ON OPTIMAL IRRIGATION SCHEDULING

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Abstract. In this paper optimal irrigation scheduling based on a dynamical model is analyzed, and global optimality is proved with the use of sufficient conditions. Krotov’s method of global bounds and Hamilton-Jacobi-Bellman formalism have been used.

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1 Introduction

One of the most significant challenges in the 21st century is water management. Economically optimal and sustainable irrigation scheduling is of growing concern given the necessary increase in crop production and the acute water scarcity. In arid and semi-arid environments water is the most important factor limiting agricultural production. Under current water scarcity conditions the available limited water must be used in the most efficient way. Much theoretical and experimental research has been devoted to this problem in recent years, see [13], [4], [5], [15], [3], [1], and many others.

[20] developed and analyzed a rather simple dynamic model of plant growth and water dynamics in soil. We shall denote this model as the STZ model. The model has two state variables: plant biomass and soil moisture. Utilizing the Pontryagin’s maximum principle, [16], the solution was found to contain a so-called singular arc, see [12]. The optimal trajectory, i.e. the optimal irrigation scheduling, contains three periods: (i) maximal irrigation up to the optimal level of soil moisture, (ii) intermediate irrigation that maintains soil moisture at the optimal level, called turnpike, and (iii) no irrigation until the end of the growth season.

An empirical example related to the control of the growth of Ornamental sunflower (Helianthus annuus var dwarf yellow) in the Arava Valley in Israel has been presented in [20]. This example has included values of all model parameters and the explicit form of all specific model functions. The optimal value of the soil moisture has been found to be significantly less than the value related to the maximal soil water capacity. This is of great practical importance. The optimal value of the harvested yield is about 10% below the maximal attainable yield. The irrigation level required to maintain water content in the root zone for maximal yield is about four times higher than the irrigation needed for optimal policy. Most of the added irrigation is wasted because of higher drainage.

The question however arises if this singular-arc solution is really optimal compared with the bang-bang solution that contains only periods with maximal rate irrigation and
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periods with no-irrigation? Following [14], [17], [18], [19] by using Green’s Theorem, [21], and [22] answered this question affirmatively.

In the present paper we take a different and presumably straightforward and clearer approach, using Krotov’s sufficient conditions of optimality, Krotov’s global bounds method, and Hamilton-Jacobi-Bellman formalism, [8], [9], [10], [11], and some facts from convex analysis.

2 The STZ model and the singular solution

According to [20] the STZ model is formulated as follows. The plant biomass is denoted as \( m(t), t \in [0, T] \), where \( T \) is the length of the growing season. Marketable yield is denoted as \( y(m) \). The water moisture of the soil [dimensionless] is denoted as \( \theta(t) \).

The plant biomass dynamics is described by the equation

\[
\frac{dm(t)}{dt} = g(\theta(t))h(m(t)).
\] (1)

The positive functions \( g \) and \( h \) are assumed to be strictly concave, and \( g \) has a maximum at some value \( \bar{\theta} \).

The evapotranspiration rate, \( ET(\theta, m) \) has the form

\[
ET(\theta, m) = \beta g(\theta)f(m),
\] (2)

where \( \beta \) is a constant positive factor, and the increasing function \( f(m) \) is constrained by

\[
0 \leq f(m) \leq 1.
\]

The rate of water drainage \( D(\theta) \) is positive, increasing and convex. The rate of irrigation flow is denoted as \( u(t) \),

\[
0 \leq u(t) \leq \bar{u}.
\]

The soil water dynamics is given by the equation

\[
Z \frac{d\theta(t)}{dt} = u(t) - \beta g(\theta)f(m) - D(\theta),
\] (3)

where \( Z \) is the root depth and \( Z\theta \) measures the total amount of water in the root zone [\( mm \)]. For simplicity, we shall assume that \( u, \beta, D \) have been divided by \( Z \), and thus \( Z \) has been eliminated. The dimensions of \( u, \beta, \) and \( D \) are now \[ \text{day}^{-1} \], \[ mm^2 g^{-1} \], and \[ ((day)^{-1}) \], respectively. Denoting the price of the crop as \( P \) and the price of the water as \( W \) the objective function, i.e. the grower’s income, will be

\[
J = Py(m(T)) - WZ \int_0^T u(t)dt \rightarrow \max,
\] (4)

or, denoting \( w = WZ/P \), the same solution will be obtained by the objective

\[
J = y(m(T)) - w \int_0^T u(t)dt \rightarrow \max
\] (5)

Utilizing Pontryagin’s maximum principle, Pontryagin et al. (1962) and denoting the costates for equations (1) and (3) as \( \lambda \) and \( \mu \), respectively, one gets

\[
H = (\mu - w)u - g(\theta)[\beta \mu f(m) - \lambda h(m)] - \mu D(\theta),
\] (6)

and the costate equations

\[
\frac{d\lambda}{dt} = -\frac{\partial H}{\partial m}, \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \theta}.
\] (7)

It follows that the optimal irrigation \( u^* \) satisfies the conditions

\[
u^* = \arg \max_u H(u), \quad 0 \leq u \leq \bar{u}
\] (8)
\[ u^* = \bar{u}, \text{ if } \mu > w. \] \tag{9} \\
\[ u^* = 0, \text{ if } \mu < w, \] \tag{10}

If \( \mu = w \) during some non-zero interval of \( t \) then the optimal control \( u^* \) is singular on that interval. [20] showed that the water content on the singular arc is constant, \( \theta = \hat{\theta} \). Thus the optimal irrigation on the singular arc is determined from the equation \( d\theta/dt = 0 \).

In addition to the initial conditions \( x(0) = x_0, \theta(0) = \theta_0 \) the transversality conditions must hold for the free state variables at \( t = T \), namely

\[ \lambda(T) = \frac{d\theta(T)}{dm}, \mu(T) = 0. \] \tag{11}

From the second of the transversality conditions, continuity of \( \mu \), and (10) it follows that on the interval prior to \( t = T \), the optimal control is \( u^* = 0 \).

Let us assume that the optimal values \( m(T) = m_f, \theta(T) = \theta_f \) are known. Then the original problem has the same optimal solution as the auxiliary problem with the objective

\[ J = \int_0^T u(t) dt \rightarrow \min \] \tag{12}

and the end conditions \( \theta(T) = \theta_f, m(T) = m_f \). In the following we shall use the problem (12) in order to prove the optimality of the singular arc.

3 Optimality of the intermediate singular arc

3.1 Krotov’s sufficient condition

Let us consider the optimal control problem:

\[ \frac{dx}{dt} = s(x, U, t), \] \tag{13}

\( x \in \mathbb{R}^n, U \in \mathbb{R}^m \), with end conditions and constraints

\[ x(t_0) = x_0, x(T) = x_f, U \in \bar{U}, \] \tag{14}

and objective

\[ J = \int_{t_0}^T s_0(x, U, t) dt \rightarrow \min \]

The pair \( x^*, U^* \), is the optimal solution of the problem if it satisfies equations (13), and the control constraints and the end conditions (14), and the following conditions, Krotov (1966), hold

\[ \max_{x, U} R(x, U, t) = R(x^*, U^*, t) = P(t) \] \tag{15}

\[ R = \frac{dV}{dt} s(x, U, t) - s_0(x, U, t) \] \tag{16}

Here max is taken pointwise for each point in \( t \). The function \( V(x, t) \) is some continuous piece-smooth function of \( x, t \) such that the integral \( \int_{t_0}^T R(x, U, t) dt \) exists. One can see that if the pair \( (x, U) \) satisfies equations (13) and (14) then

\[ R(x, U, t) = \frac{dV}{dt} - s_0(x, U, t), \] \tag{17}

where \( dV/dt \) is a full derivative with respect to the system (13). The objective \( J \) is then equal to

\[ J = V(x_f, T) - V(x_0, t_0) - \int_{t_0}^T R(t) dt \] \tag{18}

Finding the solution of the nonlinear partial differential equation (16) that generates the feasible trajectory \( x, u \) is the Hamilton-Jacobi-Bellman formalism.
3.2 Transformation of the problem

Using notation \( k(z) = f(m(z)) \) let us introduce the variable \( z(m) \), and its inverse \( m(z) \), and make a transformation in order to simplify the problem (12). Define

\[
\frac{dz}{dm} = \frac{1}{h(m)} \tag{19}
\]

\[
\frac{dz}{dt} = \frac{dx}{dm} \frac{dm}{dt} = g(\theta), \quad z(m_0) = 0. \tag{20}
\]

Clearly \( z \) is an increasing positive function of \( m \), and the final value of \( z_f \) is \( z_f(z(m_f)) \).

Hence \( z \) can be used instead of \( t \) as the independent variable, yielding

\[
\frac{d\theta}{dz} = -\beta k(z) - \frac{D(\theta)}{g(\theta)} + \frac{u}{g(\theta)} \tag{21}
\]

\[
\frac{dt}{dz} = \frac{1}{g(\theta)} \tag{22}
\]

\[ J = \int_{z_f}^{z_f} u g(\theta) dz \rightarrow \min \tag{23} \]

This is a so called gauge transformation, e.g. [7].

3.3 Using Krotov’s sufficient condition

The crucial step in Krotov’s method is to choose \( V \) appropriately. Here, \( V(\theta, t, z) \) can be chosen as

\[ V = p_0 \theta + p_t t \tag{24} \]

where \( p_0 = 1 \), and \( p_t \) is a suitable constant to be found later.

Substituting (21), (22), (23), (24) into (16) one obtains for \( p_0 = 1 \)

\[ R = -\beta k(z) - \frac{D(\theta)}{g(\theta)} + \frac{p_t}{g(\theta)} \tag{25} \]

It is now evident that \( R \) is a function of \( \theta \) and \( z \) only, and not dependent on \( u \). Hence \( \hat{\theta} = \arg \max_{\theta} R(\theta, z) = \) constant. Let us now use the notations \( D_1 = -\beta k + p_t/g(\theta) \). One can note that \( D_1 \) is concave and decreasing. We chose \( p_t \) such that \( D_1 < 0 \). We get

\[ R = -\beta k(z) + S(\theta) \tag{26} \]

It can be investigated how the value \( p_t \) affects the value of \( \hat{\theta} \). Applying the Implicit Function Theorem, [6], for the function \( \partial S(\theta, p_t)/\partial \theta = 0 \), with \( \theta = \hat{\theta} \) yields

\[
\frac{d\theta}{dp_t} = -\frac{\partial^2 S}{\partial \theta \partial p_t} \frac{\partial^2 S}{\partial \theta^2} = \frac{1}{g^2(\theta)} \frac{d^2 g}{d\theta^2}(\frac{\partial^2 S}{\partial \theta^2}) < 0 \tag{27}
\]

In this equation one must substitute the value \( \frac{\partial^2 S}{\partial \theta^2} \) from (29) below. There it is shown that this value is negative.

From here one can see that if \( S < 0 \) then \( \hat{\theta} \) decreases when \( p_t \) increases. It then follows that the growth will be slower, and hence the final time \( T \) also increases. If \( S \geq 0 \) then \( dS/d\theta < 0 \) and the max \( S \) will lie on the lower bound of \( \theta \). In the proximity of \( \theta_f \) this lower bound will be achieved with \( u = \bar{u} \) which contradicts the transversality condition (11). Thus one will have \( p_t < D_1(\theta) \) and hence \( S < 0 \). For a given value of \( \theta \) the corresponding value of \( p_t \) can be easily found from (28), see below, where we note that equation (28) is linear with respect to \( p_t \).
Generally one obtains for the point \( \theta = \bar{\theta} \), \( dS/d\theta = 0 \), that

\[
\frac{dS}{d\theta} = \frac{1}{g} \left( \frac{dD_1}{d\theta} - S \frac{dS}{d\theta} \right) = 0 \tag{28}
\]

\[
\frac{d^2S}{d\theta^2} = \frac{1}{g} \left( \frac{d^2D_1}{d\theta^2} - S \frac{d^2g}{d\theta^2} \right) + \frac{1}{g} \frac{dg}{d\theta} \left( \frac{dD_1}{d\theta} + S \frac{dS}{d\theta} \right) \tag{29}
\]

One can easily see that from the assumptions, strict concavity of \( g \) and concavity of \( D_1 \), and \( S < 0 \), it follows that the first term in the brackets in (29) is negative, and from \( dS/d\theta = 0 \), \( (28) \), it follows that the second term in the brackets is zero. Thus \( d^2S/d\theta^2 < 0 \) when \( dS/d\theta = 0 \), and also \( \partial^2R/\partial\theta^2 < 0 \). Note, that \( R \) is not dependent on \( t \). It means that function \( S \) is not necessary concave, but it has not more than a single extremum in the range \([0, \bar{\theta}]\) and this extremum is a maximum. This i.e. follows from the Mountain Pass Theorem, see [2]. Thus if the singular arc exists it is globally optimal.

Finally one can note that if \( T \) is free, then \( p_f \) can be set to zero, and the function \( R \) will have form

\[
R = -\beta k(z) - \frac{D(\theta)}{g(\theta)} \tag{30}
\]

\[\max_{\theta} [-D(\theta)/g(\theta)] \text{ corresponds to } \min_{\theta} [D(\theta)/g(\theta)] \text{ and has a clear physical meaning, because } g(\theta) \text{ represents the water efficient factor of growth and } D(\theta) \text{ represents the useless losses of water. A constant } p_f \text{ just corrects the numerator of this ratio } [D(\theta) - p_f]/g(\theta), \text{ necessitating an adjustment of the value } T.\]

4 Initial and final arcs

Here the same function \( V(z, \theta) \) as in the previous section is used and thus the trajectory still is singular, though the control is \( u = \bar{u} \), or \( u = 0 \). As in the previous section the control \( u \) does not appears in the function \( R \) and thus it has to be found from the equations of the movement along bounds of \( \theta \).

4.1 Initial arc

At the initial arc when \( \theta \) changes from \( \theta_0 \) to \( \dot{\theta} \) and \( z \) is changed from \( z = 0 \) to some unknown \( z = \bar{z} \), \( \theta(\bar{z}) = \dot{\theta} \), the function \( S(\theta) \) can not attain its unbounded maximum at \( \theta = \dot{\theta} \) because this point is in general not feasible.

Let us first consider the case when \( \theta_0 \leq \dot{\theta} \). Then the maximum will be at the upper bound \( \theta^u \) of \( \theta \). We can find this upper bound \( \theta^u \) for \( \theta \) and it will be determined by integrating equation (21) from the initial point \( z = 0, \theta(0) = \theta_0 \) with the control \( u = \bar{u} \). This is the Krotov’s method of global bounds, [8], [11].

Correspondingly when \( \theta_0 > \dot{\theta} \) the maximum will be at the lower bound \( \theta^l \) of \( \theta \) which can be found by integrating equation (21) with control \( u = 0 \) from the initial point.

Accordingly, the maximum of \( R(u, \theta, z) \) over \( \theta \) will be attained at \( \theta^u \) or at \( \theta^l \) on this interval \([0, \bar{z}]\). We should recall that for \( p_0 = 1 \) function \( R \) is not dependent on \( u \), so any appropriate \( u \) can be taken. Hover, from maximization on \( \theta \) above one can see that \( u \) is determined uniquely. The unknown value \( z = \bar{z} \) is determined by the equality \( \theta(\bar{z}) = \dot{\theta} \).

4.2 Final arc

This part of analysis is very similar to the previous one. As we have mentioned above it follows from the transversality conditions that only the case \( \theta_f \leq \dot{\theta} \) is feasible. One can note that in the case of equality the final arc disappears. The maximum of \( S(\theta) \) and hence maximum of \( R \) over \( \theta \) on the final arc will be at the upper bound \( \theta^u_f \) which will be determined by integrating equation (21) with \( u = 0 \) on the interval \([\bar{z}, \theta_f]\) from the final point backwards. The starting point of this interval \( \bar{z} \) is unknown. This value is
determined by the equality $\theta(\hat{z}) = \hat{\theta}$. Note that satisfying the final condition $t(z_f) = T$
has to be achieved by the proper choice of the constant $p_t$.

5 Conclusions

By using Krotov’s sufficient conditions of optimality, the optimal irrigation policy has been
found to be equal to that in [20]. Moreover it is proven in this paper that the singular arc is globally optimal if it exists. Thus initial irrigation up to the level of the soil water capacity is in general not optimal.

6 References


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