in [11] for the broadcast channel. Here, we have the added challenge of selecting a source coding integer matrix $\mathbf{A}_s^{dl}$ as well as distortion levels $\mathbf{D}^{dl}$.

A. One-shot Duality Algorithm

Our approach is to first choose $\mathbf{A}_c^{ul}$, $\mathbf{B}^{dl}$, $\mathbf{A}_s^{ul}$ and $\mathbf{D}^{ul}$ that maximize the sum-rate of the corresponding dual uplink channel $\mathbf{H}^{ul} = \mathbf{H}^{dl\dagger}$, then set

$$\begin{aligned}
\mathbf{A}_c^{dl} &= \mathbf{A}_c^{ul\dagger} \\
\mathbf{B}^{dl} &= \mathbf{B}^{ul\dagger} \\
\mathbf{A}_s^{dl} &= \mathbf{A}_s^{ul\dagger} \\
\mathbf{d}^{dl} &= \mathbf{C}^{ul\dagger}\mathbf{e}^{dl}
\end{aligned} \quad (34)$$

to achieve the same sum-rate on the downlink channel.

Unfortunately, optimizing $\mathbf{A}_c^{ul}$, $\mathbf{B}^{dl}$, $\mathbf{A}_s^{ul}$ and $\mathbf{D}^{ul}$ for the corresponding uplink channel is also challenging. In prior work [2], we proposed a suboptimal solution, which demonstrated good performance via simulations, and we will use this as a part of our approach. The details for our optimization algorithm is given below in Algorithm 1.

VII. Simulations

In this section, we show the performance (in terms of average sum-rate in bits/sec/Hz) of the proposed IF architecture and compare it to independent compression with successive channel encoding as well as multivariate compression with successive channel encoding. The optimization of the rate achieved by multivariate compression with dirty paper encoding is carried out jointly using the successive convex approximation algorithm proposed in [5]. (Note that this optimization must be performed over all $K!$ possible decoding orders.) For more details about multivariate compression, we refers the readers to [5].

For our simulations, we generated 500 realizations for the channel matrix $\mathbf{H}^{dl}$, each elementwise i.i.d. $\mathcal{N}(0, 1)$. We also fix the number of BSs to $L = 4$. Figure 2 shows the case

Algorithm 1 One-shot duality

procedure DUALITY($\mathbf{H}^{dl}, C_{\text{tot}}$,tol)
  Initialization: Set $\mathbf{H}^{ul} = \mathbf{H}^{dl\dagger}$, $\mathbf{V}^{ul} = \mathbf{I}$, $\mathbf{P}^{ul} = \frac{\text{SNR}}{K}\mathbf{I}$, $d_{\text{min}} = 0$ and d_{max} large enough such that $\sum_{\ell=1}^{L} R^{ul}_{s,\ell} < C_{\text{tot}}$.
  while $C_{\text{tot}} - \sum_{\ell=1}^{L} R^{ul}_{s,\ell} > \text{tol}$ **or** $\sum_{\ell=1}^{L} R^{ul}_{s,\ell} > C_{\text{tot}}$ **do**
    if $\sum_{\ell=1}^{L} R^{ul}_{s,\ell} < C_{\text{tot}}$ **then**
      $d_{\text{max}} = d^{ul}/2$.
    else
      $d_{\text{min}} = d^{ul}/2$.
    end if
    $d^{ul} = (d_{\text{min}} + d_{\text{max}})/2$.
    $\mathbf{F}^{ul}_s = \text{chol}(\mathbf{K}^{ul}_{YY}/d^{ul} + \mathbf{I})$
    $\mathbf{A}^{ul}_s = \text{LLL-reduction}(\mathbf{F}^{ul}_s)$.
    $R^{ul}_{s,\ell} = \frac{1}{2}\log^+(\|\mathbf{F}^{ul}_s \mathbf{a}^{ul}_{s,\ell}\|^2)$
  end while
  Calculate $\mathbf{C}^{ul}$ using (29).
  Calculate $\mathbf{F}^{ul}_c$ using (12).
  $\mathbf{A}^{ul}_c = \text{LLL-reduction}(\mathbf{F}^{ul}_c)$.
  Calculate $\mathbf{B}^{ul}$ using (10).
  return $(\mathbf{A}^{ul}_c, \mathbf{A}^{ul}_s, \mathbf{C}^{ul})$.
end procedure

of $L = 4$ users where we fix the total SNR $= 30$dB and plot the average sum-rate with the sum-rate of the backhaul network C_{tot}. The performance of the proposed IF scheme is quite close to that of multivariate compression combined with dirty paper coding, and has an advantage over multivariate compression with single-user decoding as well as single-user compression and channel coding. Figure 3 shows the average sum-rate against the SNR for fixed total backhaul rate $C_{\text{tot}} = 20$ for the same 4×4 channel. Again, we observe that our integer-forcing scheme is competitive with multivariate compression combined with dirty paper coding, and outperforms schemes that rely on single-user source coding and/or channel coding.

We also note that, rather than a "one-shot" algorithm, we can iterate between the uplink and downlink to optimize the parameters. However, our simulations did not show any significant performance improvement for this iterative algorithm.

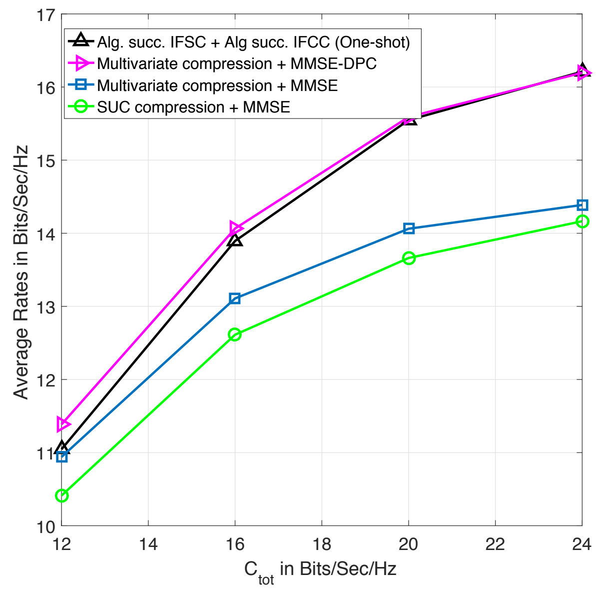

Fig. 2: The average sum-rate for $K = 4$ and SNR $= 30$dB

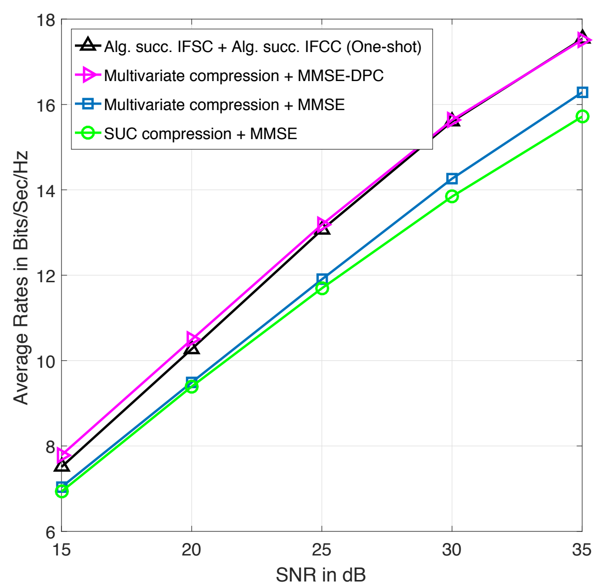

Fig. 3: The average sum-rate for $K = 4$ and $C_{\text{tot}} = 20$ Bits/Sec/Hz.

APPENDIX

A. Proof of Lemma 4

We start by multiplying (8) by $P^{ul}_k/(\sigma^{ul}_k)^2$ to get

$$P^{ul}_k = \beta^{ul}_k \left(\sum_{\ell} (\mathbf{b}^{ul\dagger}_k \mathbf{h}^{ul}_\ell v^{ul}_\ell - a^{ul}_{c,k,\ell})^2 P^{ul}_\ell + \|\mathbf{b}^{ul}_k\|^2 + \sum_i (b^{ul}_{k,i})^2 d^{ul}_i \right)$$

$$= \beta_k^{ul} \left(\sum_\ell (\mathbf{b}_k^{ul\dagger} \mathbf{h}_\ell^{ul} v_\ell^{ul} - a_{c,k,\ell}^{ul})^2 P_\ell^{ul} + \sum_i \sum_j (b_{k,i}^{ul})^2 C_{i,j}^{ul} \|\mathbf{a}_{s,j}^{ul}\|^2 \right.$$
$$\left. + \|\mathbf{b}_k^{ul}\|^2 + \sum_i \sum_j (b_{k,i}^{ul})^2 C_{i,j}^{ul} \mathbf{a}_{s,j}^{ul\dagger} (\mathbf{H}^{ul} \mathbf{V}^{ul} \mathbf{P}^{ul} \mathbf{V}^{ul\dagger} \mathbf{H}^{ul\dagger}) \mathbf{a}_{s,j}^{ul} \right)$$
$$= \beta_k^{ul} \left(\sum_\ell (\mathbf{b}_k^{ul\dagger} \mathbf{h}_\ell^{ul} v_\ell^{ul} - a_{c,k,\ell}^{ul})^2 P_\ell^{ul} + J_{k,k}^{ul} \right.$$
$$\left. + \sum_i \sum_j (b_{k,i}^{ul})^2 C_{i,j}^{ul} \mathbf{a}_{s,j}^{ul\dagger} \left(\sum_\ell \mathbf{h}_\ell^{ul} \mathbf{h}_\ell^{ul\dagger} (v_\ell^{ul})^2 P_\ell^{ul} \right) \mathbf{a}_{s,j}^{ul} \right)$$
$$= \beta_k^{ul} \sum_\ell M_{k,\ell}^{ul} P_\ell^{ul} + \beta_k^{ul} J_{k,k}^{ul} \tag{35}$$

Finally, (13) follows from the previous equation by taking $k = 1, \ldots, K$.

B. Proof of Lemma 5

Similar to Appendix A, multiplying (22) by $P_k^{dl}/(\sigma_k^{dl})^2$, we get

$$P_k^{dl} = \beta_k^{dl} \sum_\ell (\mathbf{b}_\ell^{dl\dagger} \mathbf{h}_k^{dl} v_k^{dl} - a_{c,k,\ell}^{dl})^2 P_\ell^{dl} + \beta_k^{dl} (v_k^{dl})^2$$
$$+ \beta_k^{dl} (v_k^{dl})^2 \sum_j (\mathbf{h}_k^{dl\dagger} \widetilde{\mathbf{a}}_{s,j}^{dl})^2 d_j^{dl}$$
$$= \beta_k^{dl} \sum_\ell (\mathbf{b}_\ell^{dl\dagger} \mathbf{h}_k^{dl} v_k^{dl} - a_{c,k,\ell}^{dl})^2 P_\ell^{dl} + \beta_k^{dl} (v_k^{dl})^2$$
$$+ \beta_k^{dl} \sum_j (\mathbf{h}_k^{dl\dagger} \widetilde{\mathbf{a}}_{s,j}^{dl})^2 \mathbf{c}_j^{dl\dagger} \mathbf{e}^{dl}$$
$$= \beta_k^{dl} \sum_\ell (\mathbf{b}_\ell^{dl\dagger} \mathbf{h}_k^{dl} v_k^{dl} - a_{c,k,\ell}^{dl})^2 P_\ell^{dl} + \beta_k^{dl} (v_k^{dl})^2$$
$$+ \beta_k^{dl} \sum_j \sum_i (\mathbf{h}_k^{dl\dagger} \widetilde{\mathbf{a}}_{s,j}^{dl})^2 C_{j,i}^{dl} \mathbf{b}_i^{dl\dagger} \mathbf{P}^{dl} \mathbf{b}_i^{dl}$$
$$= \beta_k^{dl} \sum_\ell (\mathbf{b}_\ell^{dl\dagger} \mathbf{h}_k^{dl} v_k^{dl} - a_{c,k,\ell}^{dl})^2 P_\ell^{dl} + \beta_k^{dl} (v_k^{dl})^2$$
$$+ \beta_k^{dl} \sum_\ell \sum_j \sum_i (\mathbf{h}_k^{dl\dagger} \widetilde{\mathbf{a}}_{s,j}^{dl})^2 C_{j,i}^{dl} (b_{i,\ell}^{dl})^2 P_\ell^{dl}$$
$$= \beta_k^{dl} \sum_\ell M_{k,\ell}^{dl} P_\ell^{dl} + \beta_k^{dl} J_{k,k}^{dl}. \tag{36}$$

Finally, (27) follows from the previous equation by taking $k = 1, \ldots, K$.

C. Proof of Lemma 7

We start by computing $d_1^{dl}, \ldots, d_L^{dl}$ such that

$$\frac{1}{2} \log \left(\frac{\mathbf{b}_\ell^{dl\dagger} \mathbf{P}^{dl} \mathbf{b}_\ell^{dl} + \mathbf{a}_{s,\ell}^{dl\dagger} \mathbf{D}^{dl} \mathbf{a}_{s,\ell}^{dl}}{d_\ell^{dl}} \right) = C_\ell^{ul}, \ell = 1, \ldots, L \tag{37}$$

where $(C_1^{ul}, \ldots, C_L^{ul})$ is the compression rate allocation achieved in the uplink. Without loss of generality[4], let $d_L^{dl} < \cdots < d_1^{dl}$. Now, we argue that the distortion levels $d_1^{dl}, \ldots, d_L^{dl}$ are achievable under some compression rate allocation for the backhaul network. This follows from

$$\sum_{\ell=1}^{L} \frac{1}{2} \log \left(\frac{\mathbf{b}_\ell^{dl\dagger} \mathbf{P}^{dl} \mathbf{b}_\ell^{dl} + \mathbf{a}_{s,\ell}^{dl\dagger} \mathbf{D}^{dl} \mathbf{a}_{s,\ell}^{dl}}{d_{\pi^{dl}(\ell)}^{dl}} \right)$$
$$= \frac{1}{2} \log \left(\frac{\prod_{\ell=1}^{L} \mathbf{b}_\ell^{dl\dagger} \mathbf{P}^{dl} \mathbf{b}_\ell^{dl} + \mathbf{a}_{s,\ell}^{dl\dagger} \mathbf{D}^{dl} \mathbf{a}_{s,\ell}^{dl}}{\prod_{\ell=1}^{L} d_{\pi^{dl}(\ell)}^{dl}} \right)$$
$$= \frac{1}{2} \log \left(\frac{\prod_{\ell=1}^{L} \mathbf{b}_\ell^{dl\dagger} \mathbf{P}^{dl} \mathbf{b}_\ell^{dl} + \mathbf{a}_{s,\ell}^{dl\dagger} \mathbf{D}^{dl} \mathbf{a}_{s,\ell}^{dl}}{\prod_{\ell=1}^{L} d_\ell^{dl}} \right)$$
$$= \sum_{\ell=1}^{L} \frac{1}{2} \log \left(\frac{\mathbf{b}_\ell^{dl\dagger} \mathbf{P}^{dl} \mathbf{b}_\ell^{dl} + \mathbf{a}_{s,\ell}^{dl\dagger} \mathbf{D}^{dl} \mathbf{a}_{s,\ell}^{dl}}{d_\ell^{dl}} \right) \tag{38}$$
$$= \sum_{\ell=1}^{L} C_\ell^{ul} \le C_{\text{tot}}, \ell = 1, \ldots, L. \tag{39}$$

where the permutation π^{dl} is chosen such that $d_{\pi^{dl}(L)}^{dl} < \cdots < d_{\pi^{dl}(1)}^{dl}$.

D. Proof of (19).

For tractability, we drop *dl* in the superscript and s in the subscript. Let us assume full-rank submatrices $\mathbf{A}_{[1:m]}$. The compression (18) can be written as

$$\boldsymbol{\lambda}_m = \left[\sum_{k=1}^{m} a_{m,k} \mathcal{Q}_{\Lambda_{F,k}} (\mathbf{v}_k + \mathbf{u}_k + \mathbf{g}_k) - \mathbf{t}_{F,m} \right] \bmod \Lambda_{C,m}. \tag{40}$$

It is worth noting that choosing $\mathbf{t}_{F,m} \in \Lambda_{F,m}$ results in $\boldsymbol{\lambda}_m \in \Lambda_{F,m} \cap \mathcal{V}(\Lambda_{C,m})$. Now it remains to show that there is a choice for $\mathbf{g}_1, \ldots, \mathbf{g}_m$ and $\mathbf{t}_{F,m} \in \Lambda_{F,m}$ such that at the m^{th} BS we recover (19).

The m^{th} BS removes the dithers and recovers

$$\left[\boldsymbol{\lambda}_m - \sum_{k=1}^{L} a_{m,k} \mathbf{u}_k \right] \bmod \Lambda_{C,m} \overset{(a)}{=}$$
$$\left[\sum_{k=1}^{m} a_{m,k} (\mathbf{v}_k + \mathbf{g}_k + \mathbf{q}_k) - \mathbf{t}_{F,m} - \sum_{k=m+1}^{L} a_{m,k} \mathbf{u}_k \right] \bmod \Lambda_{C,m}$$
$$\overset{(b)}{=} \left[\sum_{k=1}^{L} a_{m,k} (\mathbf{v}_k + \mathbf{q}_k) \right] \bmod \Lambda_{C,m} \tag{41}$$

where $\mathbf{q}_k = -[\mathbf{v}_k + \mathbf{u}_k + \mathbf{g}_k] \bmod \Lambda_{F,k}$, (a) holds from the distributive law and the Crypto lemma, (b) holds if we choose $\mathbf{g}_1, \ldots, \mathbf{g}_m$ such that

$$\sum_{k=1}^{m} a_{m,k} \mathbf{g}_k - \mathbf{t}_{F,m} = \sum_{k=m+1}^{L} a_{m,k} (\mathbf{v}_k + \mathbf{q}_k + \mathbf{u}_k). \tag{42}$$

[4]If not, we can re-index the BSs, such that the distortion levels are monotonically decreasing

which can be written as

$$\sum_{k=1}^{L} a_{m,k}\mathbf{g}_k - \mathbf{t}_{F,m} = \sum_{k=m+1}^{L} a_{m,k}\mathcal{Q}_{F,k}\left(\mathbf{v}_k+\mathbf{g}_k+\mathbf{u}_k\right). \tag{43}$$

In matrix form, for $m=1,\dots,L-1$ we can write (43) as

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,L} \\ \vdots & \ddots & \vdots \\ a_{L-1,1} & \dots & a_{L-1,L}\end{bmatrix}\begin{bmatrix}\mathbf{g}_1\\ \vdots \\ \mathbf{g}_L\end{bmatrix} - \begin{bmatrix}\mathbf{t}_{F,1}\\ \vdots\\ \mathbf{t}_{F,L-1}\end{bmatrix} \tag{44}$$

$$= \begin{bmatrix} a_{1,2} & a_{1,3} & \dots & a_{1,L}\\ 0 & a_{2,3} & & a_{2,L}\\ \vdots & & \ddots & \vdots \\ 0 & \dots & & a_{L-1,L}\end{bmatrix}\begin{bmatrix}\mathcal{Q}_{F,2}(\mathbf{v}_2+\mathbf{u}_2+\mathbf{g}_2)\\ \vdots \\ \mathcal{Q}_{F,L}(\mathbf{v}_L+\mathbf{u}_L+\mathbf{g}_L)\end{bmatrix} \tag{45}$$

To solve (44), we write $\mathbf{g}_m$ in terms of $L-m$ components

$$\mathbf{g}_m \triangleq \sum_{k=1}^{L-m}\mathbf{g}_m^{(k)} \tag{46}$$

where $\mathbf{g}_m^{(k)} = z_{m,k}\mathcal{Q}_{F,m+k}\left(\mathbf{v}_{m+k}+\mathbf{g}_{m+k}+\mathbf{u}_{m+k}\right)$ is its k^{th} component and $z_{m,k}\in\mathbb{Z}$ is going to be chosen later.

Using (46), we can write (44) as

$$\underbrace{\begin{bmatrix}a_{1,1}\\0\\ \vdots\\0\end{bmatrix}\mathbf{g}_1^{(1)}}_{\text{desired}} + \underbrace{\begin{bmatrix}0\\a_{2,1}\\ \vdots\\a_{L-1,1}\end{bmatrix}\mathbf{g}_1^{(1)}}_{\text{interference}} + \underbrace{\left(\begin{bmatrix}a_{1,1}\\a_{2,1}\\0\\ \vdots\\0\end{bmatrix}\mathbf{g}_1^{(2)} + \begin{bmatrix}a_{1,2}\\a_{2,2}\\0\\ \vdots\\0\end{bmatrix}\mathbf{g}_2^{(2)}\right)}_{\text{desired}}$$

$$+ \underbrace{\left(\begin{bmatrix}0\\0\\a_{3,1}\\ \vdots\\a_{L-1,1}\end{bmatrix}\mathbf{g}_1^{(2)} + \begin{bmatrix}0\\0\\a_{3,2}\\ \vdots\\a_{L-1,2}\end{bmatrix}\mathbf{g}_2^{(2)}\right)}_{\text{interference}} + \dots - \begin{bmatrix}\mathbf{t}_{F,1}\\ \vdots\\ \mathbf{t}_{F,L-1}\end{bmatrix}$$

$$= \begin{bmatrix}a_{1,2}\\0\\ \vdots\\0\end{bmatrix}\mathcal{Q}_{F,2}(\mathbf{v}_2+\mathbf{u}_2+\mathbf{g}_2) + \dots + \begin{bmatrix}a_{1,L}\\ \vdots\\ a_{L-1,L}\end{bmatrix}\mathcal{Q}_{F,L}(\mathbf{v}_L+\mathbf{u}_L). \tag{47}$$

Then it can be shown that for $m=1,\dots,L-1$ by choosing

$$\begin{bmatrix}\mathbf{g}_1^{(m)}\\ \vdots\\ \mathbf{g}_m^{(m)}\end{bmatrix} = \text{inv}(\mathbf{A}_{[1:m]})\mathbf{A}_{[1:m],m+1}\mathcal{Q}_{\Lambda_{F,m+1}}\left(\mathbf{v}_{m+1}+\mathbf{g}_{m+1}+\mathbf{u}_{m+1}\right) \tag{48}$$

$$\mathbf{t}_{F,m+1} = \sum_{k=1}^{m} a_{m+1,k}\mathbf{g}_k^{(m)} \in \Lambda_{F,m+1} \tag{49}$$

we get our result at the m^{th} BS. The idea behind this choice is to choose the m^{th} components $\mathbf{g}_1^{(m)},\dots,\mathbf{g}_m^{(m)}$ to match the m^{th} term on the RHS in (47), however, this will introduce the m^{th} interference term on the LHS which can be cancelled out by the terms $\mathbf{t}_{F,m+1},\dots,\mathbf{t}_{F,L-1}$ for each decompression step $m=1,\dots,L-1$. Note that $\mathbf{t}_{F,1}=0$ since there is no interference on the first decompression step and $\mathbf{g}_L=0$ since there are no missing terms in $\boldsymbol{\lambda}_L$ and it is sufficient to choose

$$\mathbf{t}_{F,L} = \sum_{k=1}^{L} a_{L,k}\mathbf{g}_k. \tag{50}$$

for (19) to hold for $m=L$.

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