POINCARÉ AND PLANCHEREL–POLYA INEQUALITIES IN HARMONIC ANALYSIS ON WEIGHTED COMBINATORIAL GRAPHS

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Abstract. We prove Poincaré and Plancherel–Polya inequalities for weighted $\ell^p$-spaces on weighted graphs in which the constants are explicitly expressed in terms of some geometric characteristics of a graph. We use a Poincaré-type inequality to obtain some new relations between geometric and spectral properties of the combinatorial Laplace operator. Several well-known graphs are considered to demonstrate that our results are reasonably sharp. The Plancherel–Polya inequalities allow for application of the frame algorithm as a method for reconstruction of Paley–Wiener functions on weighted graphs from a set of samples. The results are illustrated by developing Shannon-type sampling in the case of a line graph.

Key words. combinatorial Laplace operator, Poincaré- and Plancherel–Polya-type inequalities on weighted graphs, Paley–Wiener spaces, Shannon sampling on graphs, Hilbert frames

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1. Introduction and main results. Let $G$ denote an undirected weighted graph, with a finite or countable number of vertices $V(G)$ and weight function $w : V(G) \times V(G) \to \mathbb{R}_0^+$. $w$ is symmetric, i.e., $w(u,v) = w(v,u)$, and $w(u,u) = 0$ for all $u,v \in V(G)$. The edges of the graph are the pairs $(u,v)$ with $w(u,v) \neq 0$. Our assumption is that for every $v \in V(G)$ the following finiteness condition holds:

$$w_{V(G)}(v) = \sum_{u \in V(G)} w(u,v) < \infty.$$  

We fix a strictly positive weight $\nu : V(G) \to \mathbb{R}^+$ and let $\ell^p_\nu(G)$ denote the space of functions $f : V(G) \to \mathbb{C}$ satisfying

$$\|f\|_{p,\nu} = \left(\sum_{v \in V(G)} |f(v)|^p \nu(v)\right)^{1/p}, \quad 1 \leq p < \infty.$$  

For $\nu \equiv 1$ we write $\ell^p(G) := \ell^p_1(G)$.

In sections 1 and 2 we impose some implicit conditions on the positive weight $\nu$. In all other sections we consider only the spaces $\ell^2(G)$, i.e., the case $\nu = 1$.

Definition 1.1. The weighted gradient $p$-norm of a function $f$ on $V(G)$ is defined by

$$\|\nabla w f\|_p = \left(\sum_{u,v \in V(G)} \frac{1}{2} |f(u) - f(v)|^p w(u,v)\right)^{1/p}$$

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for $1 \leq p < \infty$. The set of all $f : G \to \mathbb{C}$ for which the weighted gradient $p$-norm is finite will be denoted as $D^p(\nabla_w)$.

Remark 1.2. The factor $\frac{1}{2}$ makes up for the fact that every edge (i.e., every unordered pair $(u, v)$) enters twice in the summation. Note also that loops, i.e., edges of the type $(u, u)$, in fact do not contribute.

The quantity $\|\nabla_w f\|_p$ can indeed be viewed as the norm of a gradient. However, since we only work with the norm, we refrain from explicitly defining the gradient; see, e.g., [11] for the construction of gradients on finite graphs.

We intend to prove Poincaré-type estimates involving weighted gradient $p$-norm. The term Poincaré-type inequality is used for estimates in which the norm of a function is estimated through its weighted gradient $p$-norm. Poincaré-type inequalities on graphs were considered in a number of papers (see [4], [5] and references therein). As it was pointed out to us by one of the referees the papers [12], [13], [21] are of some relevance to our work. However, our goals and methods are rather different from the objectives and approaches of these authors; our main interest lies in the derivation of sampling results which have possible applications in data compression and the construction of wavelet systems on graphs [3].

Before we describe the main results in more detail, let us give a brief overview of the paper. In section 2 we describe relations between the gradient and Laplacian in $\ell^2(G)$ and introduce Paley–Wiener spaces. Section 3 is a brief introduction to sampling and frames. Section 4 contains the proofs of the main results, Theorems 1.4 and 1.8. The role of the remaining sections is to explore the consequences of these results. In section 5, we relate the constants entering our estimates to already established structural constants for graph Laplacians, such as Dirichlet eigenvalues and isoperimetric constants, and study explicit examples showing that our constants are often close to optimal. Sections 6 and 7 study sampling applications, with section 6 focussing on frame-theoretic aspects. Note that some results which are relevant to the reconstruction formula (6.10) can be found in [9]. Section 7 evaluates the graph-theoretic estimates for a setting where the optimal answers are known (i.e., Shannon sampling on the integers), and shows that also in this case, our constants are close to optimal.

The central estimates of this paper will be derived from a suitable partition $\mathcal{S} = (S_m)_{m=0,\ldots,n}$ of $V(G)$, and certain quantities describing the weights associated with neighboring $S_j$. Given any subset $A \subset V(G)$ and $v \in V(G)$, we let

$$w_A(v) = \sum_{u \in A} w(u, v).$$

We note that $w_A(v) = 0$ iff there is no edge connecting $v$ and some element of $A$.

Given a finite partition $\mathcal{S} = (S_m)_{m=0,\ldots,n}$ of $V(G)$, we let

$$D_m = D_m(\mathcal{S}) = \sup_{v \in S_m} \frac{w_{S_{m+1}}(v)}{\nu(v)}$$

and

$$K_m = K_m(\mathcal{S}) = \inf_{v \in S_{m+1}} \frac{w_{S_m}(v)}{\nu(v)}.$$

The set $S_0$ is called the initial set of the partition $\mathcal{S}$, it is of primary importance for the following results.
**Definition 1.3.** Let \( \nu \) and \( w \) be as above and \( \mathcal{S} = (S_m)_{m=0, \ldots, n} \) be a finite partition of \( V(G) \), where \( n = n(S) \) is called the length of \( \mathcal{S} \). A triple \((\mathcal{S}, \nu, w)\) is called admissible from above if for every \( m = 0, \ldots, n \), the quantities \( D_m(\mathcal{S}) \) and \( K_m(\mathcal{S}) \) are finite and \( K_m(\mathcal{S}) > 0 \).

A necessary condition for admissibility is that every \( v \in S_m \) has a common edge with some \( u \in S_{m-1} \). We will write \( f_m \) for the restriction of \( f \in L^p_v(G) \) to \( S_m \).

With these notations, we can now formulate the central estimate of this paper. The definition of the constants relies on the convention that a product over an empty domain equals one by definition, and likewise, the sum over an empty domain equals zero. This applies to all products and sums for which the lower index bound exceeds the upper bound.

**Theorem 1.4.** Suppose that \( \mathcal{S} = (S_m)_{m=0, \ldots, n} \) is a partition of \( V(G) \) and the triple \((\mathcal{S}, \nu, w)\) is admissible from above. Then, for all \( 1 < p < \infty \), \( 1 < q < \infty \) with \( 1/p + 1/q = 1 \), and \( f \in D^p(\nabla_w) \) with \( f_0 = f|_{S_0} \in L^p_v(S_0) \), we have

\[
\|f\|_{p, \nu} \leq \left( \sum_{m=0}^{n-1} \frac{D_m}{K_j} \right)^{1/p} \|f_0\|_{p, \nu} + \left( \sum_{m=1}^{n} \sum_{k=1}^{m} \frac{1}{K_{k-1}} \left( \prod_{i=k}^{m-1} \frac{D_i}{K_i} \right)^{q/p} \prod_{i=k}^{m-1} \left( \frac{D_i}{K_i} \right)^{1/p} \right) \|\nabla_w f\|_p .
\]

For \( f \in D^1(\nabla_w) \) with \( f_0 \in L^p_v(S_0) \), we have

\[
\|f\|_{1, \nu} \leq \left( \sum_{m=0}^{n-1} \frac{D_m}{K_j} \right) \|f_0\|_{1, \nu} + \max_{k=1, \ldots, n} \left( \frac{1}{K_{k-1}} \sum_{m=k}^{n} \prod_{i=k}^{m-1} \frac{D_i}{K_i} \right) \|\nabla_w f\|_1 .
\]

Our main estimates below will involve the following quantity

\[
(1.2) \quad \delta_{\mathcal{S}, p} = \left\{ \begin{array}{ll}
\max_{k=1, \ldots, n} \left( \frac{1}{K_{k-1}} \sum_{m=k}^{n} \prod_{i=k}^{m-1} \frac{D_i}{K_i} \right), & p = 1, \\
\left( \sum_{m=1}^{n} \left( \sum_{k=1}^{m} \frac{1}{K_{k-1}} \left( \prod_{i=k}^{m-1} \frac{D_i}{K_i} \right)^{q/p} \right)^{1/p} \right)^{1/p}, & 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1,
\end{array} \right.
\]

where we assume the that the triple \((\mathcal{S}, \nu, w)\) is admissible from above.

We also define

\[
(1.3) \quad a_{\mathcal{S}, p} = \left( \sum_{m=0}^{n-1} \prod_{j=0}^{m} \frac{D_j}{K_j} \right)^{1/p}.
\]

**Definition 1.5.** For a given \( \epsilon > 0 \) let \( \mathcal{X}_\epsilon \subset D^p(\nabla_w) \cap \ell^p_v(G) \) denote the subset of all \( f \) fulfilling the inequality \( \|\nabla_w f\|_{p, \nu} \leq \epsilon \|f\|_{p, \nu} \) (1 \( \leq p < \infty \)).

The following corollary notes how estimates of Theorem 1.4 help characterize sampling sets.

**Corollary 1.6.** Let \( \mathcal{S} = (S_m)_{m=0, \ldots, n} \) be a partition of \( V(G) \). Assume that the triple \((\mathcal{S}, \nu, w)\) is admissible from above and satisfies

\[
\epsilon \delta_{\mathcal{S}, p} < 1.
\]
Then the following Plancherel–Polya-type inequalities hold for all \( f \in \mathcal{X}_p^\varepsilon \):

\[
(1.4) \quad \frac{1 - e\hat{S}_p,\nu}{a_{S,p}} \|f\|_{p,\nu} \leq \|f\|_{S_0,p,\nu} \leq \|f\|_{p,\nu}.
\]

The right-hand side of the last inequality is obvious. The left-hand side is a direct consequence of Theorem 1.4.

It is also desirable to obtain nontrivial upper bounds in (1.4). For this purpose, we consider a finite sequence \( \hat{S} = (S_m)_{m=0,\ldots,n} \) of disjoint subsets of \( V(G) \), where this time \( \bigcup S_m \neq V(G) \) is admitted. Again, we let \( n(\hat{S}) = n \), and call \( S_0 \) the initial set.

For \( 0 \leq m < n \) let

\[
\hat{K}_m(\hat{S}) = \hat{K}_m = \inf_{v \in S_m} \frac{w_{S_{m+1}}(v)}{\nu(v)},
\]

as well as

\[
\hat{D}_m(\hat{S}) = \hat{D}_m = \sup_{v \in S_{m+1}} \frac{w_{S_m}(v)}{\nu(v)}.
\]

**Definition 1.7.** Let \( \hat{S} = (S_m)_{m=0,\ldots,n} \), \( n = n(\hat{S}) \), be a finite sequence of disjoint subsets of \( V(G) \) (which is not necessarily a cover of \( V(G) \)). A triple \( (\hat{S},\nu,w) \) is said to be admissible from below if for any \( m = 0,\ldots,n-1 \) both quantities \( \hat{K}_m(\hat{S}) \), \( \hat{D}_m(\hat{S}) \) are positive and finite.

**Theorem 1.8.** Let \( \hat{S} = (S_m)_{m=0,\ldots,n} \), be a finite sequence of disjoint subsets of \( V(G) \), and \( n = n(\hat{S}) \). If the triple \( (\hat{S},\nu,w) \) is admissible from below then, for all \( 1 < p < \infty \), \( 1 < q < \infty \), with \( 1/p + 1/q = 1 \), and all \( f \in \mathcal{D}^p(\nabla_w) \) with \( f_0 = f|_{S_0} \in \ell_\nu^1(S_0) \), we have the inequality

\[
\|f\|_{p,\nu} + \left( \sum_{m=1}^{n} \left( \sum_{j=0}^{m-1} \frac{1}{K^j_k} \left( \prod_{i=k}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{q/p} \right) \right)^{1/p} \|\nabla_w f\|_p \|f_0\|_{p,\nu}.
\]

For \( f \in \mathcal{D}^1(\nabla_w) \) with \( f_0 \in \ell_\nu^1(S_0) \), we have

\[
\|f\|_{1,\nu} + \max_{k=0,\ldots,n-1} \frac{1}{K_k} \sum_{m=k}^{n} \left( \prod_{i=k}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right) \|\nabla_w f\|_1 \geq \left( \sum_{m=0}^{n} \frac{1}{K_m} \left( \prod_{i=m}^{n-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{q/p} \right)^{1/p} \|f_0\|_{1,\nu}.
\]

If \( \hat{S} = (S_m)_{m=0,\ldots,n} \) is a finite sequence of disjoint subsets of \( V(G) \), with \( n = n(\hat{S}) \) and such that the triple \( (\hat{S},\nu,w) \) is admissible from below then we introduce another set of constants

\[
\hat{\delta}_{\hat{S},p} = \begin{cases} 
\max_{k=0,\ldots,n} \frac{1}{K_k} \sum_{m=k}^{n} \left( \prod_{i=k}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{q/p} & p = 1, \\
\left( \sum_{m=1}^{n} \left( \sum_{k=0}^{m-1} \frac{1}{K_k} \left( \prod_{i=k}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{q/p} \right)^{p/q} \right)^{1/p} & 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1
\end{cases}
\]
and
\[ \hat{a}_{\hat{S},p} = \left( \sum_{m=0}^{n} \prod_{j=0}^{m-1} \frac{\hat{K}_j}{D_j} \right)^{1/p}. \]

Theorem 1.8 allows us to sharpen Corollary 1.6 some more. Namely, it is immediate that the following holds.

**Corollary 1.9.** Let \( \hat{S} = (S_m)_{m=0, \ldots, n} \), be a finite sequence of disjoint subsets of \( V(G) \), with \( n = n(\hat{S}) \) and such that the triple \( (\hat{S}, \nu, w) \) is admissible from below. Let the constants \( \delta_{S,p}, a_{S,p} \) be defined as in (1.2) and (1.3), associated with a suitable disjoint covering \( S \), possibly different from \( \hat{S} \), but with the same initial set \( S_0 \). Then, if \( \epsilon \delta_{S,p} < 1 \), the following Plancherel–Polya-type equivalence holds for all \( f \in X^\nu_p \):

\[
1 - \epsilon \delta_{S,p} \frac{\|f\|_{p,\nu}}{a_{S,p}} \leq \|f\|_{S,\nu} \leq \frac{1 + \epsilon \hat{\delta}_{\hat{S},p}}{\hat{a}_{\hat{S},p}} \|f\|_{p,\nu}.
\]

Theorems 1.4, 1.8, and their corollaries may be seen as a generalization and sharpening of [17, Theorem 2.1]; we will comment on the improvement in terms of sharpness when we discuss Shannon sampling on the integers in section 7. As the discussion in that section will show, it may occur that \( a_{S,p} \approx \hat{a}_{\hat{S},p} \) (e.g., for a large variety of sampling sets \( S \), the two only differ up to universal multiplicative constants). In this setting, the tightness of the estimate, i.e., the quotient of upper and lower bound, will be proportional to the quotient \( \frac{1 + \epsilon \hat{\delta}_{\hat{S},p}}{1 - \epsilon \delta_{S,p}} \). See section 3 for a precise definition of tightness, and section 6 for the significance of this quantity. In this case the constants \( a_{S,p} \) and \( \hat{a}_{\hat{S},p} \) assume the role of normalization constants that do not significantly affect the tightness of the estimate.

We also have two other consequences of our main estimates which show their direct relevance to Poincaré-type inequalities.

If a function \( f \in \ell^p(G) \) is supported on \( V \setminus S_0 \),

\[
\|f\|_{S_0} = 0,
\]

then \( \|f\|_{p,\nu} = \|f\|_{V \setminus S_0} \) and \( \|f\|_{S_0} \) is 0. In this situation Theorem 1.4 implies the following corollary.

**Corollary 1.10.** Suppose that \( S = (S_m)_{m=0, \ldots, n} \), is a partition of \( V(G) \) and the triple \( (S, \nu, w) \) is admissible from above. Then, for all \( 1 \leq p < \infty \), and \( f \in D^p(\nabla w) \) such that \( f \) is supported on \( S \), we have the following Poincaré-type inequality:

\[
\|f\|_{V \setminus S_0} \leq \delta_{S,p} \|\nabla w f\|_p.
\]

If a function \( f \in D^p(\nabla w) \) is supported on \( S_0 \),

\[
f|_{V \setminus S_0} = 0,
\]

then \( \|f\|_{p,\nu} = \|f\|_{S_0} \) and \( \|f\|_{V \setminus S_0} \) is 0. In this situation Theorem 1.8 implies the following corollary.
Let \( \hat{S} = (S_m)_{m=0,\ldots,n} \) be a finite sequence of disjoint subsets of \( V(G) \), with \( n = n(\hat{S}) \) and such that the triple \((\hat{S}, \nu, w)\) is admissible from below. Then, for all \( 1 \leq p < \infty \), and \( f \in D^p(\nabla_w) \), such that \( f|_{V\setminus S_0} = 0 \), we have the following Poincaré-type inequality:

\[
\|f|_{S_0}\|_{p,\nu} \leq \frac{\hat{\delta}_{S,p}}{a_{S,p} - 1} \|\nabla_w f\|_p .
\]

Clearly, the crux of the approach is the choice of the partition \( S \) (and \( \hat{S} \)). Note that in all sampling estimates, the sampling set \( S_0 \) is of primary importance, whereas \( S_1, \ldots, S_n \) are of a strictly auxiliary nature; they should be chosen to ensure small constants in the Poincaré estimate. The question of choosing the partition \( S \) (and \( \hat{S} \)), given the sampling set \( S_0 \), in a way that guarantees good control over constants, remains an interesting challenge. For the unweighted case, there is a natural approach via repeated closure operations, employed for similar purposes in [17]: Given \( S \subset V(G) \), we let \( c(S) = S \cup \{v \in V(G) : \exists u \in S \text{ with } u \sim v\} \), and \( b(S) = \text{cl}(S) \setminus S \). Define iteratively \( c^{m+1}(S) = \text{cl}(c^m(S)) \). Thus, if we pick \( S \) with \( c^{m-1}(S) \subseteq V(G) = c^0(S) \), then letting \( S_m = b(c^{m-1}(S)) \), for \( m \geq 1 \), and \( S_0 = S \), yields a partition \( S = \{S_0, \ldots, S_n\} \) with the property that each element of \( S_m \) is connected to at least one element of \( S_{m-1} \). In the case where the nonzero values of the weight \( w \) have a nontrivial lower bound \( \kappa > 0 \), this implies \( K_m \geq \kappa \); this applies in particular to all finite and/or unweighted graphs.

In the general weighted case this approach may not work, and it also might not be advisable even when a nontrivial lower bound \( \kappa > 0 \) is available. For example, consider a setting where \( w(u, v) > 0 \) for all \( u \neq v \), but with small values for most pairs \( (u, v) \). Here a single closure operation will always yield the whole set, i.e., one has \( V(G) = S_0 \cup b(S_0) \), but the resulting constants for this partition may be far from optimal.

It is important to realize that the quantities \( D_m, K_m \), as well as \( \hat{D}_m, \hat{K}_m \), provide an important characterization of diffusion geometry of a graph. Our inequalities reveal the role they are playing in analysis on the graph. In section 2 we introduce the Laplace operator and some of its spectral characteristics. In this way we obtain rather interesting relations between geometry of a graph and spectral properties of the Laplacian. These relations are presented in the form of frame inequalities (1.5) and (6.2). In turn, we use these frame inequalities to describe an efficient way to reconstruct Paley–Wiener functions from their values on sampling sets (Theorem 6.1).

The Plancherel–Polya inequalities, which are proved, allow for application of the frame algorithm as a method for reconstruction of Paley–Wiener functions on weighted graphs from a set of samples. In section 7 these results are illustrated in the case of a line graph.

2. The gradient, the Laplace operator, and the Paley–Wiener spaces.
In the case of a finite graph and unweighted \( \ell^2(G) \)-space where \( \nu \equiv 1 \), the weighted Laplace operator \( L_w : \ell^2(G) \rightarrow \ell^2(G) \) is introduced via

\[
(L_w f)(v) = \sum_{u \in V(G)} (f(v) - f(u))w(v, u).
\]

This graph Laplacian is a well-studied object; it is known to be a positive-semidefinite self-adjoint bounded operator.
According to Theorem 8.1 and Corollary 8.2 in \cite{8} if for an infinite graph there exists a $C > 0$ such that the degrees are uniformly bounded,

\begin{equation}
    d(u) = w_{V(G)}(u) = \sum_{v \in V(G)} w(u, v) \leq C,
\end{equation}

then the operator which is defined by (2.1) on functions with compact supports has a unique positive-semidefinite self-adjoint bounded extension $L_w$ which is acting according to (2.1).

What is really important for us is that in both of these cases for the nonnegative square root $L_{1/2}^{1/2}$ one has the equality

\begin{equation}
    \|L_{1/2}^{1/2}f\|_2 = \|\nabla_w f\|_2
\end{equation}

for all $f \in \mathcal{D}^2(\nabla_w)$. This fact is not difficult to show directly; for the finite case, see \cite{11}.

**Lemma 2.1.** For all $f \in \ell^2(G)$ contained in the domain of $L_w$, we have

\begin{equation}
    \|L_{1/2}^{1/2}f\|_2^2 = \|\nabla_w f\|_2^2.
\end{equation}

For $f \in \text{PW}_w(L_w)$, this implies

\begin{equation}
    \|\nabla_w f\|_2 = \|L_{1/2}^{1/2}f\|_2 \leq \sqrt{\omega}\|f\|_2.
\end{equation}

**Proof.** Let, as above, $d(u) = w_{V(G)}(u)$. Then we obtain

\[
    \langle f, L_w f \rangle = \sum_{u \in V(G)} f(u) \left( \sum_{v \in V(G)} (f(u) - f(v)) w(u, v) \right)
    = \sum_{u \in V(G)} \left( |f(u)|^2 d(u) - \sum_{v \in V(G)} f(u)\overline{f(v)} w(u, v) \right).
\]

In the same way

\[
    \langle f, L_w f \rangle = \langle L_w f, f \rangle
    = \sum_{u \in V(G)} \left( |f(u)|^2 d(u) - \sum_{v \in V(G)} \overline{f(u)} f(v) w(u, v) \right).
\]

Averaging these equations yields

\[
    \langle f, L_w f \rangle = \sum_{u \in V(G)} \left( |f(u)|^2 d(u) - \text{Re} \sum_{v \in V(G)} f(u)\overline{f(v)} w(u, v) \right)
    = \frac{1}{2} \sum_{u, v \in V(G)} |f(u)|^2 w(u, v) + |f(v)|^2 w(u, v) - 2\text{Re}f(u)\overline{f(v)} w(u, v)
    = \sum_{u, v \in V(G)} \frac{1}{2} |f(v) - f(u)|^2 w(u, v) = \|\nabla_w f\|_2^2.
\]

Now the first equality follows by taking the square root of $L_w$ (note that by spectral theory, $f$ is also in the domain of $L_{1/2}^{1/2}$), and (2.5) is an obvious consequence. \qed
On a general infinite weighted graph it is not so easy to define meaningful analogs of the Laplace operator (or, more generally, analogs of elliptic operators) in $\ell^2(G)$ and to prove an analog of (2.4).

Without going into details, we refer to recent papers [8], [10], [20] where one can find detailed analysis of these questions. As these papers show, there exist suitable finiteness assumptions which imply existence of a unique self-adjoint operator $L_w$ (which is not necessarily bounded) in $\ell^2(G)$ such that the equality (2.3) holds. In the following, we assume these assumptions to be fulfilled.

We use the spectral theorem for this operator $L_w$ to introduce the associated Paley–Wiener spaces.

**Definition 2.2.** $PW_\omega(L_w) \subset \ell^2(G)$ denote the image space of the projection operator $1_{[0, \omega]}(L_w)$ (to be understood in the sense of Borel functional calculus).

By using the spectral theorem one can show [14] that a function $f$ belongs to the space $PW_\omega(L_w)$ iff for every positive $t > 0$ the following inequality holds:

$$
\| L_t^w f \|_2 \leq \omega t \| f \|_2, \quad t > 0.
$$

Below we combine the notion of Paley–Wiener spaces with our main estimates.

Suppose that $S_0$ is the zero set of a nonzero function $f \in PW_\omega(L_w), \omega > 0$. Then Corollary 1.10, Lemma 2.1, and inequality (2.6) imply, for any partition $S$ with initial set $S_0$ and such that $(S, 1, w)$ is admissible from above, the following holds:

$$
\| f \|_{V \setminus S_0} \leq \delta_{S,2} \| \nabla_w f \|_2 \leq \delta_{S,2} \omega^{1/2} \| f \|_2 = \delta_{S,2} \omega^{1/2} \| f \|_{V \setminus S_0}.
$$

Thus, one has the inequality

$$
\delta_{S,2} \geq \frac{1}{\sqrt{\omega}}.
$$

At the same time Corollary 1.11 gives for the same function $f \in PW_\omega(L_w)$, if $(S, 1, w)$ is admissible from below then

$$
\| f \|_{V \setminus S_0} \leq \frac{\delta_{S,2}}{\alpha_{S,2}-1} \| \nabla_w f \|_2 \leq \frac{\delta_{S,2}}{\alpha_{S,2}-1} \omega^{1/2} \| f \|_2
$$

$$
= \frac{\delta_{S,2}}{\alpha_{S,2}-1} \omega^{1/2} \| f \|_{V \setminus S_0}.
$$

which implies the inequality

$$
\frac{\delta_{S,2}}{\alpha_{S,2}-1} \geq \frac{1}{\sqrt{\omega}}.
$$

Hence we have proved the following.

**Theorem 2.3.** Suppose that $S_0$ is the zero set of a function $f \in PW_\omega(L_w), \omega > 0$, which is not identically zero. Then for any admissible partition $S$ with initial set $S_0$ such that $(S, 1, w)$ is admissible from above and from below, the inequalities (2.8) and (2.10) hold.

**3. Sampling and Hilbert frames.** A set of vectors $\{\theta_v\}$ in a Hilbert space $H$ is called a Hilbert frame if there exist constants $A, B > 0$ such that for all $f \in H$

$$
A \| f \|_2^2 \leq \sum_v |(f, \theta_v)|^2 \leq B \| f \|_2^2.
$$
The largest $A$ and smallest $B$ are called, respectively, the lower and the upper frame bounds and the ratio $B/A$ is known as the tightness of the frame. If $A = B$ then $\{\theta_v\}$ is a tight frame, and if $A = B = 1$ it is called a Parseval frame. Parseval frames are similar in many respects to orthonormal bases. For example, if all members of a Parseval frame are unit vectors then it is an orthonormal basis.

According to the general theory of Hilbert frames \[6\], \[7\] the frame inequality (3.1) implies that there exists a dual frame $\{\Theta_v\}$ (which is not unique in general) for which the following reconstruction formula holds:

$$f = \sum_v \langle f, \theta_v \rangle \Theta_v.$$ \hspace{1cm} (3.2)

We call (1.4) and (1.5) Plancherel–Polya-type inequalities since they relate the norm of a function to its norm on a subset. In the case of a straight line similar inequalities were established by Plancherel and Polya for functions whose Fourier transform has compact support \[1\], \[18\], \[19\]. These inequalities will become frame inequalities in our subspaces $PW_\omega(L_w)$ if one replaces the delta functions $\delta_v$, $v \in S_0$, with their projections on $PW_\omega(L_w)$.

To people working in the classical sampling theory of Paley–Wiener functions such kinds of inequalities are also known as sampling inequalities or sampling estimates. In this case the set $S_0$ is called a sampling set. There are at least two reasons why such sets are of special interest. First, as the left side of (1.4) shows if $f, g, f-g \in X_p^p$ and $f$ and $g$ coincide on $S_0$ then they coincide everywhere on $V(G)$. In this sense the set $S_0$ can be called a uniqueness set (for certain functions in $X_p^p$). Second, the same left side of (1.4) shows that the extension operator from the set $S_0$ is continuous.

It is very important for the sampling theory that Paley–Wiener spaces of functions $PW_\omega(L_w), \omega > 0$, are linear subspaces of $X_p^p$. Thus, for any two $f$ and $g$ in $PW_\omega(L_w)$ their difference $f-g$ will automatically belong to $PW_\omega(L_w)$.

It is also interesting to note that the left side of (1.4) can be considered as a form of a maximum principle and in this case the sampling set $S_0$ should be treated as a kind of a boundary set.

4. Proofs of the main results.

4.1. Proof of Theorem 1.4. We retain the definitions and notations from the previous sections.

Let us consider a function $f$ in $D_w^p(\nabla)$ and introduce a family of auxiliary quantities $\varphi_{m,p}$, for $m \geq 1$, by

$$\varphi_{m,p} = \left( \frac{1}{K_{m-1}} \sum_{u \in S_m} \sum_{v \in S_{m-1}} |f(u) - f(v)|^p w(u,v) \right)^{1/p}.$$ \hspace{1cm}

Note that by assumption all denominators $K_{m-1}$ are positive. Let $f_m$ be the restriction of $f \in L_p^p(G)$ to $S_m$. We then obtain, for $m \geq 1$,

$$\|f_m\|_{p,\nu} \leq \left( \frac{D_{m-1}}{K_{m-1}} \right)^{1/p} \|f_{m-1}\|_{p,\nu} + \varphi_{m,p}.$$ \hspace{1cm} (4.1)
For the proof of that estimate, we compute

\[ \| f_m \|_{p, \nu} = \left( \sum_{u \in S_m} |f(u)|^{p} \nu(u) \right)^{1/p} \]

\[ = \left( \sum_{u \in S_m} \sum_{v \in S_{m-1}} |f(u)|^{p} \frac{w(u, v) \nu(v)}{w_{S_{m-1}}(u)} \right)^{1/p} \]

\[ \leq \left( \sum_{u \in S_m} \sum_{v \in S_{m-1}} (|f(v)| + |f(u) - f(v)|)^{p} \frac{w(u, v) \nu(u)}{w_{S_{m-1}}(u)} \right)^{1/p} \]

\[ \leq \left( \sum_{u \in S_m} \sum_{v \in S_{m-1}} |f(v)|^{p} \frac{w(u, v) \nu(u)}{w_{S_{m-1}}(u)} \right)^{1/p} \]

\[ + \left( \sum_{u \in S_m} \sum_{v \in S_{m-1}} |f(u) - f(v)|^{p} \frac{w(u, v) \nu(u)}{w_{S_{m-1}}(u)} \right)^{1/p} \]

using the triangle inequality for weighted \( \ell^p \)-norms. By the definition of \( K_{m-1} \), it is clear that the second term is \( \leq \varphi_{m,p} \). For the first term, we find

\[ \left( \sum_{u \in S_m} \sum_{v \in S_{m-1}} |f(v)|^{p} \frac{w(u, v) \nu(u)}{w_{S_{m-1}}(u)} \right)^{1/p} = \left( \sum_{v \in S_{m-1}} |f(v)|^{p} \nu(v) \sum_{u \in S_m} \frac{w(u, v) \nu(u)}{w_{S_{m-1}}(u) \nu(v)} \right)^{1/p} \]

\[ \leq \frac{D_{m-1}}{K_{m-1}} \left\| f_{m-1} \right\|_{p, \nu}, \]

where we used that for \( v \in S_{m-1} \),

\[ \sum_{u \in S_m} \frac{w(u, v) \nu(u)}{w_{S_{m-1}}(u) \nu(v)} \leq \sum_{u \in S_m} \frac{w(u, v)}{K_{m-1} \nu(v)} = \frac{w_{S_m}(v)}{K_{m-1} \nu(v)} \leq \frac{D_{m-1}}{K_{m-1}}. \]

Thus (4.1) is proved.

As a consequence, we can now estimate \( \| f_m \|_{p, \nu} \) in terms of \( \| f_0 \|_{p, \nu} \) and the \( \varphi_{m,p} \):

\[ \| f_m \|_{p, \nu} \leq \left( \prod_{j=0}^{m-1} \frac{D_j}{K_j} \right)^{1/p} \| f_0 \|_{p, \nu} + \sum_{j=1}^{m} \varphi_{j,p} \left( \prod_{i=j}^{m-1} \frac{D_i}{K_i} \right)^{1/p}, \]

\[ f \in D_w(\nabla). \]

The proof proceeds by induction over \( m \): The case \( m = 0 \) is trivial. The induction step is a straightforward application of (4.1), followed by simplification.

After these preliminary calculations, we can now estimate \( \| f \|_{p, \nu} \) for \( f \in D_w(\nabla) \). Using that \( V(G) = \bigcup S_m \) disjointly, we obtain from (4.2) via the triangle inequality...
that
\[ \|f\|_{p,\nu} = \left( \sum_{m=0}^{n} \|f_m\|_{p,\nu}^p \right)^{1/p} \]

(4.3) \[ \leq \left( \sum_{m=0}^{n-1} \prod_{j=0}^{m} D_j \|f_0\|_{p,\nu}^p \right)^{1/p} + \left( \sum_{m=1}^{n} \left( \sum_{j=1}^{m} \varphi_{j,p} \left( \prod_{i=j}^{m-1} \frac{D_i}{K_i} \right) \right)^p \right)^{1/p} \]

Note that the first summand is already as required, and it remains to estimate the second one. For this purpose, we introduce \( \tilde{\varphi}_{j,p} = K_{j-1}^{1/p} \varphi_{j,p} \), and use repeated applications of Hölder’s inequality to deduce for \( 1 < p < \infty \) that

\[
\left( \sum_{m=1}^{n} \left( \sum_{j=1}^{m} \varphi_{j,p} \left( \frac{m-1}{\prod_{i=j}^{m} K_i} \right)^{1/p} \right)^{p^{1/p}} \right)^{1/p} \]

\[
\leq \left( \sum_{m=1}^{n} \left( \sum_{j=1}^{m} \tilde{\varphi}_{j,p} \right) \cdot \left( \sum_{k=1}^{m} \frac{1}{K_{k-1}^{q/p}} \left( \prod_{i=k}^{m-1} \frac{D_i}{K_i} \right)^{q/p} \right)^{1/p} \right)^{1/p} \]

\[
\leq \left( \sum_{j=1}^{n} \tilde{\varphi}_{j,p} \cdot \left( \sum_{m=j}^{n} \left( \sum_{k=1}^{m} \frac{1}{K_{k-1}^{q/p}} \left( \prod_{i=k}^{m-1} \frac{D_i}{K_i} \right)^{q/p} \right)^{1/p} \right)^{1/p} \right) \]

where the last inequality is due to the \((1 - \infty)\)-Hölder inequality, using in addition that

\[
\tilde{\varphi}_{j,p} \rightarrow \sum_{m=j}^{n} \left( \sum_{k=1}^{m} \frac{1}{K_{k-1}^{q/p}} \left( \prod_{i=k}^{m-1} \frac{D_i}{K_i} \right)^{q/p} \right)^{p/q} \]

is increasing.

Plugging in the definition of \( \tilde{\varphi}_{j,p} \), we find that

\[
\sum_{j=1}^{n} \tilde{\varphi}_{j,p} = \sum_{j=1}^{n} \sum_{u \in S_j} \sum_{v \in S_{j-1}} |f(u) - f(v)|^p w(u, v) \leq \|\nabla w f\|_p^p
\]

by disjointness of the \( S_j \). Note that as a consequence of disjointness, for every pair \((u, v) \in V(G)^2 \) with \( w(u, v) > 0 \), there will be at most one of the pairs \((u, v), (v, u)\) appearing in the above summation. This concludes the proof for \( 1 < p < \infty \). For \( p = 1 \), the required estimate follows by a similar (easier) calculation from Hölder’s inequality.

4.2. Proof of Theorem 1.8. The proof ideas are very similar to the ones used in the previous subsection, the chief difference being a change in direction: We first
obtain lower estimates for \( \|f_{m+1}\|_{p,\nu} \) in terms of \( \|f_m\|_{p,\nu} \), and ultimately, in terms of \( \|f_0\|_{p,\nu} \). Then summation over \( m \) yields the desired results. For \( 0 \leq m < n \) and \( f \in D_{\nu}^p(\nabla) \) let

\[
\hat{\varphi}_{m,p} = \left( \frac{1}{K_m} \sum_{u \in S_m} \sum_{v \in S_{m+1}} |f(u) - f(v)|^p w(u,v) \right)^{1/p}.
\]

Essentially the same proof as for (4.1) yields, for \( 0 \leq m < n \), the inequality

\[
(4.4) \quad \|f_m\|_{p,\nu} \leq \left( \frac{\hat{D}_m}{\hat{K}_m} \right)^{1/p} \|f_{m+1}\|_{p,\nu} + \hat{\varphi}_{m,p}.
\]

Indeed, by analogous calculations we obtain

\[
\|f_m\|_{p,\nu} \leq \left( \sum_{v \in S_{m+1}} |f(v)|^p \nu(v) \sum_{u \in S_m} \frac{w(u,v)\nu(u)}{w_{S_{m+1}}(u)\nu(v)} \right)^{1/p} \leq \frac{\hat{D}_m}{\hat{K}_m} \|f_{m+1}\|_{p,\nu} + \hat{\varphi}_{m,p},
\]

which yields (4.4).

Using this observation, we prove

\[
(4.5) \quad \left( \prod_{j=0}^{m-1} \frac{\hat{K}_j}{\hat{D}_j} \right)^{1/p} \|f_0\|_{p,\nu} \leq \|f_m\|_{p,\nu} + \sum_{j=0}^{m-1} \left( \prod_{i=j}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{1/p} \hat{\varphi}_{j,p}.
\]

For this purpose we first employ (4.4) to show inductively

\[
\|f_0\|_{p,\nu} \leq \left( \prod_{j=0}^{m-1} \frac{\hat{D}_j}{\hat{K}_j} \right)^{1/p} \|f_m\|_{p,\nu} + \sum_{j=0}^{m-1} \left( \prod_{i=j}^{m-1} \frac{\hat{D}_i}{\hat{K}_i} \right)^{1/p} \hat{\varphi}_{j,p},
\]

and then divide both sides by \( (\prod_{j=0}^{m-1} \frac{\hat{D}_j}{\hat{K}_j})^{1/p} \).
We are now ready to prove the estimate: Using disjointness of the $S_m$ and the triangle inequality, we get

\[
\|f\|_{p,v} \geq \left( \sum_{m=0}^{n} \|f_m\|_{p,v}^p \right)^{1/p}
\]

\[
\geq \left( \sum_{m=0}^{n} \left( \|f_m\|_{p,v} + \sum_{j=0}^{m-1} \left( \prod_{i=j+1}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{1/p} \|\varphi_j\|_p \right)^{1/p} \right)^{1/p}
\]

\[
- \left( \sum_{m=1}^{n} \left( \sum_{j=0}^{m-1} \left( \prod_{i=j+1}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{1/p} \|\varphi_j\|_p \right)^{1/p} \right)^{1/p}
\]

\[
\leq \left( \sum_{m=1}^{n} \left( \sum_{j=0}^{m-1} \left( \prod_{i=j+1}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{1/p} \|\varphi_j\|_p \right)^{1/p} \right)^{1/p}
\]

\[
- \left( \sum_{m=1}^{n} \left( \sum_{j=0}^{m-1} \left( \prod_{i=j+1}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{1/p} \|\varphi_j\|_p \right)^{1/p} \right)^{1/p}
\]

(4.6)

Here the last inequality used (4.5). We now introduce $\tilde{\varphi}_{j,p} = \hat{K}_j^{1/p} \varphi_{j,p}$, and repeat the arguments following (4.3) to obtain $p > 1$ that

\[
\left( \sum_{m=1}^{n} \left( \sum_{j=0}^{m-1} \left( \prod_{i=j+1}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{1/p} \|\varphi_j\|_p \right)^{1/p} \right)^{1/p}
\]

\[
\leq \left( \sum_{m=1}^{n} \left( \sum_{j=0}^{m-1} \left( \prod_{i=j+1}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{1/p} \|\varphi_j\|_p \right)^{1/p} \right)^{1/p}
\]

\[
\leq \left\| \nabla_w f \right\|_p \cdot \left( \sum_{m=1}^{n} \left( \sum_{k=0}^{m-1} \frac{1}{\hat{K}_k^{q/p}} \left( \prod_{i=k+1}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)^{q/p} \right)^{1/p} \right)^{1/p}
\]

thus finishing the argument for $1 < p < \infty$. For $p = 1$, we obtain

\[
\left( \sum_{m=1}^{n} \left( \sum_{j=0}^{m-1} \left( \prod_{i=j+1}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right) \|\varphi_j\|_1 \right)^{1/p} \right)^{1/p}
\]

\[
\leq \left\| \nabla_w f \right\|_1 \cdot \max_{k=0, \ldots, n-1} \frac{1}{\hat{K}_k} \sum_{m=1}^{n} \left( \prod_{i=k+1}^{m-1} \frac{\hat{K}_i}{\hat{D}_i} \right)
\]

5. Poincaré inequalities and spectral properties of finite graphs. In this section we consider a finite connected graph $G$ of $N$ vertices. Under this assumption the Laplace operator $L_w$ has discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{N-1}$. Let $u_0, u_1, \ldots, u_{N-1}$ be a corresponding orthonormal basis of $L^2(G)$ fulfilling

\[
L_w u_j = \lambda_j u_j, \quad j = 0, \ldots, N-1.
\]
Let $\mathcal{N}[0, \omega)$ denote the number of eigenvalues of $L$ in $[0, \omega)$, counted with multiplicities. The notation $\mathcal{N}[\omega, \lambda_{N-1}]$ is used to denote a number of eigenvalues of $L$ in $[\omega, \lambda_{N-1}]$.

In this situation $PW_{\omega}(L_w)$ is the span of all eigenfunctions $u_j$ for which corresponding eigenvalues $\lambda_j$ are not greater than $\omega$.

We are using the same notation as above.

**Theorem 5.1.** Suppose that $S = \{S_0, \ldots, S_{n(S)}\}$ is an admissible partition of $V(G)$. Then the following inequality holds:

$$\mathcal{N}[0, (\delta_{S,2}^2)^{-1}] \leq |S_0|,$$

and if $|V(G)| = N$, then

$$\mathcal{N}[0, (\delta_{S,2}^2)^{-1}, \lambda_{N-1}] \geq N - |S_0|.$$

**Proof.** According to Theorem 1.4 and Lemma 2.1 if $\omega < (\delta_{S,2}^2)^{-1}$ then $S_0$ is a uniqueness set for the space $PW_{\omega}(G)$. Since the dimension of $PW_{\omega}(G)$ is exactly the number of eigenvalues (counted with multiplicities) of $L_w$ on the interval $[0, \omega)$ it means that $|S_0|$ cannot be less than the number of eigenvalues of $L_w$ on the interval $[0, \omega)$ for every $\omega < (\delta_{S,2}^2)^{-1}$. Thus,

$$\mathcal{N}[0, (\delta_{S,2}^2)^{-1}] \leq |S_0|.$$

The second inequality is obvious. $\square$

Fix a subset $S_0 \subset V(G)$ and consider all partitions $S_{S_0} = \{S_0, \ldots, S_{n(S)}\}$ of $V(G)$ for which $K_m(S) > 0$ for all $m = 0, \ldots, n(S) - 1$; then the previous theorem implies the inequalities

$$\max_{S_{S_0}} \mathcal{N}[0, (\delta_{S_{S_0},2}^2)^{-1}] \leq |S_0|,$$

and

$$\min_{S_{S_0}} \mathcal{N}[(\delta_{S_{S_0},2}^2)^{-1}, \lambda_{N-1}] \geq N - |S_0|.$$

In the simplest case $S_0 \cup bS_0 = V(G)$ we obtain the following.

**Corollary 5.2.** For any $S_0 \subset V(G)$ such that $S_0 \cup bS_0 = V(G)$ the following inequality holds:

$$\mathcal{N}[0, K_0(S_0)] \leq |S_0|,$$

and if $|V(G)| = N$, then

$$\mathcal{N}[K_0(S_0), \lambda_{N-1}] \geq N - |S_0|.$$  

The following corollary gives a lower bound for each nonzero eigenvalue.

Note that this result is “local” in the sense that any randomly chosen set $S_0 \subset V(G)$ can be used to obtain an estimate (5.7) below.

**Corollary 5.3.** Suppose that there exists an admissible partition $S_{S_0} = \{S_0, \ldots, S_{n(S)}\}$ with $|S_0| \leq k$. Then the following inequality holds:

$$\lambda_k \geq \max_{S_{S_0}} (\delta_{S_{S_0},2}^2)^{-1},$$

where max is taken over all admissible partitions $S_{S_0}$ with $|S_0| \leq k$. 

Thus, we obtain the following theorem.

If \( \lambda_k < (\delta_{S_{0,2}}^2)^{-1} \) then \( S_0 \) is a uniqueness set for the space \( PW_{\lambda_k}(G) \). Since the dimension of \( PW_{\lambda_k}(G) \) is exactly \( k + 1 \) it implies that \( |S_0| \geq k + 1 \).

From here we obtain that if \( |S_0| \leq k \) then the inequality \( \lambda_k \geq (\delta_{S_{0,2}}^2)^{-1} \) holds. The corollary is proved.

Given an \( \mathcal{M} \subset V(G) \) let us introduce \( \Lambda_{D}(\mathcal{M}) \) (“a Dirichlet” eigenvalue) as

\[
\Lambda_{D}(\mathcal{M}) = \inf_{f \in \ell^2(\mathcal{M}), \ f \neq 0} \frac{\langle f, Lw f \rangle}{\|f\|_2^2},
\]

where \( \ell^2(\mathcal{M}) \) is the set of all functions supported on \( \mathcal{M} \).

**Theorem 5.4.** For any \( S_0 \) and any admissible partition \( S_{S_0} = \{S_0, \ldots, S_{n(S)}\} \) with initial set \( S_0 \)

\[
\Lambda_{D}(V \setminus S_0) \leq \delta_{S_{S_0}, 2},
\]

and then

\[
\Lambda_{D}(V \setminus S_0) \leq \min_{S_{S_0}} \delta_{S_{S_0}, 2},
\]

where \( \min \) is taken over all admissible partition \( S_{S_0} = \{S_0, \ldots, S_{n(S)}\} \) with initial set \( S_0 \).

**Proof.** Obviously, \( \Lambda_{D}(\mathcal{M}) \) is the smallest constant such that for all functions \( f \) supported on \( \mathcal{M} \)

\[
\|f\|_2 \leq \Lambda_{D}(\mathcal{M})\|L_{w}^{1/2} f\|_2.
\]

Take an \( S_0 \) and any admissible partition \( S_{S_0} = \{S_0, \ldots, S_{n(S)}\} \) with initial set \( S_0 \). If \( f \) is supported on \( V \setminus S_0 \) then we have

\[
\|f\|_2 \leq \delta_{S_{S_0}, 2}\|L_{w}^{1/2} f\|_2
\]

and together with (5.10) it gives the inequality

\[
\Lambda_{D}(V \setminus S_0) \leq \min_{S_{S_0}} \delta_{S_{S_0}, 2}.
\]

This finishes the proof.

If \( \mathcal{M} \cup b\mathcal{M} = V(G) \) then our \( \Lambda_{D}(\mathcal{M}) \) coincides with the Dirichlet eigenvalue \( \lambda_{D}(\mathcal{M}) \) introduced in [2]. If \( \delta \) is the isoperimetric dimension of a graph \( G \) it is known [2] that there exists a constant \( C_{\delta} \) which depends just on \( \delta \) such that

\[
\lambda_{D}(\mathcal{M}) > C_{\delta} \left( \frac{1}{\text{vol} \mathcal{M}} \right)^{2/\delta}, \text{vol} \mathcal{M} = \sum_{v \in \mathcal{M}} d(v).
\]

Thus, we obtain the following theorem.

**Theorem 5.5.** If \( \delta \) is the isoperimetric dimension of the graph \( G \) and \( S_0 \cup bS_0 = V(G) \) then there exists a constant \( C_{\delta} \) which depends just on \( \delta \) such that the following inequality holds:

\[
C_{\delta} \left( \frac{1}{\text{vol} bS_0} \right)^{2/\delta} \leq \delta_{S_{S_0}, 2}, \ S_{S_0} = \{S_0, bS_0\}.
\]
We consider several concrete examples illustrating that the constants obtained in our inequalities are close to optimal.

**Example 5.6.** Suppose that $G$ is a star graph $\{v_0, v_1, \ldots, v_N\}$ whose center is $v_0$. Let $S_0$ be the vertex $\{v_0\}$. Then $K_0 = 1, D_0 = N$, and since

$$\|f\|_2 \leq \left(1 + \frac{D_0}{K_0}\right)^{1/2} \|f_0\|_2 + \frac{1}{K_0^{1/2}} \|\nabla f\|_2, \quad f_0 = f|s,$$

we have

$$\|f\|_2 \leq \sqrt{N + 1} |f(v_0)| + \|\nabla f\|_2.$$  

In particular, for the constant function $f(v_j) = 1$, $0 \leq j \leq N$, one has $\|f\|_2 = \sqrt{N + 1}$, $\|\nabla f\|_2 = 0$, and the inequality (5.14) becomes

$$\sqrt{N + 1} \leq \sqrt{N + 1}.$$  

For the same star graph $G$, $S_0 = \{v_0\}$, and

$$\delta_{S_0} = \frac{1}{K_0^{1/2}} = 1$$

we obtain according to Corollary 5.3

$$\lambda_1 \geq K_0 = 1.$$  

But for a star graph and our Laplacian, $\lambda_1 = 1$.

**Example 5.7.** Let $C_N$ be a cycle of $N$ vertices $\{v_1, \ldots, v_N\}$. Take another vertex $v_0$ and make a graph $C_N \cup \{v_0\}$ by connecting $v_0$ to each of $v_1, \ldots, v_N$. It is the so-called wheel graph.

Let $\lambda_k(N)$ be a nonzero eigenvalue of the operator $L$ on the graph $C_N$ and let $u_k$ be a corresponding orthonormal eigenfunction. Construct a function $\tilde{u}_k$ on the graph $C_N \cup \{v_0\}$ such that $\tilde{u}_k(v) = u_k(v)$ if $v \in C_N$ and $\tilde{u}_k(v_0) = 0$. Since $u_k$ is orthogonal to the constant function 1 we have that

$$\sum_{v_j \in C_N} u_k(v_j) = 0$$

and this implies that for the operator $L$ on $C_N \cup \{v_0\}$

$$L\tilde{u}_k(v_0) = 0.$$  

Clearly, for every $v_j \in C_N$ one has

$$L\tilde{u}(v_j) = Lu_k(v_j) + u(v_j) = (\lambda_k(N) + 1)u(v_j).$$

Thus,

$$L\tilde{u}_k = (\lambda_k(N) + 1) \tilde{u}_k,$$

and since $\|\tilde{u}_k\|_2 = 1$ we have that

$$\|L^{1/2}\tilde{u}_k\|_2 = (\lambda_k(N) + 1)^{1/2}.$$
Let $S_0$ be the graph $C_N = \{v_1, \ldots, v_N\}$. In this case the boundary of $S_0$ is the point
$v_0$, $K_0 = N$, $D_0 = 1$ and then for the function $\bar{u}_k$ the Theorem 1.4 implies
\[
\|\bar{u}_k\|_2 \leq \left(1 + \frac{D_0}{K_0}\right)^{1/2} \|\bar{u}_k\|_{S_0} + \frac{1}{K_0^{1/2}} \|\nabla\bar{u}_k\|_2
\]
(5.15)
\[
= \left(1 + \frac{D_0}{K_0}\right)^{1/2} \|\bar{u}_k\|_{S_0} + \sqrt{\frac{1}{K_0}} \|L^{1/2}\bar{u}_k\|_2
\]
or
\[
1 \leq \sqrt{1 + \frac{1}{N}} + \frac{\lambda_k(N) + 1}{N}.
\]
Since all eigenvalues $\lambda_k(N)$ belong to $[0, 4]$, we see that the right-hand side of the last
inequality goes to one when $N$ goes to infinity.

6. Reconstruction of Paley–Wiener functions using the frame algorithm. We are going to apply Corollary 1.9 to functions in $PW_\omega(L_w)$. Thus, we take $\sqrt{\omega}$ as the $\epsilon$ in Corollary 1.9 and we make an assumption that the following inequality holds:
\[
\omega < \frac{1}{2} \left(\sum_{m=1}^{n} \sum_{j=1}^{m} \frac{1}{K_{j-1}} \prod_{i=j}^{m-1} D_i\right)^{-1}.
\]
(6.1)
In this case we have nontrivial Plancherel–Polya-type inequalities (1.5). Let us denote
by $\theta_v$, where $v \in S$, the orthogonal projection of the Dirac measure $\delta_v$, $v \in S$, onto
the space $PW_\omega(L_w)$. Since for functions in $PW_\omega(L_w)$ one has $f(v) = \langle f, \theta_v \rangle$, $v \in S$, the
inequality (1.5) takes the form of a frame inequality in the Hilbert space $H = PW_\omega(L_w)$
\[
\left(1 - \frac{\epsilon \delta_{S,2}}{a_{S,2}}\right)^2 \|f\|_2^2 \leq \sum_{v \in S} |\langle f, \theta_v \rangle|^2 \leq \left(1 + \frac{\epsilon \delta_{S,2}}{a_{S,2}}\right)^2 \|f\|_2^2, \ \epsilon = \sqrt{\omega}
\]
(6.2)
for all $f \in PW_\omega(L_w)$. Thus there exists a dual frame (which is not unique) $\{\Theta_v\}$, $v \in S$, $\Theta_v \in PW_\omega(L_w)$, in the space $PW_\omega(L_w)$ such that for all $f \in PW_\omega(L_w)$ the
following reconstruction formula holds:
\[
f = \sum_{v \in S} f(v)\Theta_v.
\]
(6.3)
Suppose that $S_0 \subset V(G)$ is given. We want to determine sufficient conditions
for $\omega$ to ensure that $S_0$ is a sampling set for $PW_\omega(L_w)$. For this purpose, we let
$S_1 = V(G) \setminus S_0$, and compute the quantities $a_{S,2}$ and $\delta_{S,2}$ for $S = (S_0, S_1)$. This
requires computing
\[
D_0 = D_0(S) = \sup_{v \in S_0} w_{S_1}(v)
\]
and
\[
K_0 = K_0(S) = \inf_{v \in S_1} w_{S_0}(v).
\]
In order to meet the requirements of Theorem 1.4, each \( v \in S_1 \) must be connected to at least one \( v \in S_0 \). The constants in Corollary 1.6 are then computed as

\[
a_{S,2} = \left(1 + \frac{D_0}{K_0}\right)^{1/2}, \quad \delta_{S,2} = \frac{1}{K_0^{1/2}}.
\]

Thus, we have

\[
\|f\|_2 \leq \left(1 + \frac{D_0}{K_0}\right)^{1/2} \|f_0\|_2 + \frac{1}{K_0^{1/2}} \|\nabla w f\|_2
\]

\[
= \left(1 + \frac{D_0}{K_0}\right)^{1/2} \|f_0\|_2 + \sqrt{\frac{1}{K_0}} \|L^{1/2}_w f\|_2.
\]

In particular, applying (2.5) along with assumption (6.4) \( \omega < K_0 \) yields the following sampling estimate for all \( f \in PW_\omega(L_w) \):

(6.5)

\[
\|f\|_2 \leq \left(1 - \sqrt{\frac{\omega}{K_0}}\right) \left(1 + \frac{D_0}{K_0}\right)^{1/2} \|f_0\|_2, \quad f_0 = f|_{S_0}.
\]

At the same time

\[
\hat{a}_{S,2} = \left(1 + \frac{\hat{K}_0}{\hat{D}_0}\right)^{1/2}, \quad \hat{\delta}_{S,2} = \frac{1}{\hat{D}_0^{1/2}}.
\]

This yields the norm estimate

(6.6)

\[
\|f\|_2 + \frac{1}{\hat{D}_0^{1/2}} \|\nabla w f\|_2 \geq \left(1 + \frac{\hat{K}_0}{\hat{D}_0}\right)^{1/2} \|f_0\|_2, \quad f_0 = f|_{S}.
\]

If (2.5) holds, then

(6.7)

\[
\left(1 + \frac{\hat{K}_0}{\hat{D}_0}\right)^{1/2} \|f_0\|_2 \leq \|f\|_2 + \frac{1}{\hat{D}_0^{1/2}} \|\nabla w f\|_2 \leq \left(1 + \sqrt{\frac{\omega}{\hat{D}_0}}\right) \|f\|_2.
\]

After all, for functions \( f \) in \( PW_\omega(L_w) \) with \( \omega < K_0 \) we obtain the following frame inequality

(6.8)

\[
A \|f\|^2 \leq \sum_{v \in S} \|\langle f, \theta_v \rangle\|^2 \leq B \|f\|^2, \quad f_0 = f|_{S},
\]

where

(6.9)

\[
A = \frac{\left(1 - \sqrt{\frac{\omega}{K_0}}\right)^2}{1 + \frac{D_0}{K_0}}, \quad B = \frac{\left(1 + \sqrt{\frac{\omega}{\hat{D}_0}}\right)^2}{1 + \frac{\hat{K}_0}{\hat{D}_0}}.
\]

It shows that if the condition \( \omega < K_0(S_0) \) is satisfied the set \( S_0 \) is a sampling set for the space \( PW_\omega(L_w) \) and there exists a dual frame (which is not unique in general)
\{\Theta_v\}, \ v \in S, \ \Theta_v \in PW_\omega(L_w), in the space PW_\omega(L_w) such that for all \( f \in PW_\omega(L_w) \) the following reconstruction formula holds:

\[ f = \sum_{v \in S} f(v)\Theta_v. \tag{6.10} \]

However, it is not easy to find a dual frame \( \{\Theta_v\}, \ v \in S \). For this reason we are going to adopt the frame algorithm (see [7], Chap. 5) for reconstruction of functions in \( PW_\omega(L_w) \) from the set \( S_0 \). What follows is a brief description of the frame algorithm.

Let \( \{e_j\} \) be a frame in a Hilbert space \( H \) with frame bounds \( A, B \), i.e.,

\[ A\|f\|^2 \leq \sum_j |\langle f, e_j \rangle|^2 \leq B\|f\|^2, \ f \in H. \]

Given a relaxation parameter \( 0 < \gamma < \frac{2}{B} \), set \( \eta = \max\{|1 - \gamma A|, |1 - \gamma B|\} < 1 \). Let \( f_0 = 0 \) and define recursively

\[ f_n = f_{n-1} + \gamma S(f - f_{n-1}), \tag{6.11} \]

where \( S \) is the frame operator which is defined on \( H \) by the formula

\[ Sf = \sum_j \langle f, e_j \rangle e_j. \]

In particular, \( f_1 = \gamma Sf = \gamma \sum_j \langle f, e_j \rangle e_j \). Then \( \lim_{n \to \infty} f_n = f \) with a geometric rate of convergence, that is,

\[ \|f - f_n\| \leq \eta^n\|f\|. \tag{6.12} \]

Note, that for the choice \( \gamma = \frac{2}{A+B} \) the convergence factor is

\[ \eta = \frac{B - A}{A + B} \leq \frac{B/A - 1}{2}, \tag{6.13} \]

which emphasizes the interest in frames with tightness \( B/A \) close to 1.

Let us go back to our situation in which we consider a subset of vertices \( S_0 \subset V \). By \( \theta_v \), where \( v \in S_0 \), we denote the orthogonal projection of the Dirac measure \( \delta_v \), \( v \in S_0 \), onto the subspace \( PW_\omega(L_w) \). Given a relaxation parameter \( 0 < \gamma < \frac{2}{B} \), where \( B \) is defined in (6.9), consider the recurrence sequence \( g_0 = 0 \), and

\[ g_n = g_{n-1} + \gamma \sum_{v \in S_0} (f - g_{n-1})(v)\theta_v. \tag{6.14} \]

The reconstruction method for functions in \( PW_\omega(L_w) \) from their values on \( S_0 \) is the following.

**Theorem 6.1.** For \( S = (S_0, S_1) \), if the assumption (6.4) holds, then for all \( f \in PW_\omega(L_w) \) the following inequality holds for all natural \( n \):

\[ \|f - g_n\|_2 \leq \eta^n\|f\|_2, \tag{6.15} \]

where the convergence factor \( \eta \) is given by (6.13) and \( A \) and \( B \) are defined in (6.9).
7. **Sampling on the integers.** In this section, we want to gauge the precision of our sampling estimates by studying a setup for which the optimal answers are known. We consider the Cayley graph of the group $\mathbb{Z}$, associated with the symmetric system of generators given by $\{\pm 1\}$, i.e., $u \sim v$ iff $|u - v| = 1$. The sampling sets we consider are subgroups of the type $k\mathbb{Z}$, with $k \in \mathbb{Z}$. We want to identify the critical value $\omega_0 = \omega_0(k)$ with the property that for all $\omega < \omega_0$, Corollary 1.9 is applicable.

As will be shortly seen, the spectrum of the associated graph Laplacian is identical to the Fourier spectrum (up to a certain reparameterization), which will allow us to compare the value $\omega_0(k)$ to the optimal value derived from the Shannon sampling theorem.

The Fourier transform of $f \in \ell^1(\mathbb{Z})$ is defined as
\[
\hat{f}(\xi) = \sum_{k \in \mathbb{Z}} f(k)e^{-ik\xi}.
\]
The Plancherel transform of $f \in \ell^2(\mathbb{Z})$ is denoted by the same symbol. We consider $\hat{f}$ as a function on the interval $[-\pi,\pi]$. The graph Laplacian associated with the Cayley graph is a convolution product
\[
L(f) = f * C_L \quad \text{with} \quad C_L = 2\delta_0 - \delta_1 - \delta_{-1}.
\]
Hence, by the convolution theorem
\[
\widehat{L(f)}(\xi) = \hat{f}(\xi)\hat{C}_L(\xi), \quad \text{where} \quad \hat{C}_L(\xi) = 2 - 2\cos(\xi).
\]
The Fourier-analytic Paley–Wiener space is defined as
\[
PW_\omega = \{ f \in \ell^2(\mathbb{Z}) : \hat{f}(\xi) = 0 \text{ a.e. outside } [-\omega,\omega] \}.
\]
The Fourier transform $\hat{C}_L$ is a positive even function that strictly increases on $[0,\pi]$, hence (7.1), which together with the Plancherel transform, implies for $\omega \in [0,\sqrt{2}]$ that
\[
PW_\omega(L) = PW_{\omega'} , \quad \omega' = \psi(\omega).
\]
Here $\psi$ denotes the inverse map of the restriction of $\omega \mapsto 2 - 2\cos(\xi)$ to the interval $[0,\pi]$.

The Shannon sampling theorem provides a sharp characterization of the bandwidth $\omega_0(k)$.

**Theorem 7.1.** $S = k\mathbb{Z}$ is a sampling set for $PW_\omega$ iff $k\omega \leq \pi$. In this case, we have the norm equality
\[
\forall f \in PW_\omega : \frac{1}{\sqrt{k}}\|f\|_2 = \|f|_S\|_2.
\]

Let us now determine the constants entering the graph-theoretic criteria. For simplicity, we only consider $S = k\mathbb{Z}$, with $k = 2n + 1$ an odd integer, and use the partition given by $S_m = b(cl^m(S))$. Then
\[
cl^m(S) = \{\ell + k\mathbb{Z} : |\ell| \leq m \},
\]
in particular, $cl^m(S) = \mathbb{Z}$ and
\[
S_m = \{\pm m\} + k\mathbb{Z}.
\]
For \( v \in S \), there exist two \( u \in bS \) with \( u \sim v \), namely, \( u = v \pm 1 \). Thus \( d_0(v) = 2 \) and \( D_0 = 2 \). By contrast, \( k_0(v) = 1 \), which implies \( K_0 = 1 \). For \( m \geq 1 \), one easily verifies \( d_m(v) = 1 \), and thus \( D_m = 1 \), and finally \( K_m = 1 \), which results in (7.3)

\[
\delta_{S,2} = \sqrt{\sum_{m=1}^{n} \sum_{j=1}^{m} 1} = \sqrt{\frac{n(n+1)}{2}}, \quad a_{S,2} = \left( \sum_{m=0}^{n-1} \prod_{j=0}^{m-1} \frac{D_j}{K_j} \right)^{1/2} = \sqrt{2n+1} = \sqrt{k}.
\]

For the determination of the constants entering the upper bound, we note that \( \hat{k}_0(v) = 2 \) for \( v \in S \), and thus \( \hat{K}_0 = 2 \). For \( 0 < m < n \) and \( v = m' + k\ell \in S_m \) (with \( m' \in \{ \pm m \}, \ell \in \mathbb{Z} \)), there exists precisely one \( u = \text{sign}(m') + v \in S_{m+1} \), which shows that \( \hat{K}_m = 1 \) and uniquely determines \( \hat{v}_1(u) \). Finally, it is easily seen that \( \hat{D}_m = 1 \). This allows us to determine

\[
\hat{\delta}_{S,2} = \sqrt{\frac{n(n-1)}{2}}, \quad \hat{a}_{S,2} = \sqrt{k}.
\]

We can now apply Corollary 1.9 to the Fourier-analytic Paley–Wiener spaces.

**Corollary 7.2.** Let \( k \in \mathbb{Z} \) be odd, and \( S = k\mathbb{Z} \). As soon as \( \frac{k+1}{2\sqrt{2}} \sqrt{2 - 2\cos(\omega)} < 1 \), the graph-theoretic sampling estimates guarantee for all \( f \in PW_\omega \)

\[
1 - \frac{k+1}{2\sqrt{2}} \sqrt{2 - 2\cos(\omega)} \|f\|_2 \leq \|f\|_2 \leq 1 + \frac{k+1}{2\sqrt{2}} \sqrt{2 - 2\cos(\omega)} \|f\|_2.
\]

For \( (k+1)\sqrt{1 - \cos(\omega)} < 1 \), the tightness of the frame estimate is less than or equal to \( 1 + 2(k+1)\sqrt{1 - \cos(\omega)} \).

**Proof.** It is straightforward to check by the definition of Plancherel’s theorem and \( \|\nabla f\|_2 = \|L^{1/2}f\|_2 \) that Corollary 1.9 is applicable to \( PW_\omega \). The sampling estimate hence follows, since \( \frac{n(n+1)}{2} \leq \frac{(k+1)^2}{4} \). The tightness of the estimate is defined as the quotient of upper and lower bound, and it can be estimated by the quantity given in the corollary since, for all \( \epsilon < 1/2 \) we have \( \frac{1}{1 + \epsilon} < 1 + 4\epsilon \).

We stress that, while the statement concerning the sampling density can be obtained solely from Theorem 1.4, the control over the tightness is a consequence of Theorem 1.8. For the comparison of (7.5) with the equality (7.2), we first note that we are in the situation outlined after Corollary 1.9: The constants \( a_{S,2} \) and \( \hat{a}_{S,2} \) coincide for all sampling sets \( S \) considered here. Hence they are just normalization constants, and they coincide precisely with the normalization constants of the optimal sampling result (7.2). In order to compare the bandwidth conditions, we use the estimate \( (1 + \epsilon)|x| \geq \sqrt{2 - 2\cos(x)} \), for any \( \epsilon > 0 \), to obtain the sufficient condition \( (k+1)\omega < \frac{2\sqrt{2}}{4\epsilon^2} \). Thus, asymptotically, the graph-theoretic estimate requires an oversampling by a factor slightly larger than \( \frac{2}{\sqrt{2}} \approx 1.1107267 \) by comparison to the optimal rate. Furthermore the tightness of the frame estimate is \( 1 + \nu \), with \( \nu \) inversely proportional to the oversampling rate.

In the notation of section 3, the frame bounds in (7.5) allow us to estimate the parameter \( \eta \) describing the speed of convergence in the frame recovery algorithm by

\[
\eta = \frac{B - A}{B + A} \leq \frac{k+1}{2\sqrt{2}} \sqrt{2 - 2\cos(\omega)}.
\]

Hence the graph-theoretic estimates allow us to directly translate the oversampling rate to the reconstruction rate guaranteed for the frame algorithm. We note that similar observations pertain for even sampling rates, with slightly worse constants.
It is instructive to compare the sampling density with the one obtainable from the estimates in [17]. Using the constants in [17, Theorem 2.1], we obtain the sufficient density criterion
\[
\sqrt{3^k - 1} \cdot \sqrt{2 - 2\cos(\omega)} < 1,
\]
and in the absence of an upper sampling estimate improving the trivial estimate \(\|f_0\|_2 < \|f\|_2\), the tightness of the sampling estimate is given by
\[
\frac{1 + 5(3^k - 1)/2}{\sqrt{1 + 5(3^k - 1)/2}}.
\]
But this means that the prescribed sampling rate behaves as \(\log(\|\omega\|)\), rather than \(1/|\omega|\). Moreover, the tightness of the estimate grows like \(1/|\omega|\), as \(\omega \to 0\), independent of the oversampling, with poor control over the convergence rate of the frame algorithm.

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