On Transmission of Two Correlated Gaussian Memoryless Sources over a Gaussian MAC using Delay-Free Mappings

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Abstract—This paper considers delay-free, low complexity, joint source-channel coding (JSCC) for communication of two correlated Gaussian memoryless sources over a Gaussian Multiple Access Channel (GMAC). Schemes are proposed both for cooperative and distributed encoders. The cooperative case is mainly studied to get an idea about how close one can expect to get to the performance upper bound with a zero delay constraint. The main contribution of the paper are distributed JSCC schemes. Two schemes are proposed: One fully discrete scheme based on nested scalar quantization and a hybrid discrete-analog scheme based on a scalar quantizer and a linear continuous mapping. Both transmit- and received power constraints are considered. The proposed schemes are robust against noise and show promising performance which improve with increasing correlation.

Index Terms—joint source-channel coding, memoryless source-channel mappings, bivariate Gaussian, Gaussian multiple access channel.

I. INTRODUCTION

WHEN an information source is transmitted over a point-to-point communication channel, the optimal performance theoretically attainable (OPTA) can be determined by equating the channel capacity and the rate-distortion function for describing the source [1]. In the simple case of memoryless Gaussian source and channel of equal bandwidth, it is well known that uncoded transmission reaches the OPTA [2]. By uncoded transmission, we mean that the transmitter only needs to scale the source symbols to satisfy the power constraint and map the resulting samples directly onto the channel symbols. It therefore, constitutes the simplest joint source-channel coding (JSCC) scheme, due to its low complexity and zero coding delay. Separate source channel coding (SSCC), on the other hand, as summarized by the source-channel separation theorem established in Shannon’s seminal work [3].

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also reach OPTA. Optimality in this case however, requires infinitely long codewords.

In a multipoint-to-point communication system the above conclusions may no longer be applicable, even in the case of memoryless Gaussian sources and channels. Although it has been established that SSCC are sub-optimal in several cases [4]–[6], how well JSCC schemes will ultimately perform still remains an open and challenging problem. In this paper, we examine one such system. More specifically, we look at transmissions of a set of memoryless, possibly inter-correlated Gaussian sources on a memoryless Gaussian multiple access channel (GMAC). To understand the fundamental nature of our investigation and to motivate the reader, we first summarize and present recent works and other related problems regarding the system under consideration. Depending on what is the desired reconstruction at the receiver end, the existing results can be categorized into the following two groups: 1) Recovery of the sources’ common information, 2) recovery of the sources’ individual information.

A. Recovery of the common information

A common way to realize the above described multipoint-to-point system is to consider the following: a set of sensor nodes are placed to monitor a common Gaussian source with independent, memoryless, measurement noise which is also Gaussian. This simple Gaussian sensor network is said to be symmetric when the average transmit power is equal across all sensor nodes. The distortion lower bound for this case was derived in [7] and it was shown to be reachable by distributed uncoded transmission.

This communication strategy is also optimal when considering a received power constraint [5]. Distributed SCC on the other hand, turns out to be strictly sub-optimal, even if the sources are first encoded to the rate-distortion bound [8], then transmitted with channel codes achieving the GMAC capacity. It was also emphasized in [9] that collaboration between the sensor nodes does not result in a lower distortion than that given by the distortion lower bound. A variation on this problem concerns the decoding of a function of all the sensor observations. If the sum of the all the memoryless Gaussian sources is to be reconstructed, distributed uncoded transmission is again shown to be the optimal communication strategy [10].

When the transmitter power of each node is unequal, we have an asymmetric Gaussian sensor network. For this
case, the authors of [11] pointed out that distributed uncoded transmission is only nearly optimal when the total transmitted power is small or when the observation noise level is high. Further, if a sum-transmit power is considered, and if optimal power allocation by water-filling [6] is possible, then uncoded transmission can again achieve a near optimal performance.

B. Recovery of the individual information

In other applications, it may be desirable to recover the unique information acquired by each sensor, in addition to the common information. The exact distortion lower bound considering distributed encoding is not known in general. Results on this problem for the Gaussian case involving two sensor nodes transmitting over a GMAC, however, is well studied and presented in [4].

There, the authors derived the distortion lower bound by allowing full collaboration between the nodes. This approach converts the original multipoint-to-point problem into a point-to-point problem. The distortion lower bound can therefore be obtained by simply equating the rate-distortion function with the channel capacity. In terms of transmission schemes, it was shown that distributed uncoded transmission can reach the distortion lower bound, only up to a certain channel signal-to-noise ratio (SNR) depending on the intercorrelation between the two sources. In order to get close to the distortion lower bound in general, the authors then proposed a a hybrid scheme that superimposes a vector quantizer (VQ) and a linear continuous mapping (i.e. uncoded transmission). The VQ is rate optimal, i.e. infinite dimensional, and preserves the correlation between the sources onto the channel. The vector quantized sequences of each source are transmitted directly on to the channel together with an un-quantized scaled version of the source. The hybrid scheme was shown to perform close to the distortion lower bound. The authors also examined distributed SCC where each node encodes its observation to the distortion lower bound, followed by capacity achieving channel codes. Just as in the case for recovering common information, SSCC is strictly inferior to the suggested JSCC scheme.

In summary, in a symmetric Gaussian sensor network when only the common information for the sources is of interest, we can take advantage of the structure of the GMAC by using uncoded transmission distributedly to reach optimal performance. When the unique information of each source is also desired, a distributed scheme as that proposed in [4] will require infinite dimensionality to operate close to the optimal performance bound. This observation prompts an interesting question: What happens if we impose a dimensionality constraint for the case where individual sensor observations should be recovered?

The main objective of this paper is to provide an answer to this question. In particular, we are interested in finding possible practical and implementable schemes for reconstructing memoryless Gaussian sources transmitted over a GMAC. The most important constraint we impose is zero delay. That is, we require communication of the source symbols over the channel to be carried out on a symbol-to-symbol basis, as is the case with uncoded transmission. In addition, we consider fully collaborative and distributed encoders. The basic rationale behind collaborative schemes is that if, as [4] showed, a distributed scheme can come very close to the distortion lower bound at the expense of dimensionality, then tradeoffs can be expected from allowing collaboration while imposing the zero delay constraint. It is also important to keep in mind that, when assessing performance of a JSCC scheme designed under the zero delay constraint, we must expect a significant backoff from the distortion lower bound derived in [4] that is based on infinite block length. The true OPTA for finite dimensionality is a challenging problem and remains open to this day.

From a practical sensor network application perspective, one reason for wanting low dimensionality may come from applications that require low delay and complexity. For example, in an event driven sensor network for monitoring poisonous gas, all sensor measurements must be recovered at the sink in a timely manner for constructing a complete overview of the environment and taking proper action. In medical sensing applications, specifically for in-body measurements, the sensors, which may also be event driven, must be miniature in size and operate at very low power placing restrictions on complexity. Since a natural byproduct of symbol-by-symbol operation is reduced complexity, coding and transmission schemes with zero delay in this case may be particularly desirable.

The rest of the paper is organized as follows. We first present the exact problem formulation and the relevant performance bounds in Section II. We then examine the zero delay mappings for the case of full collaboration in Section III. In Section IV, which is the main contribution of this paper, we study the distributed case. Two mappings are introduced: A fully discrete mapping based on Nested Quantization (NQ) [12], and a hybrid scheme using a scalar quantizer and a limiter followed by a continuous linear mapping named Scalar Quantizer Linear Coder (SQLC). Both schemes are optimized. In Section V, the proposed schemes are simulated and compared to the performance bounds and other relevant schemes under both transmitter- and received power constraints. We conclude the paper in Section VI along with possible future research directions.

II. PROBLEM DEFINITION AND RELEVANT BOUNDS

The complete communication system is illustrated in Fig. 1.

![Fig. 1. Two inter-correlated Gaussian memoryless sources transmitted on a GMAC where each source is reconstructed at the receiver side.](image-url)

1For the reason and rationale behind preserving the inter-correlation before transmission on to GMAC, see [4].
A. Problem statement

The sensor observations $x_1$ and $x_2$ are zero mean Gaussian random variables with covariance matrix

$$C_x = \sigma^2_x \begin{bmatrix} 1 & \rho_x \\ \rho_x & 1 \end{bmatrix},$$

where $\rho_x = \mathbb{E}[x_1 x_2] / \sigma^2_x$ is the correlation between $x_1$ and $x_2$, and $\sigma^2_x$ the variance, which is the same for both variables. The joint probability density function (pdf) is thus

$$p_x(x) = p_x(x_1, x_2) = \frac{1}{2\pi \sqrt{\det(C_x)}} e^{-\frac{1}{2} x^T C_x^{-1} x} \tag{2}$$

with equal marginal distributions $p_x(x_i) \sim \mathcal{N}(0, \sigma^2_x)$. Denote the memoryless mapping functions at the encoder by $f_i(x_i)$ and channel input symbols $y_i$, where $y_i = f_i(x_i), i = 1, 2$. We define the average transmit power as $P_i = \mathbb{E}[\|y_i\|^2], i = 1, 2$. The encoded observations are communicated over a memoryless GMAC. It has a sum-capacity which is given by [4]:

$$C_{\text{sum}} = \frac{1}{2} \log_2 \left( 1 + \frac{P_1 + P_2 + 2\rho_y \sqrt{P_1 P_2}}{\sigma^2_n} \right) \tag{3}$$

where $P_i$ are as defined above, $\rho_y$ is the correlation between the channel inputs and $\sigma^2_n$ is the variance of the variable $n$ from a zero mean AWGN process with pdf denoted $p_n(n)$. We assume ideal Nyquist sampling and an ideal Nyquist channel where the sampling rate of each source is the same as the signalling rate of the channel. We also assume ideal synchronization and timing between all nodes (practical issues like these should be dealt with at a later stage). The received signal $z = f_1(x_1) + f_2(x_2) + n$ is passed through the decoding functions $g_i(z), i = 1, 2$ to reconstruct each individual source. We use the mean-squared-error distortion criterion, and define the average end-to-end distortion as:

$$D = \frac{1}{2}(D_1 + D_2) = \frac{1}{2} \mathbb{E}\{\|x_1 - \hat{x}_1\|^2 + \|x_2 - \hat{x}_2\|^2\} \tag{4}$$

We are thus interested in the minimum sum distortion point in the region stated in [4]. The objective is to determine the memoryless mapping functions $f_i$ and $g_i$, for given power constraint, such that $D$ is minimized.

B. Performance upper bound

For the above defined communication system, we consider both transmitter- and receiver-induced power constraints.

For the transmit power constraint, we restrict ourselves to the symmetric case when $P_1 = P_2 = P$. The corresponding performance upper bound, expressed in terms of signal-to-distortion ratio (SDR) is [4]:

$$\text{SDR} = \frac{\sigma^2_x}{D} = \left\{ \begin{array}{ll}
\frac{(1 + \rho_x)(1 - \rho_x^2) + \sigma^2_n}{2P(1 + \rho_x) + \sigma^2_n} & , \quad \frac{P}{\sigma^2_x} \leq \frac{\rho_x}{1 - \rho_x}, \\
\frac{\sigma^2_n(1 - \rho_x^2)}{2P(1 + \rho_x) + \sigma^2_n} & , \quad \frac{P}{\sigma^2_x} > \frac{\rho_x}{1 - \rho_x}. \end{array} \right. \tag{5}$$

Note that for the rest of the paper, the term “channel SNR” refers to $P/\sigma^2_n$, i.e. the ratio between the average power per source symbol over the channel noise variance. Observe also that distributed uncoded transmission is optimal up to $\rho_x/(1 - \rho_x^2)$ [4], i.e. the first case in (5). Here only the common information is recovered for both sources.

We are also interested in a received power constraint $P_{Rx}$, which is defined as $\mathbb{E}[\|z\|^2] \leq P_{Rx} + \sigma^2_n$. Such a constraint can be meaningful when there is a strict requirement at the receiver, for example due to regulations in terms of interference to other co-existing wireless applications. As we can see from the sum capacity of the GMAC in [3], the received power relates to the average transmit power $P_i$ through $P_{Rx} = P_1 + P_2 + 2\rho_x \sqrt{P_1 P_2}$ (a detailed proof can be found in [4]). Since $P_{Rx} = 2P(1 + \rho_x)$ when $P_1 = P_2$, the resulting performance upper bound when considering a received power constraint can be found by replacing $2P(1 + \rho_x)$ by $P_{Rx}$ in (5). We emphasize once more that these bounds are obtained by allowing full collaboration between the encoders and infinitely long codewords.

III. DELAY-FREE SCHEME WITH COOPERATIVE ENCODERS

The optimal cooperative zero delay JSCC strategy, when found, will give us bounds on the performance of any zero delay distributed scheme. The scheme we propose here has previously been optimized for Gaussian source- and channel statistics. The scheme are known to be well performing but has not been proven to be optimal for the system studied here. It will anyway give us an idea of how much one can benefit by cooperation when there is a restriction on dimensionality. Cooperative encoders can also be meaningful in a practical setting. One example is two closely located in- or on-body sensors for physiological measurements. These sensors can be wired at little cost due to proximity, while their measurements are to be communicated to the receiver via the wireless channel for better flexibility and easier access.

One scheme for zero delay JSCC in this case is to apply Shannon-Kotel’nikov mappings (S-K mappings) [13]–[15]. The inspiration for this comes from the observation that with full collaboration, the problem is similar to bandwidth compression with a factor of 2 as long as $\rho_x < 1$.

S-K mappings are direct mappings between source- and channel space realized by parametric functions. For bandwidth compression, the parametric function resides in the source space as a subset, with the channel signal acting as the parameter(s). For the system in Fig. 1, the S-K mapping is a parametric curve in two dimensions, and the parameter corresponds to the single dimension of the channel.

For the boundary case of $\rho_x = 0$, a curve shown to be well performing for a bi-variate Gaussian is the Archimedes spiral [13] as depicted in Fig. 3 $\Delta$, which is the distance between the spiral arms, can be used to adapt the spiral to different channel SNR. The encoder projects the relevant source vectors onto the closest point on the curve in Euclidean distance $(\bar{x}_1, \bar{x}_2) = (g_1(\bar{z}), g_2(\bar{z}))$, where

$$\bar{z}_{opt} = \arg \min_{\bar{z}} (x_1 - g_1(\bar{z}))^2 + (x_2 - g_2(\bar{z}))^2, \quad \bar{z}$$

and $\bar{z}$ it the curve’s parameter before addition of noise. The approximated point $(\bar{x}_1, \bar{x}_2)$ on the spiral is transmitted over the channel through a conveniently chosen memoryless function $f_i$. At the receiver, the noise corrupted parameter value $\hat{z}$ is mapped through the inverse S-K mapping $g(z) = [g_1(z), g_2(z)]$ to reconstruct the source vector.
A. Nested Quantization

NQ consists of two quantizers, one for each encoder. For sequential decoding to be possible, one of the quantizers must be placed/nested in between the centroids of the other in such a way that the sum preformed by the GMAC does not act as a many-to-one mapping, i.e. that we can identify each centroid at the decoder and thereby split the interfering sources apart.

Without loss of generality, we choose encoder 2 to be nested in between encoder 1. We denote the quantization and transmission process by:

\[ z = f_1(x_1) + f_2(x_2) + n = q_\Delta(x_1) + \alpha(\ell_{\pm c}[q_\Delta(x_2)]), \]  

where \( q_\Delta \) represents a uniform mid-rise quantizer with step-size \( \Delta \), \( \ell_{\pm c} \) denotes the limitation to a certain (integer) value \( c \) and \( \alpha \) is an attenuation factor.

For a given \( \Delta \), \( c \) and \( \alpha \) must be chosen small enough so that encoder 2 can be placed/nested in between encoder 1 (see Fig. 3 below). This can be better understood by looking at the encoding process in more detail. We first quantize both sources

\[ q_\Delta(x_m) = \left\lfloor \frac{x_m}{\Delta} \right\rfloor = m(x_m), m = 1, 2 \] 

that is

\[ \tilde{i}_2 : \mathbb{R} \rightarrow \left\{ -\frac{c-1}{2}, -\frac{c-3}{2} , \ldots, -\frac{1}{2} \right\} \] 

We then attenuate \( \tilde{i}_2 \) by a factor \( \alpha \) in order to place the second encoder in between the first.

The basic concept is illustrated in Fig. 3. Note that the star representing the placement of the centroids of encoder 1 does not show up in the real channel signal. They are included for better understanding of the encoding operations. When

\[ \rho_x = 0; \quad C=7 \]

\[ \rho_x = 0.9; \quad C=15 \]

the correlation is low (\( \rho_x \) close to 0), all centroids of encoder 2 lie between two centroids of encoder 1. When the correlation is high (\( \rho_x \geq 0.7 \)), the centroids of encoder two will be nested in between the centroids of encoder 1. Note that not all centroids are re-used for every interval as is the case.

Unfortunately the S-K mappings discussed in the previous section do not apply for the distributed case in general. This is because the projection of the closest point on the curve (corresponding to the one-dimensional channel space) becomes impossible without knowing both source values.

For the zero delay JSCC scheme to work distributively, we have to ensure that the summed sources can be split apart at the receiver. A classic approach for solving this problem is to decode the sources sequentially at the receiver. One possible approach is the Nested Quantizers (NQ) introduced in [12].

IV. DISTRIBUTED ZERO DELAY SCHEMES

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when \( \rho_x \approx 0 \). The correlation between the sources makes it possible to increase the resolution of both quantizers. Hence overall fidelity improves when \( \rho_x \) increases. A more involved explanation is given in Section IV-C.

At the decoder, we first make a maximum likelihood (ML) estimate of the indices of the outer quantizer, this is followed by estimation of the inner quantizer index using the recovered first index. An MMSE estimate is then made using both estimated indices. Mathematically, the ML estimate of the first index is

\[
j_1(z) = \arg \min_{j \in \mathbb{Z}} \| f_1(j \Delta) (1 + \alpha \rho_z) - z \|^2, \quad (11)
\]

where the factor \((1 + \alpha \rho_z)\) takes into account that the mass midpoint for each channel segment (what lies between two decision borders for encoder one) shown in Fig. 3 changes with \( \rho_z \). That is, given that \( i_1(x_1) \) is transmitted, the mass midpoint for the relevant channel segment is

\[
i_1(x_1) + E\{\alpha x_2|i_1(x_1)\} = i_1(x_1) + \alpha \rho_x i_1(x_1), \quad (12)
\]

where we used the relation \( E\{x_2|x_1\} = \rho_x x_1 \) for correlated Gaussian random variables [19, p. 233]. Given the first index, one can recover the second by:

\[
\hat{j}_2(z) = \arg \min_{j \in \mathbb{Z}} \| z - f_1(j_1(z) \Delta) - f_2(j \Delta) \|^2 \quad (13)
\]

\( |j| \leq (c - 2)/2 \) since the inner quantizer is limited at \( \pm c \). The MMSE estimate of the two sources is then

\[
\hat{x}_1(z) = E\{x_1|j_1(z), j_2(z)\}, \quad \hat{x}_2(z) = E\{x_2|j_1(z), j_2(z)\} \quad (14)
\]

The average transmit power from each encoder is

\[
\begin{align*}
P_1 &= E[f_1(x_1)^2] = \sum_{i_1=\infty}^{\infty} Pr(i_1) i_1^2 \\
P_2 &= E[f_2(x_2)^2] = \alpha^2 \sum_{i_2=-(c-2)/2}^{(c-2)/2} Pr(i_2) i_2^2
\end{align*} \quad (15)
\]

To design the optimal NQ, we need to determine \( \Delta, \alpha, \) and \( c \), which minimize \( D \) with respect to the given power constraints. The two encoders are asymmetric, which result in \( P_1 \) being strictly greater than \( P_2 \), as on can see from (15). Optimization for equal transmit power is for this reason not advantageous. We can, however, bypass this problem by first optimizing the NQ scheme with a received power constraint \( P_{\text{Rx}} \) (the optimal NQ will then result in \( P_1 \neq P_2 \)) then use time sharing between the optimized codecs such that the transmit power of each source averaged over long source symbols are approximately equal. Recall that \( P_{\text{Rx}} \) is related to \( P_1, P_2 \) by \( P_{\text{Rx}} = P_1 + P_2 + 2 \rho_x \sqrt{P_1 P_2} \). We get the following optimization problem:

\[
\min_{\Delta, \alpha, c: (P_1 + P_2 + 2 \rho_x \sqrt{P_1 P_2}) \leq P_{\text{Rx}}} D, \quad (16)
\]

where \( P_{\text{Rx}} \) is as defined in Section II.

The average end-to-end distortion \( D \) can as shown in the Appendix, be split into three terms. I.e.

\[
D = \frac{1}{2} \sum_{m=1}^{2} D_m = \frac{1}{2} \sum_{m=1}^{2} E\{(x_m - \tilde{x}_m)^2\}
\]

\[
= \frac{1}{2} \sum_{m=1}^{2} (D_{q,m} + D_{c,m} + D_{n,m}),
\]

where \( D_{q,m} \) is the quantization distortion given by

\[
D_{q,m} = \int \int p_x(x_1, x_2) (x_m - \tilde{x}_m(i_1(x_1), i_2(x_2)))^2 dx_1 dx_2,
\]

\( (18) \)

\( D_{c,m} \) is the clipping distortion given by

\[
D_{c,m} = \int \int p_x(x_1, x_2) (\tilde{x}_m(i_1(x_1), i_2(x_2)) - \tilde{x}_m(i_1, i_2))^2 dx_1 dx_2
\]

\( (19) \)

\( D_{n,m} \) is distortion from channel noise given by

\[
D_{n,m} = \int \int p_x(x_1, x_2)p(z|i_1, i_2) \frac{Pr(i_m)}{Pr(i_1, i_2)}(\tilde{x}_m(i_1(x_1), i_2(x_2)) - \tilde{x}_m(i_1, i_2))^2 dxdy \]

\( (20) \)

In (18)-(20) \( \tilde{x}_m \) is as previously defined, \( \tilde{x}_m = E[x_m|i_1, i_2] \) denotes the quantized source(s), \( \tilde{x}_m = E[\tilde{x}_m|i_1, i_2] \) the quantized and limited source(s) and \( Pr(\cdot) \) the probability for the event inside the parentheses. \( \tilde{e}_2(i_2) \) denotes limitation of the inner quantizer. The optimization \( (16) \) must be done numerically due to the nature of \( (18)-(20) \).

We can postulate that better performance may be possible, if different step-size is allowed for each quantizer, i.e. \( \Delta_1 \neq \Delta_2 \). One case we are particularly interested in is when \( \Delta_2 \to 0 \). That is, we retain the encoder 1 and encode source 2 with a continuous mapping. Benefit of such a hybrid scheme can be understood intuitively as follows: suppose that the quantizer that encodes source 1 has a large enough distance between its centroids compared to the channel noise, such that jumps across the quantization intervals (shown in Fig. 3) do not occur. Then, for the second source, we are essentially transmitting a memoryless Gaussian source over an AWGN channel with attenuation \( \alpha \) and limitation to a certain real value \( \pm \kappa \) (determined by the relation between \( \Delta \) and \( c \)). Since the source has a sampling rate equal to the channels signalling rate, if we disregard the limitation, then as mentioned in the introduction, uncoded transmission is optimal. We present in detail the resulting scheme, Scalar Quantizer Linear Coder (SQLC) in the following section.

B. Hybrid discrete-analog scheme: SQLC

Encoder 1 is the same as for NQ, i.e. a uniform mid-rise quantizer \( f_1(x_1) = q_\Delta(x_1) \), as given in (8). Let centroid \( i \) be
denoted $q_i$. For encoder 2, we now have $f_2 = \alpha(\ell_{±\kappa}[x_2])$, which is a limiter that clips the amplitude to $±\kappa \in \mathbb{R}_+$, followed by a scaling factor $\alpha$. Notice that this scheme is similar to the Hybrid scheme proposed in [4]. A crucial difference is that we use practically realizable scalar quantizers instead of a rate optimal VQ. The SQLC also bares resemblance to the Hybrid Digital-Analog (HDA) scheme proposed in [20], except that here we have two encoders where one is fully discrete and the other is fully analog. Like for the NQ, we must ensure that the sum of the quantized and the linear coded value is uniquely decodable. In order to do so, the second encoded source must be placed between the centroids of encoder 1 without overlapping with its “neighbors”. The concept is depicted in Fig. 4 for the $\rho_z = 0$. Note that Fig. 5 is a quantized version of the channel space structure shown in Fig. 4.

The received signal is:

$$z = f_1(x_1) + f_2(x_2) + n = q_\Delta(x_1) + \alpha(\ell_{±\kappa}[x_2]) + n, \quad (21)$$

The decoders operate in a similar way as for the NQ, and decodes the sources sequentially. We have

$$\hat{x}_1 = g_1(z) = q_j(z), \quad \text{where}$$

$$j(z) = \arg \min_{j \in \mathbb{Z}} \|q_j(\Delta)(1 + \alpha \rho_x) - z\|^2, \quad (22)$$

followed by

$$\hat{x}_2 = g_2(z) = \beta(z - g_1(z)), \quad (23)$$

where $\beta$ is an amplification factor. The factor $(1 + \alpha \rho_x)$ is included for the same reason as for the NQ, i.e. the mass midpoints of the channel segments shown in Fig. 4 depends on $\rho_x$. Why this is so can be seen geometrically by studying Fig. 4 and Fig. 5(a) where the centroid of the first encoder is no longer lying at the origin of the second encoder. The distance between two neighboring centroids for encoder one at the GMAC output is therefore $d = \Delta(1 + \alpha \rho_x)$.

Similar to the NQ, we formulate the optimization problem with a received power constraint:

$$\min_{\alpha,\beta,\Delta,k;P_1, P_2; z_1, z_2} D, \quad (24)$$

where $D$ is the average end-to-end distortion and $P_{\text{Rx}}$ is as defined earlier.

To calculate $D$, we divide the average end-to-end distortion into the following five contributions:

1. quantization distortion and channel distortion for source 1 and clipping distortion, channel distortion and anomalous distortion for source 2. We begin with evaluating the channel output pdf, which is needed for the distortion calculations.

Channel output pdf: The pdf depends on $\rho_x$. As $\rho_x$ increases, the limitation to $±\kappa$ become more and more insignificant since the correlation itself effectively limits the signal. That is, when $\rho_x$ increases, the conditional pdf $p(x_2|x_1)$ narrows. Then given a certain $q_i$, the correlation itself “limits” $x_2$ when $\rho_x$ gets close to one. This can be seen by studying Fig. 6 and Fig. 7 together.

Assume first that $\rho_x$ is small enough for the limitation to $±\kappa$ to be significant. Further let $y_2 = f_2(x_2)$, and $u(\cdot)$ be the Heaviside function. It is straightforward to show (using e.g. [19]) that the pdf of $y_2$ is

$$p_{y_2}(y_2) = \frac{1}{\sqrt{2\pi \alpha \sigma_x}} e^{-\frac{y_2^2}{2\alpha \sigma_x^2}} u(y_2 - \alpha \kappa)u(-y_2 + \alpha \kappa)$$

$$+ p_0 \delta(y_2 - \alpha \kappa) + \delta(y_2 + \alpha \kappa),$$

where

$$p_0 = \int_{-\infty}^{\alpha \kappa} p_e(x_2) \, dx_2. \quad (25)$$

When $y_1$ and $y_2$ are summed over GMAC, the resulting pdf is centered at the transmitted centroid from encoder 1 (see Fig. 4). The distribution after addition of noise is given by $p_{y_2} \ast p_n$ [19, 181-182]. Let $z_2$ denote the received signal when source 1 is subtracted. Assuming that the noise will not confuse the centroids of encoder 1, then

$$p_{z_2}(z_2) = p_{y_2} \ast p_n = \frac{1}{2\pi \alpha \sigma_x \sigma_n} \int_{-\alpha \kappa}^{\alpha \kappa} e^{-\frac{y_2^2}{2\alpha \sigma_x^2} + \frac{z_2^2}{2\sigma_n^2}} \, dy$$

$$+ p_0 (p_n(z_2 - \alpha \kappa) + p_n(z_2 + \alpha \kappa)). \quad (26)$$

The pdf for each channel segment is as in (27) until $\rho_x$ becomes close enough to one ($\rho_x \approx 0.7$) such that the clipping to $±\kappa$ becomes negligible. Further, $y_2$ will no longer be repeated on each segment, but spread over several of the channel segments. This means that the channel segments shown in Fig. 4 are no longer exact copies of each other but contains different “parts” of $y_2$ (see Fig. 4 and Fig. 5). This is the same effect as we saw for the NQ in Fig. 5.

Now let $y_2 = \alpha x_2$. Given that $q_i$ was transmitted it is easy to show, using the expression for $p(x_2|x_1)$ [19, p.223], that

$$p(y_2|q_i) = \frac{1}{\sigma_x \alpha \sqrt{2\pi(1 - \rho_x^2)}} e^{-\frac{(y_2 - q_i(1 + \alpha \rho_x))^2}{2\alpha \sigma_x^2(1 - \rho_x^2)}}. \quad (28)$$

After addition of noise, the resulting pdf becomes

$$p_{z_2}(z_2) = p(y_2|q_i) \ast p_n(n) = \frac{1}{\sqrt{2\pi(n^2 + \sigma_n^2)}} e^{-\frac{(q_i(1 + \alpha \rho_x) - z_2)^2}{2n^2 + \sigma_n^2}}. \quad (29)$$

The pdf in (27) is valid when $\gamma \leq 2\kappa$ while (29) is valid when $\gamma > 2\kappa$. $\gamma$ is shown in Fig. 5(b) and is the error we

This is an approach inspired by Kotel'nikov [21, pp.62-98].
get on source 2 when two neighboring channel segments are interchanged. \( \gamma \) is calculated in Section 11B.2.

The total channel output pdf can be found as a weighted sum of (29) or shifted and weighted versions of (27) depending on which of them is valid. The weighting is done with the probabilities for the centroids from encoder 1.

2) Distortion and power calculation for source 2: The distortion for source 2 is divided into three contributions: clipping distortion, channel distortion and anomalous distortion.

Clipping distortion: Source 2 must be limited to a certain level \( \pm \kappa \) in order to achieve non-intersecting channel segments. The distortion from clipping is given by

\[
\bar{\varepsilon}_c^2 = 2 \int_{-\infty}^{\infty} (x_2 - \kappa)^2 p_x(x_2) dx_2.
\]

This term will be significant when \( p_x \) is small, but becomes negligible when \( p_x \) gets close to one.

Channel distortion: The effect of the channel noise on source 2 can be divided into weak (additive/thermic) noise, and anomalous distortion resulting from a threshold effect [24].

First, we consider the effect of weak noise, which we name channel distortion. The effect of channel noise will be determined from \( \alpha \) and \( \beta \). Let \( \tilde{x}_2 \) denote the clipped source.

\[
\bar{\varepsilon}_c^2 = E\{(\tilde{x}_2 - \tilde{\tilde{x}}_2)^2\} = E\{(\tilde{x}_2 - (\alpha \tilde{x}_2 + n)\beta)^2\} \approx \sigma_c^2 (1 - \alpha \beta)^2 + \beta^2 \sigma_n^2
\]

where the last approximation comes from assuming that \( E\{\tilde{x}_2\} = \sigma_c^2 \). For simplicity we use this approximation during optimization.

Anomalous distortion: When the wrong centroid for source 1 is detected, large decoding errors result for source 2. In the worst case, positive values get mapped to negative and vice versa. This can be seen from Fig. 5. The magnitude of this error depends on whether \( \gamma \leq 2\kappa \) or not.

Consider first that \( \gamma \leq 2\kappa \): The probability that anomalies happen is equal for each segment and given by

\[
p_{th1} = \Pr\left\{ y_2 + n \geq \frac{\Delta}{2} (1 + \alpha p_x) \right\}
= 2 \int_{\frac{\Delta}{2}(1 + \alpha p_x)}^{\infty} p_{z_2}(z_2)dz_2,
\]

where \( p_{z_2}(z_2) \) is given in (27). The integral in (32) must be solved numerically. The error that occurs is bounded by \( (2\kappa)^2 \), since \( \kappa \) is detected as \(-\kappa\) when the neighbors in the channel space first start to overlap.

Now consider the case when \( \gamma > 2\kappa \). It can be shown that the probability for anomalies is the same for each channel segment also for this case (given \( q_i \), then (29) is just shifted accordingly and so the probability stays the same), and we get

\[
p_{th2} = 2 \int_{\frac{\Delta}{2}(1 + \alpha p_x)}^{\infty} p_{z_2}(z_2)dz_2,
\]

where \( p_{z_2}(z_2) \) is given in (29). We need to determine the error \( \gamma \), when the anomalous errors first start to happen. Since we consider a uniform quantizer, \( \gamma \) is upper bounded by the jumps we get between the segments which are closest to the origin in the channel space. We can therefore, use the parallelogram shown in Fig. 5(b) to determine the maximal error \( \gamma \). Let \( \lambda = \sigma_c^2 (1 - p_x) \), i.e. the smallest eigenvalue of the covariance matrix \( \Pi \). We have

\[
l_1 = 2b \frac{\lambda}{\cos \phi} = 2b \sigma_x \sqrt{2(1 - p_x)},
\]

where \( b \) is a constant determining the width of the ellipse containing most of the probability mass, \( b \approx 4 \) is a good choice as can be seen from a scatter-plot of \( \mathbf{x} \). Since the relevant parallelogram (when \( \gamma \leq 2\kappa \)) consists of a square and two triangles with both legs equal to \( \Delta \), we get

\[
\gamma = 2b \sigma_x \sqrt{2(1 - p_x)} - \Delta.
\]

Since \( p_{th1} \) and \( p_{th2} \) are the same no matter which \( q_i \) is transmitted, we arrive at the resulting anomalous distortion

\[
\bar{\varepsilon}_a^2 \leq \begin{cases}
4p_{th1}\kappa^2, & \gamma \leq 2\kappa, \\
p_{th2}\gamma^2, & \gamma > 2\kappa.
\end{cases}
\]

Power: The transmit power of source 2 is

\[
P_2 = \int_{-\infty}^{\infty} y_2^2 p_{y_2}(y_2)dy_2 = \int_{-\alpha \kappa}^{\alpha \kappa} y_2^2 p_{y_2}(y_2)dy_2 + 2p_o \sigma_n^2 \alpha^2 \kappa^2.
\]

3) Distortion and power calculation for source 1: The distortion for source 1 is divided into two contributions: Quantization distortion and channel distortion.

Quantization distortion: We assuming a large enough number of quantization levels so that the quantization distortion will consist of granular noise only, then

\[
\bar{\varepsilon}_q^2 = 2 \sum_{i=1}^{\frac{N_x - 1}{\Delta}} \int_{(i-1)\Delta}^{i\Delta} \left( x_1 - (i - 1)\Delta - \frac{\Delta}{2} \right)^2 p_x(x_1)dx_1
\]

Channel distortion: Distortion from channel noise for source 1, \( \bar{\varepsilon}_{C_1} \), only occurs when the signal from source 2 and
the channel noise together are larger than $\Delta/2$. Since we are only interested in finding the optimal parameters for designing the SQLC, we simplify the analysis by considering jumps to the nearest neighboring centroids only. The probability for this event is then the same as that in the case of the anomalous distortion for source 2 calculated above. Further, the distortion we get when two neighboring centroids are exchanged is $\Delta^2$. And so

$$
\bar{\varepsilon}_{C1}^2 = \Delta^2 \begin{cases} 
\rho_{th1}, & \gamma \leq 2\kappa, \\
\rho_{th2}, & \gamma > 2\kappa.
\end{cases}
$$

(39)

Power: The average transmit power for source 1 is:

$$
P_1 = 2\sum_{i=1}^{N_q} Pr(q_i) \left( (i - 1)\Delta + \frac{\Delta}{2} \right)^2,
$$

(40)

where

$$
Pr(q_i) = \int_{(i-1)\Delta}^{i\Delta} p_x(x_1)dx_1.
$$

(41)

4) Optimization: From the above analysis, the average end-to-end distortion for the SQLC is given by

$$
D(\alpha, \beta, \Delta, \kappa) = \frac{1}{2}(D_1(\Delta, \alpha, \kappa) + D_2(\alpha, \beta, \Delta, \kappa))
$$

(42)

where

$$
D_1(\alpha, \Delta, \kappa) = \bar{\varepsilon}_{C1}^2(\Delta) + \bar{\varepsilon}_{C2}^2(\alpha, \Delta, \kappa),
$$

$$
D_2(\alpha, \beta, \Delta, \kappa) = \bar{\varepsilon}_{C2}^2(\beta) + \bar{\varepsilon}_{C1}^2(\alpha, \beta) + \bar{\varepsilon}_{C1}^2(\alpha, \Delta, \kappa).
$$

(43)

The optimization problem in (24) can now be solved by using (42), (40) and (37). Similar to NQ, numerical optimization is needed. This is due to the integral in (37).

The resulting optimized parameters as a function of channel SNR is given in Fig. 6 for $\rho_x = 0$ and $\rho_x = 0.95$ when $\sigma_x = 1$. Notice that when $\rho_x = 0.95$ we get a smaller $\Delta$ and a larger $\alpha$ for a given channel SNR compared to the $\rho_x = 0$ case. This implies improved fidelity (SDR) when $\rho_x$ increases. The reason why $\kappa$ increase quickly (seen in the upper left corner in Fig. 6(a)) comes from the fact that it becomes irrelevant when $\rho_x$ is close to one (since $p(x_2|x_1)$ narrows). Notice that $\beta$ is almost identical to $1/\alpha$ times the Wiener factor [23 pp. 344-378] $P_2/(P_2 + \sigma_n^2)$, for both cases. It is therefore rather simple to find decoder 2 when the encoder is known. All the curves in Fig. 6 can be described by mathematical expressions using nonlinear curve fitting to polynomials and/or exponential functions. With channel state information available one can adapt the encoders to varying channel conditions by using these functions.

C. Comparison of NQ and SQLC

Through the analysis and optimization, we can see that as perceived in Section IV-A, NQ can be roughly interpreted as a quantizes version of the SQLC scheme, at least at high channel SNR. They both follow the same underlying principles, i.e. we can use similar geometrical arguments to explain the NQ as we did to explain the SQLC.

When $\rho_x \approx 0$, the number of quantization levels determined by $c$ must be finite. At low channel SNR $c$ must be small and as the channel SNR grows $c$ can be made larger and larger. The reason for this is that all centroids of quantizer 2 is placed between two centroids of encoder 1 on the channel for low values of $\rho_x$. See the $\rho_x = 0$ case shown in Fig. 3 and further picture a quantized version of Fig. 4. When $\rho_x$ gets larger ($\rho_x \approx 0.8$), $c$ can be made much larger for a given channel SNR since the narrowing of $p(x_2|x_1)$ effectively “limits” the signal for a given $i_1$. This can be understood by picturing a quantized version of Fig. 5. When $\rho_x \rightarrow 1$ then $c \rightarrow \infty$. This is exactly the same effect as seen for the SQLC scheme shown...
in Fig. 3(a) only that now each segment is quantized. For NQ it means that just a few of the centroids for quantizer 2 is placed between two centroids of encoder 1. This is illustrated in Fig. 3 for $\rho = 0.9$. The resolution for both quantizers can therefore be increased when $\rho_x$ increase for a given channel SNR.

Although the SQLC has better performance than NQ in terms of SDR performance, as will be evident from Section V, NQ is advantageous if we consider an extension of this scheme to more than two sources, or if a fully discrete system must be constructed. If one wish to construct a similar scheme for $M$ sources transmitted on a GMAC, then the $M - 1$ encoders should be nested quantizers, whereas the innermost source could be continuous. The advantage of having one continuous encoder will, however, get less and less as $M$ increase.

V. SIMULATIONS

We compare the optimized distributed schemes NQ, SQLC and the S-K mappings to the performance upper bound discussed in Section II-B. Both the SSCC bound and the optimal distributed linear scheme (both derived in [4]) are also presented as reference.

A. Received power constraint

Simulation results are shown in Fig. 7 for $\rho_x = 0$ and $\rho_x = 0.95$. When $\rho_x = 0$ the SQLC is around 2.5-3 dB away from OPTA and the NQ inferior to the SQLC by approximately 0.5-1.5 dB. Both schemes are significantly better than the linear scheme but inferior to the S-K mapping which is around 1dB from OPTA for all channel SNRs. Another important characteristic to notice is that the NQ and SQLC are robust to noise, as can be seen from the robustness plot for the $\rho_x = 0$. The proposed schemes are also robust for other values of $\rho_x$.

When $\rho_x$ gets larger, the performance of all schemes improve in terms of SDR. The gap to OPTA in terms of SDR, however, becomes larger for the S-K mappings (2.5-3dB), SQLC ($\approx$5dB) and NQ ($\approx$7dB). The contrary is true for the linear scheme. Considering that the mappings are delay free, the performance is still quite good. It is also interesting to note that both the SQLC and NQ improves with increasing $\rho_x$ without any modification of the encoders. The gain from...
increasing correlation as a function of $\rho_x$ is shown in Fig. 8(a) for the SQLC scheme at 30 dB channel SNR.

Note also that there is no noticeable gain before $\rho_x \approx 0.7$, whereas the gain gets significant when $\rho_x \to 1$. The gap to the performance upper bound as a function of $\rho_x$ is plotted for some of the schemes in Fig. 8(b) for 30 dB channel SNR. Here SQLC, NQ and the linear scheme all reach OPTA in the limit $\rho_x \to 1$, while the optimal SSCC scheme does not. The reason why the SQLC and NQ reach OPTA is that when $\rho_x \to 1$, $\Delta \to 0$, and $\kappa, c \to \infty$. The result is then a linear scheme which is optimal when $\rho_x = 1$ [7].

B. Equal transmitter power constraints

Under an equal transmit power constraint, we expect a loss in performance for SQLC and NQ, since they have asymmetric encoders. We only examine the SQLC, as a similar effect is observed for the NQ. To simulate equal transmit power, we adopt a time sharing transmission, where each source is coded with one encoder half of the time. Averaged over large number of source symbols, we have $P_1 = P_2 = P$.

Fig. 9 shows the loss in performance when having an equal transmit power constraint compared to a received power constraint as a function of $\rho_x$ at 30 dB channel SNR. Notice that the loss from imposing equal transmit power constraint becomes less as $\rho_x$ increases. The reason for this is that the two encoders become more “similar” as $\rho_x$ increases. In the limit $\rho_x \to 1$ the two encoders become the same and no loss is observed relative to the total received power case.

For other channel SNR values the trend is that the loss decrease slightly with decreasing values of SNR whereas it increases slightly with increasing SNR.

VI. CONCLUDING REMARKS AND FUTURE RESEARCH

In this paper, delay free joint source-to-channel mappings for two memoryless Gaussian sources communicated over a Gaussian multiple access channel (GMAC) are proposed, both for cooperative and distributed encoders. All schemes show promising performance.

The cooperative scheme is based on Shannon-Kotel’nikov (S-K) mappings, since the problem studied is similar to that for bandwidth compression. The S-K mappings utilized in this paper are optimized for the relevant source and channel distributions, and they therefore give an indication on how close to the upper performance bound one can expect to get when a zero delay constraint is imposed. The S-K mappings are shown to be only 1 to 3 dB inferior to the performance upper bound. It is evident that full cooperation clearly benefits a zero-delay scheme.

For distributed transmission, we proposed a fully discrete mapping based on nested quantization (NQ) and a hybrid discrete-analog scheme (SQLC). Both schemes are noise robust. The SQLC has a performance 2.5 to 5 dB inferior to the performance upper bound, considering a received power constraint and the NQ is inferior to the SQLC with about 0.5 to 2dB. The NQ and SQLC are strictly inferior to the S-K mappings due to less degrees of freedom in constructing encoders. The S-K mapping, is among other things, performing better space-filling of the source space than what is possible with a distributed system, leading to a smaller total distortion. Since the NQ and SQLC have nonsymmetric encoders, their performance deteriorates somewhat when we impose an equal transmit power constraint at each node. The deterioration is $\approx 0$ to 1 dB depending on $\rho_x$ as well as the channel SNR.

Interestingly, both NQ and the SQLC improve with increasing $\rho_x$ without changing the structure of the encoders and decoder, and when $\rho_x \to 1$ both schemes reach the upper performance bound.

Further research should consider multiple sources ($> 2$) as well as different source statistics and different attenuation for each sub-channel of the GMAC. A natural approach would be a combination of the NQ and SQLC scheme. It would also be beneficial to determine if there exist smarter ways of doing distributed coding for two sources than the SQLC in the zero delay case. That is, can we close the gap to cooperative encoding?

Identifying the optimal performance of a communication system with finite dimensionality constraint is clearly crucial in assessing the performance of the proposed JSCC schemes. It is however, a very difficult problem since many of the standard tools in information theory rely on infinitely long codewords. A generalization of work such as [24] could be an important step towards finding the performance bounds under such constraints.

APPENDIX

Let $\tilde{x}_m$ denote the quantized version of $x_m$ and $\bar{x}_m$ the quantized and limited version of $x_m$. The distortion terms $D_{\tilde{m}}$, ...
\begin{align*}
\text{For the first cross term we get} & \\
\sum_{i_1, i_2} \int \int \int_{x_1, x_2 \in i_1, i_2} p_{x}(x_1, x_2) Pr(j_1, j_2 | i_1, i_2) \\
& \times (x_m - \tilde{x}_m(i_1, i_2)) (\tilde{x}_m(i_1, \tilde{i}_2(i_2)) - \tilde{x}_m(j_1, j_2)) dx_1 dx_2 \\
& = \sum_{i_1, i_2, j_1, j_2} Pr(j_1, j_2 | i_1, i_2) (\tilde{x}_m(i_1, \tilde{i}_2(i_2)) - \tilde{x}_m(i_1, i_2)) dx_1 dx_2 = 0. \\
& \text{(45)}
\end{align*}

Finally, studying the third cross term we get
\begin{align*}
E((\tilde{x}_m - \tilde{x}_m) (\tilde{x}_m - \tilde{x}_m)) = \\
\sum_{i_1, i_2, j_1, j_2} \int \int_{x_1, x_2 \in i_1, i_2} p_{x}(x_1, x_2) Pr(j_1, j_2 | i_1, i_2) \\
& \times (\tilde{x}_m(i_1, \tilde{i}_2(i_2)) - \tilde{x}_m(j_1, j_2)) dx_1 dx_2 \\
& = \sum_{i_1, i_2, j_1, j_2} Pr(i_1, i_2) Pr(j_1, j_2 | i_1, i_2) \\
& \times (\tilde{x}_m(i_1, i_2) - \tilde{x}_m(i_1, \tilde{i}_2(i_2))) (\tilde{x}_m(i_1, \tilde{i}_2(i_2)) - \tilde{x}_m(j_1, j_2)) \\
& = \sum_{i_1, i_2} Pr(i_1, i_2) (\tilde{x}_m(i_1, i_2) - \tilde{x}_m(i_1, \tilde{i}_2(i_2))) \\
& \sum_{j_1, j_2} Pr(j_1, j_2 | i_1, i_2) (\tilde{x}_m(i_1, \tilde{i}_2(i_2)) - \tilde{x}_m(j_1, j_2)) = 0. \\
& \text{(46)}
\end{align*}

due to the choice of \( \tilde{x}_m \). Hence, we conclude that
\begin{align*}
D_m = D_{q,m} + D_{c,m} + D_{n,m}. \\
& \text{(48)}
\end{align*}