FUZZY ANTI-NORMED LINEAR SPACE

Iqbal H. Jebril \( ^a \), T. K. Samanta \( ^b \)

\( ^a \) Department of Mathematics, King Faisal University (SAUDI ARABIA)
\( ^b \) Department of Mathematics, Uluberia College, West Bengal (INDIA)

\( ^a \) iqbal501@hotmail.com , \( ^b \) mump_tapas5@yahoo.co.in

ABSTRACT

Felbin’s definition of fuzzy norms on a linear space [6] corresponds to a pair of which one is a fuzzy norm and fuzzy anti-norm in sense of Bag and Samanta [3]. In the present paper, fuzzy anti-norm on a linear space is studied and some results are introduced in fuzzy anti-norms on a linear space. Lastly, we have introduced the definition of intuitionistic fuzzy normed linear space.

Key words: Fuzzy anti-norm, fuzzy norm, Intuitionistic fuzzy norm

1. INTRODUCTION

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The idea of fuzzy norm was initiated by katsaras in [4]. Felbin [6] defined a fuzzy norm on a linear space whose associated fuzzy metric is of kaleva-Seikkala type [8]. Cheng and Mordeson [5] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [7].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [7]. They also studied some properties of the fuzzy norm in [2] and [3].

Bag and Samanta discussed the notions of convergent sequence and Cauchy sequence in fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [4], Felbin [6], and Bag and Samanta [1].

In this paper after an introduction of fuzzy norms, we introduce a fuzzy anti-norm linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [3] and investigate their important properties. Then we shall introduce the notions of convergent sequence, Cauchy sequence in fuzzy anti-normed linear space. We also introduce the concept of compact subset and bounded subset in fuzzy anti-normed linear space. Lastly, we have introduced the definition of intuitionistic fuzzy normed linear space.

In [6], Felbin introduced the concept of a fuzzy norm based on a kaleva-Seikkala type [8] of fuzzy metric using the notion of fuzzy number. Let \( X \) be a vector space over \( R \) (set of real numbers). Let \( \| \cdot \| : X \rightarrow R \) \( (I) \) be a mapping and let the mappings \( L, U : [0,1] \times [0,1] \rightarrow [0,1] \) be symmetric, non-decreasing in both arguments and satisfying \( L(0,0) = 0 \) and \( U(1,1) = 1 \). Write \( \| x \|_\alpha = \left[ \frac{\| x \|}{\| x \|_\alpha} \right] \) for \( x \in X, 0 < \alpha \leq 1 \) and suppose for all \( x \in X, x \neq 0 \) there exists \( \alpha_0 \in (0,1] \) independent of \( x \), such that for all \( \alpha \leq \alpha_0 \),

(A) \( \| x \|_\alpha < \infty \),

(B) \( \inf \| x \|_\alpha > 0 \).

The quadruple \( (X, \| \cdot \| , L, U) \) is called a Felbin-fuzzy normed linear space and \( \| \cdot \| \) is a Felbin-fuzzy norm if:

(i) \( \| x \| = \bar{0} \) if and only if \( x = \bar{0} \) (the null vector),

(ii) \( \| rx \| = |r| \| x \|, x \in X, r \in R \)

(iii) for all \( x, y \in X \),

\( (a) \) Whenever \( s \leq \| x \|, t \leq \| y \| \) and \( s + t \leq \| x + y \| \),

\( \| x + y \| (s + t) \geq L \left( \| x \| (s), \| y \| (t) \right) \).
(b) whenever \( s \geq \|x\|, t \geq \|y\| \), and \( s + t \geq \|x + y\| \),
\[ \|x + y\|(s + t) \leq U(\|x\|(s), \|y\|(t)). \]

### 2. FUZZY NORMS ON A LINEAR SPACE

This section is devoted to a collection of basic definitions and results which will be needed in the sequel.

**Definition 2.1.** Let \( X \) be a linear space over a real field \( F \) (field of real/complex numbers). A fuzzy subset \( N \) of \( X \times R \) is called a fuzzy norm on \( X \) if the following conditions are satisfied for all \( x, y \in X \).

(N1) For all \( t \in R \) with \( t \leq 0, N(x, t) = 0 \),

(N2) For all \( t \in R \) with \( t > 0, N(x, t) = 1 \) if and only if \( x = 0 \),

(N3) For all \( t \in R \) with \( t > 0, N(cx, t) = N(x, t/|c|) \) if \( c \neq 0, c \in F \),

(N4) For all \( s, t \in R, N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\} \),

(N5) \( N(x, \cdot) \) is a non-decreasing function of \( R \) and \( \lim_{t \to \infty} N(x, t) = 1 \).

Then \( N \) said to be a fuzzy norm on a linear space \( X \) and the pair \( (X, N) \) is said to be a fuzzy normed linear space or in short FNLS.

The following condition of fuzzy norm \( N \) will be required later on.

(N6) \( N(x, t) > 0, \forall t > 0 \) implies \( x = 0 \).

**Example 2.2.** Let \( (X, \| \|) \) be a normed linear space. Define

\[ N(x, t) = \begin{cases} \frac{t}{t + \|x\|}, & \text{when } t > 0, t \in R, x \in X, \\ 0, & \text{when } t \leq 0. \end{cases} \]

Then \( (U, N) \) is an FNLS.

**Example 2.3.** Let \( (X, \| \|) \) be a normed linear space. Define

\[ N(x, t) = \begin{cases} 0, & \text{if } t \leq \|x\|, t \in R, x \in X, \\ 1, & \text{if } t > \|x\|, t \in R, x \in X. \end{cases} \]

Then \( (U, N) \) is an FNLS.

**Theorem 2.4.** Let \( (X, N) \) be a fuzzy normed linear space. Define \( \|x\|_\alpha = \inf \{t : N(x, t) \geq \alpha\}; \alpha \in (0, 1) \). Then \( \{\| \|_\alpha : \alpha \in (0, 1)\} \) is an ascending family of norms on \( X \). These norms are called \( \alpha \) - norms on \( X \) corresponding to fuzzy norm on \( X \).

**Theorem 2.5.** Let \( \{\| \|_\alpha : \alpha \in (0, 1)\} \) be an ascending family of norms on linear space \( X \). Define a function \( N' : X \times R \to [0, 1] \) as:

\[ N'(x, t) = \begin{cases} \sup\{\alpha \in (0, 1) : \|x\|_\alpha \leq t\}, & \text{when } (x, t) \neq 0, \\ 0, & \text{when } (x, t) = 0. \end{cases} \]

Then \( N' \) is a fuzzy norm on \( X \).
If the index set \((0, 1)\) of the family of crisp norms \(\left\{ \| \alpha \| : \alpha \in (0, 1) \right\}\) of Theorem 2.5 is extended to \((0, 1)\) then a fuzzy norm \(N\) is generated, satisfying an additional property that \(N(x, .)\) attains the value 1 at some finite value \(t\).

**Theorem 2.6.**[1] Let \(\left\{ \| \alpha \| : \alpha \in (0, 1) \right\}\) be a descending family of norms on a linear space \(X\). Now define a function \(N' : X \times [0, 1] \rightarrow [0, 1]\) as
\[
N'(x, t) = \begin{cases} 
\sup \{ \| \alpha \| : \alpha \in (0, 1) \; \text{with} \; \| x \| \leq t \}, & \text{when} \; (x, t) \neq 0, \\
0, & \text{when} \; (x, t) = 0.
\end{cases}
\]
Then
(a) \(N'\) is a fuzzy norm on \(X\).
(b) For each \(x \in X\), \(\exists t = t(x) > 0\) such that \(N'(x, s) = 1, \forall s \geq t\).

**Definition 2.7.**[1] Let \((X, N)\) be a fuzzy normed linear space. Let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is said to be convergent if \(\exists x \in X\) such that \(\lim_{n \to \infty} N(x - x_n, t) = 1, \forall t > 0\).

**Definition 2.8.**[1] Let \((X, N)\) be a fuzzy normed linear space. Let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is said to be a Cauchy sequence if \(\lim_{n \to \infty} N(x_{n+p} - x_n, t) = 1, \forall t > 0\) and \(\{x_n\}, p = 1, 2, 3, \ldots \ldots\)

**Definition 2.9.**[1] A subset of a fuzzy normed linear space \((U, N^*)\) is said to be bounded if and only if \(\exists t > 0\) and \(0 < r < 1\) such that \(N^*(x, t) > 1-r, \forall x \in A\).

**Definition 2.10.**[1] A subset \(A\) of a fuzzy normed linear space \((U, N^*)\) is said to be compact if any sequence \(\{x_n\}\) in \(A\) has a subsequence converging to an element of \(A\).

3. FUZZY ANTI-NORMS ON A LINEAR SPACE

In this section we introduce the notion of fuzzy anti-normed linear space and investigate their important properties.

**Definition 3.1.** Let \(U\) be a linear space over a real field \(F\). A fuzzy subset \(N^*\) of \(X \times R\) such that for all \(x, u \in U\) and \(c \in F\):

(N\(^*\)1) For all \(t \in R\) with \(t \leq 0\), \(N^*(x, t) = 1\); 
(N\(^*\)2) For all \(t \in R\) with \(t > 0\), \(N^*(x, t) = 0\) if and only if \(x = 0\); 
(N\(^*\)3) For all \(t \in R\) with \(t > 0\), \(N^*(cx, t) = N^*(x, t / |c|)\) if \(c \neq 0, c \in F\); 
(N\(^*\)4) For all \(s, t \in R\), \(N^*(x + u, s + t) \leq \max\{N^*(x, s), N^*(u, t)\}\); 
(N\(^*\)5) \(N^*(x, t)\) is a non-increasing function of \(t \in R\) and \(\lim_{t \to \infty} N^*(x, t) = 0\).

Then \(N\) is said to be a fuzzy anti-norm on a linear space \(U\) and the pair \((U, N^*)\) is called a fuzzy anti-normed linear space or in short Fa-NLS.
The following condition of fuzzy norm $N$, will be required later on.

(N6) For all $t \in R$ with $t > 0$, $N^*(x,t) < 1$ implies $x = 0$.

**Example 3.2.** Let $(U, \| \|)$ be a normed linear space. Define

$$N^*(x,t) = \begin{cases} \frac{\|x\|}{t + \|x\|}, & \text{when } t > 0, \ t \in R, \ x \in U, \\ 1, & \text{when } t \leq 0. \end{cases}$$

Then $(U, N^*)$ is an Fa-NLS.

**Proof:** Now we have to show that $N^*(x,t)$ is a fuzzy anti-norm in $U$.

(N1) For all $t \in R$. If $t \leq 0$, we have by definition $N^*(x,t) = 1$.

(N2) For all $t \in R$ with $t > 0$, $N^*(x,t) = 0 \Leftrightarrow \frac{\|x\|}{t + \|x\|} = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = 0$.

(N3) For all $t \in R$ with $t > 0$, and $c \neq 0 \in F$ (field of real/complex numbers), we get

$$N^*(cx,t) = \frac{\|cx\|}{t + \|cx\|} = \frac{|c|\|x\|}{t + |c|\|x\|} = \frac{\|x\|}{\frac{t}{|c|} + \|x\|} = N^*(x, \frac{t}{|c|}).$$

(N4) For all $s,t \in R$ and $x,u \in U$. We have to show that $N^*(x+u,s+t) \leq \max \{N^*(x,s),N^*(u,t)\}$ if

(a) $s+t < 0$, (b) $s=t=0$, (c) $s+t > 0; s > 0, t < 0$; $s < 0, t > 0$, then in these cases the relation is obvious. If (d) $s > 0,t > 0,s + t > 0$. Then, assume that

$$N^*(x,s) \leq N^*(u,t) \Rightarrow \frac{\|x\|}{s + \|x\|} \leq \frac{\|u\|}{t + \|u\|} \Rightarrow \|x\|(t + \|x\|) \leq \|u\|(s + \|x\|) \Rightarrow t\|x\| \leq s\|u\|...{(1)}$$

Now,

$$\frac{\|x+u\|}{s+t+\|x+u\|} \leq \frac{\|x\| + \|u\|}{s+t + \|x\| + \|u\|} = \frac{\|x\| - s\|u\|}{(s+t+\|x\|+\|u\|)(t+\|x\|)}.$$

By (1),

$$\frac{\|x+u\|}{s+t+\|x+u\|} \leq \frac{\|x\| + \|u\|}{s+t + \|x\| + \|u\|} \leq \frac{\|x\|}{s + \|x\|}.$$

Hence $N^*(x+u,s+t) \leq \max \{N^*(x,s),N^*(u,t)\}$.

(N5) If $t_1 < t_2 \leq 0$, then we have $N^*(x,t_1) = N^*(x,t_2) = 0$. If $t_1 < t_2 \leq 0$ then

$$\frac{\|x\|}{t_1 + \|x\|} - \frac{\|x\|}{t_2 + \|x\|} = \frac{\|x\|(t_2 - t_1)}{(t_1 + \|x\|)(t_2 + \|x\|)} > 0 \Rightarrow N^*(x,t_1) \geq N^*(x,t_2).$$

Thus $N^*(x, \cdot)$ is a non-increasing of $R$. Again if $x \neq 0$ then

$$\lim_{t \to \infty} N^*(x,t) = \lim_{t \to \infty} \frac{\|x\|}{t + \|x\|} = 0.$$

IF $x = 0$ then

$$\lim_{t \to \infty} N^*(x,t) = \lim_{t \to \infty} N^*(0,t) = 0.$$

Thus $\lim_{t \to \infty} N^*(x,t) = 1, \forall x \in U$. Hence $(U, N^*)$ is an Fa-NLS.
Example 3.3. Let \( (U, \| \|) \) be a normed linear space. Define \( N^*: X \times R \to [0,1] \) by
\[
N^*(x,t) = \begin{cases} 
0, & \text{if } t \| x \|, \ t \in R, x \in U, \\
1, & \text{if } t \leq \| x \|, \ t \in R, x \in U.
\end{cases}
\]

Then \( (U, N^*) \) is an Fa-NLS.

Proof: It can be easily verified that \( (U, N^*) \) is Fa-NLS.

Remark 3.4.[3] \( N^* \) is a fuzzy anti-norm on \( U \) iff \( 1 - N^* \) is a fuzzy norm on \( U \).

Lemma 3.5. Let \( (U, N^*) \) be a Fa-NLS. Then \( N^*(x-y,t) = N^*(y-x,t) \) for all \( x, y \in U \) and \( t \in (0,\infty) \).

Proof: For \( x, y \in U \) and \( t \in (0,\infty) \),
\[
N^*(x-y,t) = N^*(-(y-x),t) = N^*(y-x,t) = N^*(y-x,t).
\]

Definition 3.6.[3] Let \( N^* \) be a fuzzy anti-norm on \( U \) satisfying (N*6). Define
\[
\| x \|_{\alpha}^* = \inf \{ t > 0 : N^*(x,t) < \alpha, \ \alpha \in (0,1) \}.
\]

Lemma 3.7. Let \( (U, N^*) \) be a Fa-NLS. For each \( \alpha \in (0,1) \) and \( x \in U \). Then we have
(i) \( \| x \|_{\alpha_1}^* \geq \| x \|_{\alpha_2}^* \) for \( 0 < \alpha_1 < \alpha_2 \leq 1 \).
(ii) \( \| cx \|_{\alpha}^* = |c| \| x \|_{\alpha}^* \), for any scalar \( c \).
(iii) \( \| x + y \|_{\alpha}^* \leq \| x \|_{\alpha}^* + \| y \|_{\alpha}^* \).

Proof: (i) For \( 0 < \alpha_1 < \alpha_2 \leq 1 \), we note
\[
\inf \{ t > 0 : N^*(x,t) < \alpha_1 \} \geq \inf \{ t > 0 : N^*(x,t) < \alpha_2 \}
\]
\[
\Rightarrow \| x \|_{\alpha_1}^* \geq \| x \|_{\alpha_2}^*.
\]

(ii) For any scalar \( c \) and \( \forall \alpha \in (0,1) \),
\[
\| cx \|_{\alpha}^* = \inf \{ t > 0 : N^*(cx,t) < \alpha, \alpha \in (0,1) \} = \inf \{ t > 0 : N^*(x,t/|c|) < \alpha, \alpha \in (0,1) \} = |c| \| x \|_{\alpha}^*.
\]

(iii) For all \( \alpha \in (0,1) \),
\[
\| x \|_{\alpha}^* + \| y \|_{\alpha}^* = \inf \{ t > 0 : N^*(x,t) < \alpha \} + \inf \{ s > 0 : N^*(y,s) < \alpha \}
\]
\[
\geq \inf \{ t + s > 0 : N^*(x,t) < \alpha, N^*(y,s) < \alpha \} = \| x + y \|_{\alpha}^*.
\]

Theorem 3.8. Let \( (U, N^*) \) be a Fa-NLS. Then \( \{ \| \|_\alpha: \alpha \in (0,1) \} \) is a decreasing family of norms on \( U \).

Proof: By Lemma 3.7 it can be easily verified that.

Theorem 3.9. Let \( \{ \| \|_\alpha: \alpha \in (0,1) \} \) be a decreasing family of norms on linear space \( U \). Now define a function \( N^*_1: U \times R \to [0,1] \) as
\[ N_1^*(x,t) = \begin{cases} \inf \left\{ \alpha \in (0,1] : \|x\|_\alpha \leq t \right\}, & \text{when } (x,t) \neq 0, \\ 1, & \text{when } (x,t) = 0. \end{cases} \]

Then

(a) \( N_1^* \) is a fuzzy anti-norm on \( U \).

(b) For each \( x \in U \), \( \exists r(x) > 0 \) such that \( N_1^*(x,t) = 1 \).

**Proof:** Now we have to show that \( N_1^* \) is a fuzzy anti-norm on \( X \).

(N*1) (a) \( \forall t \in R \) with \( t < 0 \), \( \inf \left\{ \alpha \in (0,1] : \|x\|_\alpha \leq t \right\} = \phi, \forall x \in U \), we have \( N_1^*(x,t) = \inf \left\{ \alpha \in (0,1] : \|x\|_\alpha \leq t \right\} = 1 \),

(b) For \( t = 0 \) and \( x \neq 0 \), \( \inf \left\{ \alpha \in (0,1] : \|x\|_\alpha \leq t \right\} = \phi, \forall x \in U \), we have \( N_1^*(x,t) = 1 \),

(c) For \( t = 0 \) and \( x = 0 \) then from the definition \( N_1^*(x,t) = 1 \).

Thus \( \forall t \in R \) with \( t \leq 0, N_1^*(x,t) = 1, \forall x \in U \).

(N*2) \( \forall t \in R \) with \( t > 0, N_1^*(x,t) = 0 \). Choose any \( \varepsilon \in (0,1) \). Then for any \( t > 0, \exists \alpha_1 \in (\varepsilon,1] \) such that \( \|x\|_\alpha \leq t \), and hence \( \|x\|_\alpha \leq t \). Since \( t > 0 \) is arbitrary, this implies that \( \|x\|_\alpha = 0 \) then \( x = 0 \).

If \( x = 0 \) then for \( t > 0, N_1^*(0,t) = \inf \left\{ \alpha \in (0,1] : \|0\|_\alpha \leq t \right\} = 0 \). Thus for all \( t \in R \) with \( t > 0, N_1^*(x,t) = 0 \) if and only if \( x = 0 \).

(N*3) For all \( t \in R \) with \( t > 0 \), and \( c \neq 0 \) we have

\[ N_1^*(cx,t) = \inf \left\{ \alpha \in (0,1] : \|cx\|_\alpha \leq t \right\} = \inf \left\{ \alpha \in (0,1] : \|x\|_c \leq \frac{t}{|c|} \right\} = N_1^*\left(\frac{x}{|c|},\frac{t}{|c|}\right), \forall x \in U. \]

(N*4) We have to show that \( \forall s,t \in R \) and \( \forall x,u \in U \), \( N_1^*(x+u,s+t) \leq \max \left\{ N_1^*(x,s), N_1^*(u,t) \right\} \).

Suppose that \( \forall s,t \in R \) and \( \forall x,u \in U \), \( N_1^*(x+u,s+t) > \max \left\{ N_1^*(x,s), N_1^*(u,t) \right\} \).

Choose \( k \) such that \( N_1^*(x+u,s+t) > k > \max \left\{ N_1^*(x,s), N_1^*(u,t) \right\} \). Now

\[ N_1^*(x+u,s+t) > k \Rightarrow \inf \left\{ \alpha \in (0,1] : \|x+u\|_\alpha \leq s+t \right\} > k \Rightarrow \|x+u\|_\alpha \leq s+t \Rightarrow \|x\|_k + \|u\|_k > s+t. \]

Again,

\[ k > \max \left\{ N_1^*(x,s), N_1^*(u,t) \right\} \Rightarrow k > N_1^*(x,s) \text{ and } k > N_1^*(u,t) \Rightarrow \|x\|_k \leq s \text{ and } \|u\|_k \leq t \]

\[ \Rightarrow \|x\|_k + \|u\|_k \leq s + t. \]

Thus \( s + t < \|x\|_k + \|u\|_k \leq s + t \), a contradiction.

Hence \( N_1^*(x+u,s+t) \leq \max \left\{ N_1^*(x,s), N_1^*(u,t) \right\} \).
(N5) Let \( x \in U, \alpha(0,1) \). Now \( t > \|x\|_\alpha \Rightarrow N_1^*(x,t) = \inf \{ \beta \in (0,1]: \|x\|_\beta \leq t \} \leq \alpha \). So \( \lim_{t \to \infty} N^*(x,t) = 0 \). Next we verify that \( N^*(x,\cdot) \) is a non-increasing function of \( R \). If \( t_1 < t_2 \leq 0 \), then \( N^*(x,t_1) = N^*(x,t_2) = 1, \forall x \in U \), if \( t_2 > t_1 > 0 \) then
\[
\{ \alpha \in (0,1]: \|x\|_\alpha \leq t_1 \} \subseteq \{ \alpha \in (0,1]: \|x\|_\alpha \leq t_2 \} \Rightarrow \inf \{ \alpha \in (0,1]: \|x\|_\alpha \leq t_1 \} \leq \inf \{ \alpha \in (0,1]: \|x\|_\alpha \leq t_2 \} \Rightarrow N_1^*(x,t_2) \leq N_1^*(x,t_1).
\]
Thus \( N_1^*(x,\cdot) \) is a non-increasing function of \( R \) and hence \( N_1^* \) is a fuzzy anti-norm on \( U \).

(b) For each \( x \neq 0 \), \( \|x\|_\alpha > 0 \). Thus \( \exists r = r(x) > 0 \) such that
\[
\|x\|_\alpha > r(x) > 0 \Rightarrow \|x\|_\alpha > r(x), \forall \alpha \in (0,1] \Rightarrow \inf \{ \alpha \in (0,1]: \|x\|_\alpha \leq t \} = 1 \Rightarrow N_1^*(x,t) = 1.
\]

Definition 3.10. Let \( (U, N^*) \) be a Fa-NLS. A sequence \( \{x_n\} \) in \( U \) is said to be convergent to \( x \in U \) if given \( t > 0, 0 < r < 1 \) there exists an integer \( n_0 \in N \) such that \( N^*(x_n - x,t) < r, \) for all \( n \geq n_0 \).

Theorem 3.11. In a Fa-NLS \( (U, N^*) \), a sequence \( \{x_n\} \) converges to \( x \in U \) if and only if
\[
\lim_{n \to \infty} N^*(x_n - x,t) = 0, \forall t > 0.
\]

Proof: Fix \( t > 0 \). Suppose \( \{x_n\} \) converges to \( x \in U \) Then for a given \( r, 0 < r < 1 \), there exists an integer \( n_0 \in N \) such that \( N^*(x_n - x,t) < r, \) for all \( n \geq n_0 \), and hence \( N^*(x_n - x,t) \to 0, \) as \( n \to \infty \).

Conversely, if for each \( t > 0, N^*(x_n - x,t) \to 0, \) as \( n \to \infty \), then for every \( r, 0 < r < 1 \), there exists an integer \( n_0 \) such that \( N^*(x_n - x,t) < r, \) for all \( n \geq n_0 \). Hence \( \{x_n\} \) converges to \( x \) in \( U \).

Definition 3.12. Let \( (U, N^*) \) be a Fa-NLS. A sequence \( \{x_n\} \) in \( U \), is said to be Cauchy sequence if given \( t > 0, 0 < r < 1 \) there exists an integer \( n_0 \in N \) such that \( N^*(x_{n+p} - x_n,t) < r, \) for all \( n \geq n_0 \).

Theorem 3.13. In a Fa-NLS \( (U, N^*) \), \( \{x_n\} \) is an Cauchy sequence in \( U \) if and only if
\[
\lim_{n \to \infty} N^*(x_{n+p} - x_n,t) = 0, \quad p = 1, 2, 3, \ldots, \text{ and } t > 0.
\]

Proof: Fix \( t > 0 \). Suppose \( \{x_n\} \) is a Cauchy sequence in \( U \). Then for a given \( r, 0 < r < 1 \) and \( p = 1, 2, 3, \ldots \) there exists an integer \( n_0 \in N \) such that \( N^*(x_{n+p} - x_n,t) < r, \) \( n \geq n_0 \) and hence \( N^*(x_{n+p} - x_n,t) \to 0, \) as \( n \to \infty \).

Conversely, if for each \( t > 0 \) and \( p = 1, 2, 3, \ldots, N^*(x_{n+p} - x_n,t) \to 0, \) as \( n \to \infty \), then for every \( r, 0 < r < 1 \), there exists an integer \( n_0 \) such that \( N^*(x_{n+p} - x_n,t) < r, \) for all \( n \geq n_0 \). Hence \( \{x_n\} \) Cauchy sequence in \( U \).

Theorem 3.14. If a sequence \( \{x_n\} \) in a Fa-NLS \( (U, N^*) \) is convergent, its limit is unique.

Proof: Let \( \{x_n\} \) *– converges to \( x \) and \( y \). Also let \( s, t \in R^+ \).
Now \( \lim_{n \to \infty} N^*(x_n - x, t) = 0 \) and \( \lim_{n \to \infty} N^*(x_n - y, t) = 0 \)
\[
N^*(x - y, t + s) = N^*(x - x_n + x_n - y, t + s) \leq \max \left\{ N^*(x - x_n, t), N^*(x_n - y, s) \right\}
= \max \left\{ N^*(x_n - x, t), N^*(x_n - y, s) \right\}.
\]
Taking limit, we have
\[
N^*(x - y, t + s) \leq \max \left\{ \lim_{n \to \infty} N^*(x_n - x, t), \lim_{n \to \infty} N^*(x_n - y, s) \right\}
\Rightarrow N^*(x - y, t + s) = 0, \forall s, t > 0 \Rightarrow x - y = 0 \Rightarrow x = y.
\]

Theorem 3.15. In a Fa-NLS \((U, N^*)\) every subsequence of a convergent sequence converges to the limit of sequence.

Proof: Obvious.

Theorem 3.16. Let \( L \) be a linear space, \( N^* \) be a fuzzy anti-norm on \( L \), \( \hat{N} = (1 - N^*) \) be a fuzzy norm on \( L \). Then

a) \( \{x_n\} \) is an convergent sequence in \((L, N^*)\) if and only if \( \{x_n\} \) is a convergent sequence in \((L, \hat{N})\).

b) \( \{x_n\} \) is an Cauchy sequence in \((L, N^*)\) if and only if \( \{x_n\} \) is a Cauchy sequence in \((L, \hat{N})\).

Proof:

a) \( \{x_n\} \) is an convergent sequence in \((L, N^*)\)
\[\iff \lim_{n \to \infty} N^*(x_n - x, t) = 0, \text{ for all } t > 0,\]
\[\iff \lim_{n \to \infty} \hat{N}(x_n - x, t) = 1, \text{ for all } t > 0,\]
\[\iff \{x_n\} \text{ is a convergent sequence in } (L, \hat{N}).\]

b) Let \( \{x_n\} \) be an Cauchy sequence in \((L, N^*)\)
\[\iff \lim_{n \to \infty} N^*(x_{n+p} - x_n, t) = 0, \text{ for all } t > 0, \text{ and } p = 1,2,3,...\]
\[\iff \lim_{n \to \infty} \hat{N}(x_{n+p} - x_n, t) = 1, \text{ for all } t > 0, \text{ and } p = 1,2,3,...\]
\[\iff \{x_n\} \text{ is a Cauchy sequence in } (L, \hat{N}).\]

Theorem 3.17. In a Fa-NLS \((U, N^*)\), every convergent sequence is a Cauchy sequence.

Proof: Let \( \{x_n\} \) be a convergent sequence in the a-NLS \((U, N^*)\) then
\[
\lim_{n \to \infty} N^*(x_n - x, t) = 0, \text{ for all } t > 0. \text{ Let } s, t \in R^+ \text{ and } p = 1,2,3,...., \text{ we have}
\[
N^*(x_{n+p} - x_n, s + t) = N^*(x_{n+p} - x + x - x_n, s + t) \leq \max \left\{ N^*(x_{n+p} - x, s), N^*(x_n - x, t) \right\}
= \max \left\{ N^*(x_{n+p} - x, s), N^*(x_n - x, t) \right\}.
\]
Taking limit, we have
\[
\lim_{n \to \infty} N^*(x_{n+p} - x_n, s + t) = \max \left\{ \lim_{n \to \infty} N^*(x_{n+p} - x, s), \lim_{n \to \infty} N^*(x_n - x, t) \right\} = 0.
\]
\[\Rightarrow \lim_{n \to \infty} N^*(x_{n+p} - x_n, s + t) = 0, \forall s, t \in R^+ \text{ and } p = 1,2,3,....\]
Thus, \( \{x_n\} \) is the Cauchy sequence in Fa-NLS \((U, N^*)\) The converse of the above theorem is not necessarily true. This is justified by the following example.
Example 3.18. Let $(X, \| \|)$ be a normed linear space and $N^*: X \times R \rightarrow [0,1]$. Define

$$N^*(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|}, & \text{when } t > 0, t \in R, \\ 1, & \text{when } t \leq 0. \end{cases}$$

Then $(X, N^*)$ is a Fa-NLS (see Example 3.2). Let $\{x_n\}$ be a sequence in $X$, then

a) $\{x_n\}$ is a Cauchy sequence in $(X, \| \|)$ if and only if $\{x_n\}$ is an Cauchy sequence in $(X, N^*)$.

b) $\{x_n\}$ is a convergent sequence in $(X, \| \|)$ if and only if $\{x_n\}$ is an convergent sequence in $(X, N^*)$.

Proof: a) Let $\{x_n\}$ be a Cauchy sequence in $(X, \| \|)$. Then

$$\lim_{n \to \infty} \|x_n - x_{n+p}\| = 0, \text{ for all } p = 1, 2, 3, \ldots.$$

$$\lim_{n \to \infty} N^*(x_n - x_{n+p}) = \lim_{n \to \infty} \frac{\|x_n - x_{n+p}\|}{t + \|x_n - x_{n+p}\|} = 0, \text{ for all } t > 0.$$

$$\lim_{n \to \infty} N^*(x_n - x_{n+p}) = 0.$$

$$\{x_n\} \text{ is an Cauchy sequence in } (X, N^*).$$

b) $\{x_n\}$ is a convergent sequence in $(X, \| \|)$. Then

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

$$\lim_{n \to \infty} N^*(x_n - x) = \lim_{n \to \infty} \frac{\|x_n - x\|}{t + \|x_n - x\|} = 0, \text{ for all } t > 0.$$

$$\lim_{n \to \infty} N^*(x_n - x) = 0.$$

$$\{x_n\} \text{ is an convergent sequence in } (X, N^*).$$

Remark 3.19. If there exist a normed linear space $(X, \| \|)$ which is not complete, then the fuzzy anti-norm induced by such a crisp norm $\| \|$ on an incomplete linear space $X$, is an incomplete Fa-NLS.

Definition 3.20. Let $(U, N^*)$ be a Fa-NLS. A subset $B$ of $U$ is said to be closed if for any sequence $\{x_n\}$ in $B$ converges to $x \in B$, that is, $\lim_{n \to \infty} N^*(x_n - x) = 0, \forall t > 0$ implies that $x \in B$.

Definition 3.21. Let $(U, N^*)$ be a Fa-NLS. A subset $W$ of $U$ is said to be the closure of $B \subseteq W$ if for any $w \in W$, there exists a sequence $\{x_n\}$ in $B$ such that $\lim_{n \to \infty} N^*(x_n - x, t) = 0, \forall t \in R^+$, we denote the set $W$ by $\overline{B}$.

Definition 3.22. A subset $A$ of a Fa-NLS is said to be bounded if and only if $\exists t > 0$ and $0 < r < 1$ such that $N^*(x, t) < r, \forall x \in A.$
Definition 3.23. Let \((U, N^*)\) be a Fa-NLS. A subset \(A\) of a Fa-NLS is said to be compact if any sequence \(\{x_n\}\) in \(A\) has a subsequence converging to an element of \(A\).

Theorem 3.24. Let \((U, N^*)\) be a Fa-NLS. Every Cauchy sequence in \((U, N^*)\) is bounded.

Proof: Let \(\{x_n\}\) be a Cauchy sequence in the Fa-NLS \((U, N^*)\). Then
\[
\lim_{n \to \infty} N^* (x_n - x_{n+p}, t) = 0, \quad p = 1, 2, 3, \ldots, \text{ and } t > 0.
\]
Choose a fixed \(\alpha_0, \quad 0 < \alpha_0 < 1\). Then
\[
\lim_{n \to \infty} N^* (x_n - x_{n+p}, t) = 0 < -\alpha_0, \quad \forall t > 0, p = 1, 2, 3, \ldots \Rightarrow \text{ For } t' > 0, \exists n'_0 = n'_0 (t').
\]
Such that
\[
\lim_{n \to \infty} N^* (x_n - x_{n+p}, t) < 1 - \alpha_0, \quad \forall n \geq n'_0, p = 1, 2, 3, \ldots
\]
Since \(\lim_{n \to \infty} N^* (x_n, t) = 0\), we have for each \(x_i, \exists t'_i > 0\) such that
\[
N^* (x_i, t'_i) < 1 - \alpha_0, \quad \forall t > t'_i, i = 1, 2, 3, \ldots
\]
Let \(t'_0 = t' + \max \{t'_1, t'_2, \ldots, t'_{n'_0}\}\). Then,
\[
N^* (x_n, t'_0) \leq N^* (x_n, t'_1 + t'_{n'_0}) = N^* (x_n - x_{n'_0} + x_{n'_0}, t'_1 + t'_{n'_0})
\]
\[
\leq \max \{N^* (x_n - x, t'_1), N^* (x_{n'_0}, t'_{n'_0})\} = (1 - \alpha_0), \quad \forall n \geq n'_0.
\]
i.e. \(N^* (x_n, t'_0) \leq (1 - \alpha_0), \quad \forall n \geq n'_0\). Therefore \(\{x_n\}\) is bounded in \((U, N^*)\).

4. INTUITIONISTIC FUZZY NORM

In this section we redefine the notion of fuzzy normed linear space using t-norm and fuzzy anti-normed linear space using t-conorm then we introduce the definition of intuitionistic fuzzy norm over a linear space.

Definition 4.1.[9]. A binary operation \(*: [0,1] \times [0,1] \rightarrow [0,1]\) is continuous t-norm if \(*\) is it satisfies the following conditions:
(a) \(*\) is commutative and associative;
(b) \(*\) is continuous;
(c) \(a * 1 = a\) for all \(a \in [0,1]\);
(d) \(a * b \leq c * d\) whenever \(a \leq c,\) and \(b \leq d,\) and \(a, b, c, d \in [0,1]\).

Examples of continuous t-norm are \(a * b = ab, a * b = \min \{a, b\}\) and \(a * b = \max \{a + b - 1, 0\}\).

Definition 4.2. Let \(X\) be a linear space over a real field \(F\) (field of real/complex numbers). A fuzzy subset \(N\) of \(X \times R\) (set of real numbers) is called a fuzzy norm on \(X\) if the following conditions, are satisfied for all \(x, y \in X\):

\[(N1)\] For all \(t \in R\) with \(t \leq 0, N (x, t) = 0,\)
\[(N2)\] For all \(t \in R, N (x, t) = 1\) if and only if \(x = 0,\)
\[(N3)\] For all \(t \in R\) with \(t > 0, N (cx, t) = N (x, t/|c|)\) if \(c \neq 0, c \in F,\)
\[(N4)\] For all \(s, t \in R\), \(N (x + y, s + t) \geq N (x, s) * N (y, t),\)
\[(N5)\] \(N (x, \cdot)\) is a non-decreasing function of \(R\) and \(\lim_{t \to \infty} N (x, t) = 1.\)

Then \(N\) is said to be a fuzzy \(*\)–norm on a linear space \(X\).
Definition 4.3. [9]. A binary operation $\odot : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if it satisfies the following conditions:

(a) $\odot$ is commutative and associative;
(b) $\odot$ is continuous;
(c) $a \odot 0 = a$ for all $a \in [0,1]$;
(d) $a \odot b \leq c \odot d$ whenever $a \leq c$, and $b \leq d$, and $a, b, c, d \in [0,1]$.

Examples of continuous t-conorm are $a \odot b = \min\{a + b, 1\}, a \odot b = \max\{a, b\}$ and $a \odot b = a + b - ab$.

Definition 4.4. Let $U$ be a linear space over a real field $F$. A fuzzy subset $M$ of $X \times R$ such that for all $x, u \in U$ and $c \in F$:

(N*1) For all $t \in R$ with $t \leq 0, M (x, t) = 1$;
(N*2) For all $t \in R$ with $t > 0, M (x, t) = 0$ if and only if $x = 0$;
(N*3) For all $t \in R$ with $t > 0, M (cx, t) = M (x, t/k)$ if $c \neq 0, c \in F$;
(N*4) For all $s, t \in R, M (x + u, s + t) \leq M (x, s) \odot N^* (u, t)$;
(N*5) $M (x, t)$ is a non-increasing function of $t \in R$ and $\lim_{t \to \infty} M (x, t) = 0$.

Then $M$ is said to be a fuzzy $\odot$-antinorm on a linear space $U$.

Definition 4.5. [10] Let $E$ be any set. An intuitionistic fuzzy set $A$ of $E$ is an object of the form $A = \{(x, \mu_A (x), \nu_A (x)) : x \in E\}$, where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ denotes the degree of membership and the non-membership of the element $x \in E$ respectively and for every $x \in E$, $0 \leq \mu_A (x) + \nu_A (x) \leq 1$.

Definition 4.6. Let $*$ be a continuous t-norm, $\odot$ be a continuous t-conorm and $V$ be a linear space over the field $F (= R$ or $C$). An intuitionistic fuzzy norm on $V$ is an object of the form $A = \{(x, N(x, t), M(x, t)) : (x, t) \in V \times R^+\}$, where $N, M$ are fuzzy sets on $V \times R^+$, $N$ denotes the degree of membership and $M$ denotes the degree of non-membership $(x, t) \in V \times R^+$ satisfying the following conditions:

(i) $N$ is a fuzzy $* -$ norm on a linear space $V$.
(ii) $M$ is a fuzzy $\odot -$ antinorm on a linear space $V$.
(iii) $N (x, t) + M (x, t) \leq 1, \forall (x, t) \in V \times R^+$.

Example 4.7. Let $(V = R, \|\|)$ be a normed linear space where $\|x\| = |x|, \forall x \in R$. Define $a * b = \min\{a, b\}$ and $a \odot b = \max\{a, b\}$ for all $a, b \in [0,1]$. Also define $N (x, t) = \frac{t}{t + k |x|}$ and $M (x, t) = \frac{k |x|}{t + k |x|}$ where $k > 0$. We now consider $A = A = \{(x, t), N (x, t), M (x, t)) : (x, t) \in V \times R^+\}$. Here $A$ is an intuitionistic fuzzy norm on $V$.

Proof: Obviously follows from the calculation of the example 2.2 and example 3.2.
Definition 4.8. [11]. If $A$ is an intuitionistic fuzzy norm on $V$ (a linear space over the field $F = R$ or $C$) then $(V, A)$ is called an intuitionistic fuzzy normed linear space or in short IFNLS.

Though there are the concepts of fuzzy inner product spaces [12] but the concept of fuzzy norm could not be induced by these concepts of fuzzy inner product. So, one can develop the concept of fuzzy inner product which can induce the concept of fuzzy norm. Also, one can develop the concept of anti fuzzy inner product which can induce the concept of anti fuzzy norm.

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