Robbins Algebras Are Boolean: A Revision of McCune's Computer-Generated Solution of Robbins Problem

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In the early 1930s, Robbins asked whether a certain equation together with commutativity and associativity of the union operation was sufficient to characterize Boolean algebras. In 1992, Winker reduced this to the problem of proving the solvability of another equation from Robbins' axioms. In October 1996, William McCune confirmed Winker's condition with the help of the automated theorem prover EQP. In this paper we give a simplified presentation of the proof discovered by EQP.

1. INTRODUCTION

Searching for simple axiomatizations of the theory of Boolean algebras, Huntington [3, 4] proposed in 1933 an axiom system consisting of the commutativity and associativity of $\cup$ together with the equation

$$x \cup y \cup z = x.$$

Then Robbins conjectured soon thereafter that this equation can be replaced by

$$x \cup y \cup x \cup y = x.$$

Algebras satisfying Robbins' axion system became known as near-Boolean algebras or Robbins algebras.

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Winker [6] proved that the solvability of the equation

\[ x \cup \bar{y} = \bar{y} \]

in each Robbins algebra would imply that Robbins algebras are Boolean. This is referred to as Winker's second condition.

In October 1996, McCune [5] proved with the automated theorem prover EQP that each Robbins algebra in fact satisfies this condition. Due to the complexity of the formulas used by EQP, the mathematical contents of the output of the EQP theorem prover were hard to grasp. With the kind permission of William McCune I give below a simplified presentation of that proof. The Appendix contains some information on the relation between the proof given here and the output of EQP and other deductive systems. I should like to emphasize that the priority for discovering the proof is due to the EQP system.

2. EACH ROBBINS ALGEBRA SATISFIES WINKER'S SECOND CONDITION

We start from the following axiom system for Robbins algebras.

1. \( x \cup y = y \cup x \).
2. \( (x \cup y) \cup z = x \cup (y \cup z') \).
3. \( x \cup \cdots \cup x = nx \).

The last equation is called Robbins equation. We introduce the following abbreviations.

- \( \delta(x, y) = \bar{x} \cup \bar{y} \).
- \( x \cup \cdots \cup x = nx \).

We shall prove the following theorem. Its assertion is known as Winker's second condition. From this condition it follows, according to [6], that each Robbins algebra is Boolean.

**Theorem 1 (Winker's Second Condition).** In each Robbins algebra there are \( x, y \) such that \( x \cup \bar{y} = \bar{y} \).

Before we start the proof, we collect some useful techniques, which are either a consequence of Robbins' axiom or of the abbreviations we made.

The left-hand side of Robbins equation is obviously equal to \( \delta(x \cup y, \delta(x, y)) \). We call this term the \( \delta \)-expansion of \( y \) by \( x \) and we note that

\[ \delta(x \cup y, \delta(x, y)) = y. \]
A major technique of the proof is to build $\delta$-expansions in such a way that the $\delta$-term occurring as its second argument can be simplified by equations that had been obtained earlier also by $\delta$-expansion.

Other useful term transformations are an immediate consequence of our abbreviations. For example, we can lift complements in $\delta$-terms by the rule

$$\delta(\bar{x} \cup y, z) = \delta(x, y) \cup \bar{z}$$

since both sides are equal to $\bar{x} \cup y \cup \bar{z}$. We call this transformation \textit{c-lifting}. Note that c-lifting preserves the order of arguments.

Arguments of $\delta$-terms can be exchanged by the rule

$$\delta(x, \bar{y}) = \delta(y, \bar{x}),$$

which is also an immediate consequence of the definition of $\delta$. When we use the expression \textit{exchange of arguments}, we shall refer to applications of this equation.

Also the \textit{left argument substitution principle}

$$\bar{x}_1 = \bar{x}_2 \rightarrow \delta(x_1, y) = \delta(x_2, y)$$

is easy to see.

Let us now fix an arbitrary element $a$. Let $a_0 = \bar{a} \cup \bar{a}$ and let $a_n = a_0 \cup na$. We are done if we can show that $\bar{a}_1 \cup \bar{a}_3 = \bar{a}_3$. This will be proved by analyzing the behavior of the $\delta$-function on $a, a_0, \ldots, a_3$.

First we observe that we can shift indices in unions:

$$a_n \cup a_m = a_k \cup a_l \quad \text{if} \quad n + m = k + l.$$  \hspace{1cm} (1)

We note, moreover, that

$$\delta(a, a) = a_0.$$  \hspace{1cm} (2)

This implies that $\delta(2a, a_0) = \delta(2a \cup a_0, \delta(a, a))$ is the $\delta$-expansion of $a_0$ by $2a$. Hence also

$$\delta(a_2, a) = a_0.$$  \hspace{1cm} (3)
We shall prove a series of similar equations. We start with

**LEMMA 1.** \( \delta(a_3, a_0) = a \).

**Proof.** We expand \( a \) by \( a_2 \): 
\[
\delta(a \cup a_2, \delta(a_2, a)) = \delta(a_3, a_0) \quad \text{by (3)}.
\]
Q.E.D.

Our next aim is to show that—similar to (3)—also

**LEMMA 2.** \( \delta(a_3, a) = a_0 \).

**Proof.** We expand \( a \) by \( \overline{a_3} \cup a_0 \):
\[
a = \delta\left(\overline{a_3} \cup a_0 \cup a, \delta\left(\overline{a_3} \cup a_0, a\right)\right)
= \delta\left(\overline{a_3} \cup a_1, \delta\left(\overline{a_3} \cup a_0, a\right)\right)
= \delta\left(\overline{a_3} \cup a_1, \delta\left(a_3, a_0\right) \cup a\right) \quad \text{by c-lifting},
= \delta\left(\overline{a_3} \cup a_1, \overline{2a}\right) \quad \text{by Lemma 1}.
\]

This \( \delta \)-term occurs in the expansion of \( \overline{2a} \) by \( \overline{a_3} \cup a_1 \):
\[
\overline{2a} = \delta\left(\overline{a_3} \cup a_1 \cup \overline{2a}, \delta\left(\overline{a_3} \cup a_1, \overline{2a}\right)\right)
= \delta\left(\overline{a_3} \cup a_1 \cup \overline{2a}, a\right).
\]

Thus, by the left argument substitution principle, we can replace \( 2a \) with \( \overline{a} \cup a \) in left arguments of \( \delta \). Especially, by (2),
\[
a = \delta\left(\overline{a} \cup a, a_0\right). \tag{4}
\]

\( \alpha \) occurs also in the \( \delta \)-expansion of \( \overline{a_3} \) by \( a_1 \cup \overline{2a} \):
\[
\overline{a_3} = \delta\left(\alpha, \delta\left(a_1 \cup \overline{2a}, \overline{a_3}\right)\right)
= \delta\left(\alpha, \delta\left(a_3, \delta\left(2a, a_1\right)\right)\right) \quad \text{by argument exchange}
= \delta\left(\alpha, a_1\right) \quad \text{by \( \delta \)-expansion since \( a_3 = a_1 \cup 2a \)}
= \overline{a} \cup \overline{a_1}.
\]
Hence $a_3$ can be replaced as the left argument of $\delta$ by $\overline{a} \cup a_1$. Thus we obtain

$$\delta(a_3, a) = \delta(\overline{a} \cup a_1, a)$$
$$= \delta(\overline{a} \cup a_0 \cup a, \delta(\overline{a} \cup a, a_0)) \quad \text{by (4)},$$
$$\delta(a_3, a) = a_0 \quad \text{by } \delta\text{-expansion.} \quad \text{Q.E.D.}$$

Now we investigate properties of $a_1 \cup a_3$ and its complement.

**Lemma 3.** $\delta(a_1 \cup a_3, a) = a_0$.

*Proof.* We expand $a_0$ by $a_4$:

$$a_0 = \delta(a_4 \cup a_0, \delta(a_4, a_0))$$
$$= \delta(a_3 \cup a_1, \delta(a_3 \cup a, \delta(a_3, a))) \quad \text{by Lemma 2},$$
$$a_0 = \delta(a_3 \cup a_1, a) \quad \text{by } \delta\text{-expansion.} \quad \text{Q.E.D.}$$

**Lemma 4.** $\delta(a_1 \cup a_2, a) = a_0$.

*Proof.* Expand $a_0$ by $a_4$: $a_0 = \delta(a_0 \cup a_3, \delta(a_3, a)) = \delta(a_1 \cup a_2, a)$ by Lemma 1. \text{Q.E.D.}

**Lemma 5.** $\delta(a_1 \cup a_3, a_0) = a$.

*Proof.* Expand $a$ by $a_1 \cup a_2$: $a = \delta(a_1 \cup a_2 \cup a, \delta(a_1 \cup a_2, a)) = \delta(a_1 \cup a_3, a_0)$ by Lemma 4. \text{Q.E.D.}

Let $\beta = \overline{a_1} \cup \overline{a_3} \cup a \cup \overline{a_3}$.

**Lemma 6.** $\delta(\beta, a) = \overline{a_3}$.

*Proof.* $\beta$ occurs in the $\delta$-expansion of $\overline{a_3}$ by $\overline{a_1} \cup \overline{a_3} \cup a$:

$$\overline{a_3} = \delta(\beta, \delta(\overline{a_1} \cup \overline{a_3} \cup a, \overline{a_3}))$$
$$= \delta(\beta, \delta(a_3, \delta(a_1 \cup a_3, a))) \quad \text{by exchange of arguments},$$
$$= \delta(\beta, \delta(a_3, a_0)) \quad \text{by Lemma 3},$$
$$\overline{a_3} = \delta(\beta, a) \quad \text{by Lemma 1}. \quad \text{Q.E.D.}$$
We are done when we have proved 

**Lemma 7.** \( \delta(\beta, a) = \overline{a_1 \cup a_3} \).

**Proof.** \( \beta \) occurs also in the \( \delta \)-expansion of \( \overline{a_1 \cup a_3} \) by \( a \cup \overline{a_3} \):

\[
\overline{a_1 \cup a_3} = \delta(\beta, \overline{a_1 \cup a_3}) = \delta(\beta, \delta(a \cup \overline{a_3}, \overline{a_1 \cup a_3})) = \delta(\beta, \delta(a \cup \overline{a_3}, \overline{a_1 \cup a_3})) = \delta(\beta, a) \] by exchange of arguments

\[
= \delta(\beta, \delta(a \cup \overline{a_3}, a_0)) \] by Lemma 2,

\[
\overline{a_1 \cup a_3} = \delta(\beta, a) \] by Lemma 5. \( \) Q.E.D.

Lemmas 6 and 7 together imply Winker's second condition in the theorem.

### 3. APPENDIX: FROM EQP TO THE CURRENT PROOF

The EQP system printed a rough sketch of the proof. This was used by William McCune to guide his theorem prover Otter in order to produce a detailed proof. Otter's proof contained fairly complex terms which were hard to understand or even to print in a readable format.

Several authors tried to find simpler representations of this proof [1, 5, 2]. The ILF system, developed by the author and his associates, produced several presentations of Otter's proof automatically. It converted Otter's indirect proof into a direct proof and restructured it for better readability. For example, it removed numerous applications of associativity and commutativity and made the relevant substitutions explicit. ILF's automated analysis of Otter's proof revealed the relevance of the \( \delta \)-function in this proof. Further simplification was obtained by replacing the general formulas proved by Otter by the instances which were really needed to prove the final results. This led to the introduction of the constants \( a_i \) used in the proof above.

### REFERENCES

1. S. Burris, An anthropomorphized version of the McCune's machine proof that Robbins algebras are Boolean, preprint.