A COUNTABLE DENSE HOMOGENEOUS SET OF REALS OF SIZE $\aleph_1$

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Abstract. We prove there is a countable dense homogeneous subspace of $\mathbb{R}$ of size $\aleph_1$. The proof involves an absoluteness argument using an extension of $L_{\omega_1\omega}(\mathbb{Q})$ obtained by adding predicates for Borel sets.

A separable topological space $X$ is countable dense homogeneous (CDH) if given any two countable dense subsets $D, D' \subseteq X$ there is a homeomorphism $h$ of $X$ such that $h[D] = D'$. The main purpose of this note is to show the following.

Theorem 1. There is a countable dense homogeneous set of reals $X$ of size $\aleph_1$. Moreover, $X$ can be chosen to be a $\lambda$-set.

Recall that a set of reals is a $\lambda$-set if all of its countable subsets are relatively $G_\delta$, and therefore it cannot be completely metrizable. Theorem 1 and this remark solve problems 390 and 389 from [4]. Our construction necessarily uses the Axiom of Choice. In [6] it was shown that under sufficient large cardinal assumptions every CDH metric space in $L(\mathbb{R})$ is completely metrizable. Our proof of Theorem 1 uses Keisler’s completeness theorem for logic $L_{\omega_1\omega}(\mathbb{Q})$ (see §2), and the secondary purpose of this note is stating a somewhat general method for proving absoluteness of the existence of an uncountable set of reals properties of which are described using Borel sets as parameters.

1. A MEAGER COUNTABLE DENSE HOMOGENEOUS SET

Recall that every compact zero-dimensional subset of $\mathbb{R}$ without isolated points is homeomorphic (even isomorphic as linearly ordered sets) to the Cantor set.

Lemma 1.1. There is an uncountable $F_\sigma$ set $F$ containing the rationals $\mathbb{Q}$ and an $F_\sigma$ equivalence relation $E \subseteq F \times F$ with all equivalence classes countable dense subsets of $\mathbb{R}$, such that for every dense $A \subseteq \mathbb{Q}$ there is a homeomorphism $h : F \to F$ satisfying

1. $h(\mathbb{Q}) = A$ and
2. $h(x) E x$ for every $x \in F$.

Proof. Let $F = \mathbb{Q} \cup D \cup \bigcup_{n \in \omega} F_n$, where $\mathbb{Q}$ and $D$ are disjoint countable dense subsets of $\mathbb{R}$ and $\{F_n : n \in \omega\}$ is a family of pairwise disjoint copies of the Cantor
set disjoint from both $\mathbb{Q}$ and $D$ and such that every nonempty open set contains one of the $F_n$s. Denote by $C$ the set of all relatively clopen subsets of all the Cantor sets $F_n$. For every pair $U, W \in C$ fix $h_{U,W}: U \rightarrow W$ an increasing homeomorphism. Let $F$ be the (countable) family of all compositions of finitely many functions of the type $h_{U,W}$ and their inverses. Then define $xEy$ if and only if $x, y \in Q \cup D$ or $y = h(x)$ for some $h \in F$. The relation $E$ is then obviously an equivalence relation with countable and dense equivalence classes and it is $F_\sigma$ as it is a countable union of compact sets.

Let $A \subseteq Q$ be dense. Enumerate $C$ as $\{A_n : n \in \omega\}$, $Q$ as $\{q_n : n \in \omega\}$, $D$ as $\{d_n : n \in \omega\}$, $D \cup (Q \setminus A)$ as $\{c_n : n \in \omega\}$ and $A$ as $\{a_n : n \in \omega\}$. Using the back-and-forth argument of Cantor, construct the homeomorphism $h: F \rightarrow F$ as an increasing union of strictly increasing partial homeomorphisms $h_n, n \in \omega$, so that, for every $n \in \omega$:

1. $h_n$ extends $h_{n-1}$,
2. $\text{dom}(h_n)$ consists of a finite subset of $Q \cup D$ and a finite union of elements of $C$,
3. $\text{range}(h_n)$ consists of a finite subset of $Q \cup D$ and a finite union of elements of $C$,
4. $h_n$ restricted to $\text{dom}(h_n) \setminus (Q \cup D)$ is covered by finitely many elements of $F$,
5. $h_n(q) \in A$ for every $q \in Q \cap \text{dom}(h_n),$
6. $h_n(d) \in D \cup (Q \setminus A)$ for every $d \in D \cap \text{dom}(h_n),$
7. $\{q_m : m \leq n\} \cup \{d_m : m \leq n\} \cup \{\bigcup \{A_m : m \leq n\} \subseteq \text{dom}(h_n),$
8. $\{c_m : m \leq n\} \cup \{e_m : m \leq n\} \cup \{\bigcup \{A_m : m \leq n\} \subseteq \text{range}(h_n).$

Then $h = \bigcup_{n \in \omega} h_n$ is the desired homeomorphism of $F$. \hfill $\Box$

Recall that if $E$ is an equivalence relation then a set $X$ is $E$-saturated if for all $x \in X$ we have $x \in X$ if and only if $y \in X$.

**Lemma 1.2.** Assume $Q, D, F, E$ and $F$ are as in Lemma 1.1 and its proof. If $X \subseteq F$ is an $E$-saturated set such that for every countable $B \subseteq X$ there is an $E$-saturated $A \subseteq X$ containing $B$ and a homeomorphism $h: X \rightarrow X$ satisfying $h[A] = Q$, then $X$ is countable dense homogeneous.

**Proof.** Fix a countable dense subset $B$ of $X$. Let $g$ be an autohomeomorphism of $X$ such that $g^{-1}(Q)$ is an $E$-saturated set containing $B$. Then $A = g[B]$ is a dense subset of $Q$. By Lemma 1.1 there is an autohomeomorphism $h$ of $F$ such that $h[Q] = A$ and $h(x)Ex$ for every $x \in F$. Therefore $h \upharpoonright X$ is an autohomeomorphism of $X$. Then $H = h^{-1} \circ g$ is an autohomeomorphism of $X$ such that $H[B] = Q$ as required. \hfill $\Box$

2. **Absoluteness**

Recall that $L_{\omega_1 \omega}(Q)$ is an extension of the first-order logic that allows countable disjunctions and has quantifier $Qx$, ‘there exists uncountably many.’ It is well-known that completeness of this logic is useful for proving that the existence of certain objects of size $\aleph_1$ is absolute between models of ZFC (see [7, 1, 3, 5, 9]).

Let $L^{B}_{\omega_1 \omega}(Q)$ be the extension of $L_{\omega_1 \omega}(Q)$ allowing countably many Borel predicates in the following sense. For some Borel sets $A_n \subseteq (\mathbb{N}^\mathbb{N})^{k_n}$ ($n \in \mathbb{N}$) and Borel functions $f_n: (\mathbb{N}^\mathbb{N})^{k_n} \rightarrow \mathbb{N}^\mathbb{N}$ ($n \in \mathbb{N}$), we have relation and function symbols $A_n$.
and $f_n$ of matching arity, and for $b_n \in \mathbb{N}^n$ ($n \in \mathbb{N}$) we have constant symbols $b_n$ ($n \in \mathbb{N}$).

If $\phi$ is a sentence of $L^B_{\omega_1}(Q)$, we say that a model $X$ of $\phi$ (with universe $X$) is correct if

1. each $A_n$ is interpreted as $A_n \cap X^{k_n}$, each $f_n$ is interpreted as $f_n \upharpoonright X^{l_n}$, each $b_n$ is interpreted as $b_n$, and
2. if $A_n$ is countable then $A_n \subseteq X$.

A model of an $L_{\omega_1}(Q)$ sentence is standard if it interprets $Qx$ as ‘there exist uncountably many. Recall that a linear order is $\omega_1$-like if it is uncountable yet each of its initial segments is countable.

**Theorem 2.** An $L^B_{\omega_1}(Q)$-sentence $\phi$ has a correct model if and only if it has a correct model in some forcing extension $\mathcal{V}^\mathcal{P}$ of the universe $\mathcal{V}$.

Let us postpone the proof of Theorem 2 for a moment. Fix an $L^B_{\omega_1}(Q)$-sentence $\phi$. We shall define an $L_{\omega_1}(Q)$ sentence $\phi^M$ as follows. (For simplicity we shall treat only the case when we have only one Borel set, $A \subseteq \mathbb{N}^n$; a standard coding argument shows that the general case with infinitely many Borel sets, functions and constants is really not any more general.) First, the language of $\phi$ is expanded by adding new symbols $N, M, \{c_n : n \in \mathbb{N}\}, B$ and $\{N_s : s \in \omega^{<\omega}\}$. Let $\phi_0$ be the conjunction of sentences stating the following:

1. $(\forall x)N(x) \Leftrightarrow \bigvee_{n \in \mathbb{N}} x = c_n$,
2. $(\forall x)B(x) \Leftrightarrow \bigvee_{x \in N^{<\omega}} x = N_s$,
3. axioms of formal arithmetic for $c_n$ ($n \in \mathbb{N}$),
4. first-order properties of basic open sets $[s] = \{x \in \mathbb{N}^n : s \subseteq x\}$ for $N_s$ ($s \in \omega^{<\omega}$),
5. if $M(x)$, then $x \in N_s$ for exactly one $s$ of length $n$ for all $n$, and moreover \{$s : x \in N_s\}$ forms a chain (all this can clearly be stated in $L_{\omega_1}$).

Since $A$ is a Borel set, we can fix arithmetic formulas $\psi_0(x, y)$ and $\psi_1(x, y)$ such that $x \in A \iff (\forall y)\psi_0(x, y) \iff (\exists y)\psi_1(x, y)$. Let $\phi_i$ ($i < 2$) be the translation of $\psi_i$ into the language of $N_s$ ($s \in \omega^{<\omega}$). Replace each occurrence of $A(x)$ in $\phi$ by $M(x) \wedge (\forall y)\phi_0(x, y)$, and let $\phi^M$ be the conjunction of thus modified $\phi$, $\phi_0$, and $(\forall x)(\exists y)\phi_0(x, y) \vee (\exists y)\neg\phi_1(x, y)$.

**Lemma 2.1.** An $L^B_{\omega_1}(Q)$ sentence $\phi$ has a correct model if and only if $\phi^M$ has a standard model.

**Proof.** Assume $\phi$ has a correct model $X = (X, A, \ldots)$. Extend its universe by adding all natural numbers, basic open subsets of $\mathbb{N}^n$, and the set $Y$ of ‘witnesses’ defined as follows. If $x \in X \cap A$, pick $y_x$ such that $\phi_0(x, y_x)$ holds. If $x \in X \setminus A$, pick $y_x$ such that $\neg\phi_1(x, y_x)$ holds. Let $Y = \{y_x : x \in X\}$. Finally interpret $M$ as $X$. It is clear that thus obtained model is a standard model of $\phi^M$.

Now assume $\phi^M$ has a standard model, $3 = (Z, A', \ldots)$. Let $X = \{x \in Z : 3 \models M(x)\}$, and let $\bar{X}$ be the reduction of $(X, A' \cap X, \ldots)$ to the language of $\phi$. We only need to check that $A$ is interpreted as $A' \cap X$. Note that $3 \models \phi_i(x, y)$ iff $\phi_i(x, y)$ holds, for $i < 2$. For every $x \in X$ we either have $3 \models \phi_0(x, y)$ or $3 \models \neg\phi_1(x, y)$ for some $y$. If $3 \models \phi_0(x, y)$ for some $y$, then $3 \models A(x)$ and $x \in A$. On the other hand, if $3 \models \phi_1(x, y)$ for some $y$, then $3 \models \neg A(x)$ and $x \notin A$. \qed
Proof of Theorem 2. By Lemma 2.1 \( \phi \) has a correct model if and only if \( \phi^M \) has a standard model. By Keisler’s completeness theorem for \( L_{\omega_1\omega}(Q) \) ([8]), \( \phi^M \) has a standard model if and only if it is not inconsistent in the proof system described in [8]. However, if \( \phi^M \) is inconsistent in \( V \), then it would remain such in the extension. If \( \phi^M \) has a model \( X \) in \( V \), then \( X \) is a weak model (see [8]) of \( \phi^M \) in \( V \), and again by Keisler’s theorem \( \phi^M \) has a standard model in \( V \) as well. □

In the following lemma \( A, B, C, D \) are unary relation symbols, \( h \) is a unary function symbol and \( f \) is a binary function symbol. We say that a property is expressible in \( L_{\omega_1\omega}(Q) \) if there is a sentence of \( L_{\omega_1\omega}(Q) \) such that in each of its correct models the interpretations \( A, B, C, D, f, h \) of these predicates satisfy the stated property.

Lemma 2.2. The following properties are expressible in \( L_{\omega_1\omega}(Q) \).

1. \( A \) is countable.
2. A binary relation \( < \) is an \( \omega_1 \)-like linear order.
3. \( h: A \to B \) is a surjection.
4. \( h: A \to B \) is a continuous function.
5. \( h: A \to B \) is a homeomorphism.
6. \( h: A \to B \) and it satisfies \( h[C] = D \).
7. \( f(x, \cdot): A \to B \) is a homeomorphism for every \( x \).
8. \( x \) is in the closure of \( A \).
9. \( A \) is a dense subset of \( B \).
10. \( A \) is a relatively open subset of \( B \).
11. \( A \) is a relatively \( G_\delta \) subset of \( B \).
12. \( B \) has a countable dense subset \( K \) that is relatively \( G_\delta \) in \( B \).
13. \( X \) is \( E \)-saturated, for a given Borel equivalence relation \( E \) all of whose equivalence classes are countable.

Proof. Items (3) and (6) are first-order definable, and (1) and (2) are straightforward to define using \( Qx \).

For (4), (5) and (8) one only needs to observe that since we have a standard model of \( L_{\omega_1\omega}(Q) \), quantifiers such as \((\forall \epsilon > 0)(\exists \delta > 0)\) are evaluated correctly. Item (7) is immediate from the preceding items, and (10) and (9) are immediate from (8). For (11), introduce new predicates \( A_n (n \in \mathbb{N}) \) and require that each \( A_n \) is a relatively open set of \( B \) and \( A = \bigcap_n A_n \).

To see (12), add a predicate for \( A \) and then use (1), (11), (2) and (9).

Let \( E \) be as in (13). It is well-known that there are Borel functions \( f_n (n \in \mathbb{N}) \) such that \( xEy \) if and only if \( (\exists n)x = f_n(y) \), hence for (13) we only need to add names for \( f_n (n \in \mathbb{N}) \) to our language. □

3. Proof of Theorem 1

Assume \( Q, D, F, E \) and \( \mathcal{F} = \{ g_n : n \in \mathbb{N} \} \) are as in Lemma 1.1 and its proof. By Lemma 1.2, an uncountable \( E \)-saturated \( X \subseteq F \) with an \( \omega_1 \)-like ordering \( < \) such that

1. Each \( E \)-equivalence class is an interval in \( < \).
2. There is a function \( H: X \times X \to X \) such that for every \( x \in X \):
   a. \( H(x, \cdot) \) is an autohomeomorphism of \( X \),
(b) \( H(x, y) \in \mathbb{Q} \) if and only if \( y < x \)

will be countable dense homogeneous. By Lemma 2.2, the existence of \( X \) and \( H \)
can be expressed in \( L^{\omega_1}_{\omega}(\mathbb{Q}) \), and by Theorem 2 it suffices to show that \( X \) exists
in some forcing extension. In order to assure that \( X \) is uncountable, we will force with a ccc poset. In [2] it was proved that if \( \{ C_\alpha : \alpha < \omega_1 \} \) and \( \{ D_\alpha : \alpha < \omega_1 \} \)
are two families of pairwise disjoint countable dense subsets of \( \mathbb{R} \) then a ccc forcing
adds a homeomorphism \( h : \bigcup_{\alpha < \omega_1} C_\alpha \rightarrow \bigcup_{\alpha < \omega_1} D_\alpha \) such that \( h[C_\alpha] = D_\alpha \) for
every \( \alpha < \omega_1 \). Therefore, if we pick any \( \omega_1 \) sequence of equivalence classes so that
the first one is \( \mathbb{Q} \cup D \) and well-order their union \( X \) in type \( \omega_1 \) then a standard ccc
forcing such that MA holds in the extension adds \( H \) with the required properties.

Since \( \mathbb{Q} \) is a relatively \( G_\delta \) subset of \( \mathbb{F} \), it is a countable dense and relatively \( G_\delta \)
subset of \( X \). By the countable dense homogeneity, \( X \) is a \( \lambda \)-set.

References


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