Dynamic Harmonic Balance Principle and Analysis of Rocking Block Motions

Igor M. Boiko

Abstract—The harmonic balance (HB) principle is a powerful and convenient tool for finding periodic solutions in nonlinear systems. In the present paper, this principle is extended to transient processes in systems with one single-valued odd-symmetric nonlinearity and linear plant not having zeros in the transfer function, and named the dynamic HB. Based on the dynamic HB, first the equations for the amplitude, frequency, and amplitude decay of an oscillatory process in the Lure system are derived. It is then applied to analysis of rocking block decaying motions. An example is provided.

I. INTRODUCTION

Harmonic balance principle is a convenient tool for finding parameters of self-excited periodic motions. Due to this convenience, it is widely used in many areas of science and engineering. For a system with one nonlinearity and linear dynamics (Lure system), it can be illustrated by drawing the Nyquist plot of the linear dynamics and the plot of the negative reciprocal of the describing function (DF) [1] of the nonlinearity in the complex plane and finding the point of intersection of the two plots, which would correspond to the self-excited periodic motion in the system. Therefore, the harmonic balance principle treats the system as a loop connection of the linear dynamics and of the nonlinearity. It is also possible to reformulate the harmonic balance, so that the format of the system analyzed is not a loop connection but the denominator of the closed-loop system. This would imply a different interpretation of the harmonic balance, which would allow one to extend the harmonic balance principle to analysis of not only self-excited periodic motions but also other types of oscillatory motions.

I. Boiko is with The Petroleum Institute, P.O. Box 2533, Abu Dhabi, U.A.E. (email: i.boiko@ieee.org).
One of the types of the systems that exhibit vanishing oscillatory motion of variable frequency is the conventional and second-order sliding mode (SM) control system. There are a number of second-order SM (SOSM) algorithms available now, the most popular of which are “twisting”, “super-twisting”, “twisting-as-a-filter” [2], [3], “sub-optimal” [4], [5]. These algorithms provide finite-time convergence and because of this feature are used in both control and observation [6]-[8]. An overview of finite-time convergent second-order SM algorithms and algorithms with asymptotic convergence is presented in [9]. Another example of a transient oscillatory motion is the transient processes in oscillatory mechanical systems. A number of examples are given in [10], [11]. The same mechanical system, as of the present paper, but in a different setup was considered in [12]. A number of examples of different areas engineering and physics can be found where transient oscillatory motions of variable frequency occur. Therefore, some common approach to the problem of the convergence rate assessment, including qualitative (finite-time or asymptotic) and quantitative assessment, would be of high importance.

The availability of the model of the system revealing transient oscillations might seem to be sufficient for analysis and not requiring development of methods other than simulations. However, the frequency-domain approach to assessment of convergence rate would provide a number of advantages over the direct solution of the system differential equations. The most important one is the possibility of explanation of the mechanism of the frequency and amplitude change during the transient. This analysis can also lead to the formulation of the criteria of finite-time and asymptotic convergence in terms of frequency-domain characteristics.

In the present paper, a frequency-domain approach to analysis of transient oscillations is presented, which is suitable for analysis of high-order nonlinear systems. The harmonic balance principle is extended to the case of transient oscillations and named as the dynamic harmonic balance principle. The paper is organized as follows. At first the harmonic balance principle is considered. Then a Lure system with a high-order plant is analyzed with the use of the approach proposed with the use of the dynamic harmonic balance involving quasi-static approach to the frequency of the oscillations. Such
characteristics as frequency and amplitude of oscillations as functions of time are derived. After that the condition of the full dynamic harmonic balance is derived. Finally, the proposed approach is applied to analysis of the decaying motions of increasing frequency of the rocking block. Example of analysis and simulations supporting the proposed theory are provided.

II. HARMONIC BALANCE FOR TRANSIENT OSCILLATIONS

Consider the system that includes linear dynamics given by the following equations:

\[ \dot{x} = Ax + Bu \]
\[ y = Cx, \]  

(1)

where \( x \in R^n \), \( y \in R^1 \), \( u \in R^1 \), \( A \in R^{n \times n} \), \( B \in R^{n \times 1} \), \( C \in R^{1 \times n} \), and a single-valued odd-symmetric nonlinearity \( f(y) \) with zero mean:

\[ u = -f(y), \]  

(2)

We shall refer to (1) as to the linear part of the system. One can see that the system (1), (2) is a Lure system. The transfer function of the linear part is \( W_l(s) = C(I - A)^{-1}B \), which can also be presented as a ratio of two polynomials \( W_l(s) = P(s)/Q(s) \). Assume also an autonomous mode, so that the input to the nonlinearity is the output of the linear dynamics, and the output of the nonlinearity is the input to the linear dynamics. Self-excited periodic motions in the system can be found through the use of the HB principle. The conventional HB condition (for periodic motion) is formulated as

\[ W_l(j\Omega)N(a) = -1, \]  

(3)

where \( \Omega \) is the frequency and \( a \) is the amplitude of the self-excited periodic motion at the input to the nonlinearity, \( N(a) \) is the describing function of the nonlinearity, which does not depend on the frequency because the nonlinearity is a single-valued function. If the linear part has relative degree higher than two, then the Nyquist plot of system (1) has a point of intersection with the real axis at some finite frequency and, therefore, equation (3) has a solution. We now find the closed-loop transfer function \( W_{cl}(s) \) of system (1), (2) using the replacement of the nonlinearity with the DF \( u = -N(a) \cdot y \):
\[ W_{cl}(s) = \frac{W_i(s)N(a)}{1 + W_i(s)N(a)} = \frac{P(s)N(a)}{Q(s) + P(s)N(a)} \] (4)

Let us note that (3) is equivalent to

\[ R(a, j\Omega) = Q(j\Omega) + P(j\Omega)N(a) = 0, \] (5)

which means that the denominator of the closed-loop transfer function turns into zero when the frequency and the amplitude become equal to the frequency and the amplitude of the periodic motion. Equation (5) is also sometimes used for finding a periodic solution via algebraic methods. However, equation (5) usually is not attributed to the denominator of the closed-loop transfer function but considered a direct result of (3). Assuming that \( R(a, s) \) can be represented in the following form

\[ R(a, s) = (s - s_1)(s - s_2)\cdots(s - s_n), \] where \( s_i \) are roots of the characteristic polynomial, we must conclude that there must be at least one pair of complex conjugate root with zero real parts. It would imply the existence of the conservative component in \( W_{cl}(s) \). Indeed, we can consider the existence of non-vanishing oscillations as a result of the existence of the component \((s^2 + \rho^2)\) in the denominator of \( W_{cl}(s) \), where \( \rho \) is a parameter that depends on the amplitude \( a \). However, one can notice that even if a damped oscillation occurs, so that there exists a pair of complex conjugate roots \( s_i, s_{i+1} \) then \((s - s_i)(s - s_{i+1}) = 0\), and the characteristic polynomial becomes zero, with \( s = \sigma \pm j\Omega \), where \( \sigma \) is the decay (Note: strictly speaking, we have a decaying oscillation only if \( \sigma < 0 \); yet we will refer to this variable as to the decay even if \( \sigma \geq 0 \)).

We can view the harmonic balance condition (3) as the realization of the regeneration principle, according to which \( y(t) = L^{-1}[L[u(t)]W_i(s)] \), and \( u(t) = f(y(t)) \). We can now use the regeneration principle approach to a transient oscillation, in which we consider both \( u(t) \) and \( y(t) \) sinusoidal signals with exponentially decaying (or growing) amplitude with constant value of the decay, and formulate and prove the following property.

**Theorem 1.** With the input signal to the linear dynamics given by the transfer function \( W_i(s) \) being the
harmonic signal with decaying amplitude \( u(t) = e^{\sigma} \sin(\Omega t) \), the output of the linear dynamics is also a harmonic signal of the same frequency and decay \( y(t) = ae^{\sigma} \sin(\Omega t + \varphi) \).

**Proof.** It follows from the property of the Laplace transform that \( L[e^{-a} f(t)] = F(s + a) \). Therefore, for the system input \( u(t) \), the Laplace transform will be \( L[u(t)] = \Omega / [(s - \sigma)^2 + \Omega^2] \), which will result in the system output (in the Laplace domain) \( Y(s) = \Omega W_i(s) / [(s - \sigma)^2 + \Omega^2] \). The substitution \( s' = s - \sigma \) yields \( Y(s') = \Omega W_i(s' + \sigma) / [(s')^2 + \Omega^2] \), which means that \( y'(t) = L^{-1}[Y(s')] \) is a sinusoid of frequency \( \Omega \), amplitude \( |W_i(\sigma + j\Omega)| \), and having the phase shift \( \arg W_i(\sigma + j\Omega) \). In turn, the output signal is \( y(t) = e^{\sigma} y'(t) \), i.e. a decaying sinusoid.

Therefore, for our analysis of propagation of the decaying sinusoids through linear dynamics we can use the same transfer functions, in which the Laplace variable should be replaced with \( (\sigma + j\Omega) \).

The describing function \( N \) in the case of a transient oscillation may become a function of not only amplitude but of its derivatives too [13] (we disregard possible dependence of the DF on the frequency). Considering that conditions (3) and (5) are equivalent, and the equality of the denominator of the closed-loop transfer function to zero (for some \( s \)) implies the fulfillment of (3), we can rewrite (3) for the transient oscillation as follows.

\[
N(a, \dot{a},...)W_i(\sigma + j\Omega) = -1, \tag{6}
\]

The use of the derivatives of the amplitude as arguments of the DF is inconvenient because it results in the necessity of consideration of additional variables (derivatives of the amplitude) which are not present otherwise. It is more convenient to consider \( \sigma \) and its derivatives than the derivatives of the amplitude. Also, we limit our consideration of the describing function arguments to the first derivative of the amplitude (or equivalently, to \( \sigma \)) only, which will correspond to the use of the regeneration principle for decaying sinusoids. We show below that for many nonlinearities the DF is a function of the amplitude only – like in the conventional DF analysis. Therefore, we can write the condition of the
existence of a transient or steady oscillation as follows:

\[ N(a, \sigma)W_i(\sigma + j\Omega) = -1, \quad (7) \]

where the decay and frequency are considered constant on some small interval of time (theoretically on infinitely small interval of time, and practically on one step of integration when using a numeric method), while the amplitude is considered exponentially changing on this interval (due to the constant decay). Joining solutions for these intervals, we obtain the frequency and the decay as functions of time. Therefore, we assume that the decay and the frequency are slowly or quasi-statically changing.

Equation (7) is referred to in [14] as the dynamic harmonic balance condition (equation). However, it does not account for the derivative of the frequency and we shall further refer to (7) as dynamic harmonic balance quasi-static w.r.t. frequency.

Assume now that the characteristic polynomial of the closed-loop system (with parametric dependence on the amplitude of the oscillations) has a pair of complex conjugate roots with negative real parts. Then a vanishing oscillation of certain frequency and amplitude occurs. The idea of considering equations of vanishing oscillations is similar to the one of the Krylov-Bogoliubov method [15]. However, the latter can only deal with small “deviations” from the harmonic oscillator and is limited to second-order systems. In the present approach, the “equivalent damping” is not limited to small values. Let us consider instantaneous values of the frequency, amplitude and decay and formulate the dynamic HB quasi-static w.r.t. frequency and decay as follows.

At every time, a single-frequency mode transient oscillation can be described as a process of variable (instantaneous) frequency, amplitude and decay, which must satisfy equation (7).

Note: In (7) and the formulation given above, we consider only transient oscillations with zero mean and single-frequency mode when the characteristic polynomial (5) has only one pair of complex conjugate roots.

The describing function of an arbitrary nonlinearity for non-harmonic input signal should be computed as suggested in [16] (see also [17]).
\[ N(a, \sigma, \omega) = q(a, \sigma, \omega) + q'(a, \sigma, \omega) \frac{s - \sigma}{\omega}, \]  
\[ N(a_0)W_i(j\Omega_0) = -1, \]  
where \( s = \frac{d}{dt} \), \( q = \frac{1}{2\pi} \int_0^{2\pi} f[a \sin \Psi, a(\sigma \sin \Psi + \omega \cos \Psi)] \sin \Psi \, d\Psi, \)  
\[ q' = \frac{1}{2\pi} \int_0^{2\pi} f[a \sin \Psi, a(\sigma \sin \Psi + \omega \cos \Psi)] \cos \Psi \, d\Psi, \]

which accounts for the dependence on the frequency also. It should be noted that for single-valued or hysteric symmetric nonlinearities formula (8) produces conventional describing function expressions that are commonly used for harmonic inputs.

The overall motion can now be obtained from the dynamics HB as follows:

\[ y(t) = a(t)e^{\sigma(t)t} \sin \Psi(t), \]  
where \( a(t), \sigma(t) \) are obtained from the following differential equation (\( \sigma(t) \) is supposed to be expressed through \( a(t) \) using other equations, which is given in details below):

\[ \dot{a}(t) = a(t)\sigma(t), \quad a(0) = a_0, \]  
and \( \Psi(t) \) is the phase computed as follows: \( \Psi(t) = \int_0^t \Omega(\tau)d\tau + \phi \), where \( \Omega(t) \) is obtained from (7), \( \phi \) is selected to satisfy initial conditions.

### III. ANALYSIS OF MOTIONS IN THE VICINITY OF A PERIODIC SOLUTION

Carry out frequency-domain analysis of the transient process of the convergence to the periodic motion in the vicinity of a periodic solution in system (1), (2), using the dynamic harmonic balance condition (7). We can write the conventional harmonic balance condition, which can also be obtained from (7) when \( \sigma = 0 \), as follows:

\[ N(a_0)W_i(j\Omega_0) = -1, \]  
where \( \Omega_0 \) and \( a_0 \) are the frequency and the amplitude of the periodic solution. Write the dynamics harmonic balance condition for the increments from the periodic solution:
\[ N(a_0 + \Delta a, \sigma) W_i(\sigma + j(\Omega_0 + \Delta \Omega)) = -1, \] (12)

We now take the derivative from both sides of (12) with respect to \( \Delta a \) (or \( a \)) in the point \( a = a_0 \):

\[
\left( \frac{\partial N(a, \sigma)}{\partial a} \right)_{a=a_0} + \left( \frac{\partial N(a, \sigma)}{\partial \sigma} \right)_{\sigma=0} \frac{d \sigma}{d a} \bigg|_{a=a_0} W_i(j\Omega_0) + N(a_0) \frac{d W_i(s)}{d s} \bigg|_{s=j\Omega_0} \frac{d s}{d a} \bigg|_{a=a_0} = 0, \] (13)

At first limit our analysis only to the nonlinearities the describing function of which does not depend on \( \sigma \) (for example, the ideal relay nonlinearity). Later the same analysis can be applied to nonlinearities that depend on \( \sigma \). Express the derivative \( \frac{d s}{d a} \bigg|_{a=a_0} \) from equation (13):

\[
\frac{d s}{d a} \bigg|_{a=a_0} = -\frac{\frac{d N(a)}{d a} \bigg|_{a=a_0} W_i(j\Omega_0)}{N(a_0) \frac{d W_i(s)}{d s} \bigg|_{s=j\Omega_0}}. \] (14)

Considering that \( s = \sigma + j\Omega \), we can rewrite equation (14) as follows:

\[
\frac{d \sigma}{d a} \bigg|_{a=a_0} + j \frac{d \Omega}{d a} \bigg|_{a=a_0} = -\frac{\frac{d N(a)}{d a} \bigg|_{a=a_0} W_i(j\Omega_0)}{N(a_0) \frac{d W_i(s)}{d s} \bigg|_{s=j\Omega_0}}. \] (15)

Equation (15) is a complex equation. It can be split into two equations for the real and imaginary parts. However, only real parts of (15) give an equation that has a solution. Once it is solved and \( a(t) \) is found, \( \Omega(t) \) can be found too. Considering that

\[
\frac{1}{W_i(s)} \frac{d W_i(s)}{d s} \bigg|_{s=j\Omega_0} = \frac{d \ln W_i(s)}{d s} \bigg|_{s=j\Omega_0} = \frac{d \arg W_i(j\omega)}{d \omega} \bigg|_{\omega=\Omega_0} - j \frac{d \ln W_i(j\omega)}{d \omega} \bigg|_{\omega=\Omega_0},
\]

and

\[
-\frac{1}{N(a)} \frac{d N(a)}{d a} \bigg|_{a=a_0} = \frac{d \ln \tilde{N}(a)}{d a} \bigg|_{a=a_0} = \frac{d \ln \tilde{N}(a)}{d a} \bigg|_{a=a_0} + j \frac{d \arg \tilde{N}(a)}{d a} \bigg|_{a=a_0},
\]
where \( \tilde{N}(a) = -N^{-1}(a) \), we can write for the real part of (15):

\[
\frac{d\sigma}{da} = \text{Re} \left\{ \frac{d\ln|\tilde{N}(a)| + j\frac{d\arg \tilde{N}(a)}{d\omega}}{d\omega} \frac{d\arg W_i(j\omega)}{d\omega} - j \frac{d\ln|W_i(j\omega)|}{d\omega} \right\}.
\]

which can be rewritten as follows (we skip for brevity the notation of the point in which the derivative is taken):

\[
\frac{d\sigma}{da} = \frac{d\ln|\tilde{N}(a)|}{d\omega} \frac{d\arg W_i(j\omega)}{d\omega} - \frac{d\arg \tilde{N}(a)}{d\omega} \frac{d\ln|W_i(j\omega)|}{d\omega}
\]

\[
- \left( \frac{d\arg W_i(j\omega)}{d\omega} \right)^2 + \left( \frac{d\ln|W_i(j\omega)|}{d\omega} \right)^2
\]

As a “side” product of our analysis, stability of a periodic solution can be assessed as follows:

\[
\frac{d\sigma}{da}_{a=a_0} < 0.
\]

The left-hand side of (16) can be rewritten as follows: \( \frac{d\sigma}{da} = \frac{1}{a} \left[ \dot{a} - \frac{\ddot{a}}{a} \right] \) and, therefore, equation (16) is a second-order differential equation. Because the right-hand side of (16) contains only one unknown variable \( a \), this equation can be numerically solved and convergence of the amplitude to the constant value can be obtained as a function of time.

IV. DYNAMIC HARMONIC BALANCE INCLUDING FREQUENCY RATE OF CHANGE

In many cases, such as analysis of motions in the vicinity of a periodic solution, the dynamic HB quasi-static w.r.t. to frequency derivative is quite capable of providing a precise result because the frequency changes insignificantly and the derivative of the frequency can be neglected. In some other cases, such as analysis of convergence of systems with second-order sliding modes [18] and finite-time convergence, the system experiences significant changes of the instantaneous frequency of the oscillations. In this situation the quasi-static approach to the account of the oscillations frequency may
result in the loss of accuracy of analysis. Therefore, inclusion of the frequency derivative would be desirable in such cases.

Consider the following illustrative example. Let the linear part be the second-order dynamics and the controller be the ideal relay with the amplitude \( h \). The describing function of the relay is \( N(a) = \frac{4h}{\pi a} \). If we assume that the input to the nonlinearity is a decaying sinusoid of constant frequency \( y(t) = a_0 e^{\sigma t} \sin(\omega t) \) then the control amplitude should be \( a_u = aN(a) \), the derivative of the control amplitude \( \dot{a}_u = \frac{\partial N}{\partial a} \dot{a} + Na \dot{a} \), and the decay \( \sigma_u = \frac{\dot{a}_u}{a_u} = -\frac{a}{\pi a} \dot{a} + \frac{4h}{\pi a} \dot{a} = 0 \). The result showing that \( \sigma_u = 0 \) is quite predictable because the output of the relay controller is an oscillation of constant amplitude, which features zero decay. However, it is well known (see [1], [13], for example) that the output of the linear part is oscillations of decaying amplitude and growing frequency. Because the input to the linear dynamics has zero decay, we can conclude that the decay in the signal \( y(t) \) is a result not of the decay in the control \( u(t) \) but because of the variable frequency of \( u(t) \). Moreover, the nonlinearity may change the decay according to

\[
\sigma_u = \frac{\dot{a}_u}{a_u} = \frac{\partial N}{\partial a} \dot{a} + Na \dot{a} = \frac{\partial \ln N}{\partial a} \dot{a} + \sigma = \left( \frac{\partial \ln N}{\partial a} + 1 \right) \sigma ,
\]

and this change is offset by the decay change due to the frequency variation at the signal propagation through the linear part. This situation is not covered by the equation (7) that assumes constant frequency, which requires the development of a different model.

Another example is the example of the propagation of a sinusoid of nearly constant amplitude and increasing frequency through the integrator. We assume that the signal of ideally constant amplitude is

\[
u^*(t) = \cos \left( \omega_0 t + \frac{1}{2} \dot{\omega}_0 t^2 \right),
\]

propagation of which through the integrator produces signal
\[ y^*(t) \approx y(t) = \frac{1}{\omega_0 + \dot{\omega}_0 t} \sin \left( \omega_0 t + \frac{1}{2} \dot{\omega}_0 t^2 \right) = \frac{1}{\omega} \sin \Psi \] (the approximate equality is valid if \( \dot{\omega}_0 \) is small enough). We note that the instantaneous phase of the signal is \( \Psi(t) = \omega_0 t + \frac{1}{2} \dot{\omega}_0 t^2 \). Therefore, the instantaneous frequency is \( \omega(t) = \dot{\Psi}(t) = \omega_0 + \dot{\omega}_0 t \), and the instantaneous derivative of the frequency is \( \ddot{\Psi}(t) = \ddot{\omega}_0 = \text{const} \). Find the derivative of \( y(t) \) as follow:

\[ \dot{y}(t) = \cos \left( \omega_0 t + \frac{1}{2} \dot{\omega}_0 t^2 \right) + \frac{\dot{\omega}_0}{(\omega_0 + \omega_0 t)^2} \sin \left( \omega_0 t + \frac{1}{2} \dot{\omega}_0 t^2 \right) = \cos \Psi + \frac{\dot{\omega}_0}{\omega} \sin \Psi \]

Therefore, introduces the phase lag less than 90° and the equations of the integrator can be represented by the conventional integrator having an additional feedback, which accounts for the effect of the variable frequency.

Therefore, the frequency response of the integrator to a decaying sinusoid of variable frequency is

\[ W(\sigma + j\omega, \dot{\omega}_0) = \frac{1}{\sigma + \dot{\omega}_0 + j\omega} \] . From this formula, one can see that if \( \sigma = -\frac{\dot{\omega}_0}{\omega^2} \), then the phase lag introduced by the integrator \( \varphi = \arg W(\sigma + j\omega, \dot{\omega}_0) \) is 90°. The amplitude characteristic of the integrator is given by \( M = |W(\sigma + j\omega, \dot{\omega}_0)| \), and the decay introduced by the integrator due to variable frequency is

\[ \sigma(t) = \frac{\dot{a}}{a} = -\frac{\dot{\omega}_0}{\omega} \] (given the amplitude of \( y(t) \) being \( a = \frac{1}{\omega_0 + \dot{\omega}_0 t} \)).

To analyze propagation of a decaying sinusoid of variable frequency through higher-order linear dynamics we shall use the notion of the rotating phasor. Let the output of the linear dynamics \( y(t) \) be given by

\[ \tilde{y}(t) = a(t)e^{j\Psi(t)} = e^{\ln a(t)}e^{j\Psi(t)} = e^{\ln a(t) + j\Psi(t)} \],

where \( a \) represents the length of the phasor and \( \Psi \) represents the angle between the phasor and the real axis. We can associate either the real or the imaginary part of \( \tilde{y}(t) \) with the real signal \( y(t) \). We assume
that the transfer function of the linear part does not have any zeros:

\[ W_i(s) = \frac{1}{Q(s)} = \frac{1}{\left(a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n\right)} \]

In the present paper we limit our analysis to plants not containing zeros in the transfer function. Given this transfer function, we can write for \( y(t) \):

\[ u = a_0 y + a_1 \dot{y} + a_2 \ddot{y} + \ldots + a_n y^{(n)} \tag{19} \]

We can find the derivatives of \( y(t) \) as follows:

\[
\begin{align*}
\dot{y}(t) &= e^{ln\sigma + j\omega t} \left( \frac{d\ln a}{dt} + j\dot{\psi} \right) = e^{ln\sigma + j\omega t} \left( \frac{1}{a} + j\dot{\psi} \right) = e^{ln\sigma + j\omega t} \left( \sigma + j\omega \right) y, \\
\ddot{y}(t) &= (\sigma + j\omega)\ddot{y} + (\sigma + j\omega)\dot{y} = \left[(\sigma + j\omega) + (\sigma + j\omega)^2\right] y
\end{align*}
\]

Considering that we disregard all derivatives of the decay and higher than first derivatives of the frequency in our model we can rewrite the last formula as follows:

\[ \ddot{y}(t) \approx \left[(\sigma + j\omega)^2 + j\dot{\omega}\right] y \tag{21} \]

and the formula for the third and forth derivatives as follows:

\[
\begin{align*}
\dddot{y}(t) &\approx \left[2(\sigma + j\omega)(\sigma + j\dot{\omega}) + j\ddot{\omega}\right] y + \left[(\sigma + j\omega)^2 + j\dot{\omega}\right] (\sigma + j\omega) y \\
&\approx \left[j3\dot{\omega} + (\sigma + j\omega)^2\right] y + \left[(\sigma + j\omega)^3 + j3\dot{\omega}(\sigma + j\omega)\right] y \\
\dddot{y}(t) &\approx \left[(\sigma + j\omega)^4 + j6\dot{\omega}(\sigma + j\omega) - 3\omega^2\right] y \tag{22}
\end{align*}
\]

We can continue with taking further derivatives. It is worth noting that the formulas above are organized the way that they have a term \((\sigma + j\omega)\) to the respective power and the term which is the product of \(\dot{\omega}\) and another multiplier. Therefore, we can write for \( y(t) \):

\[ \bar{u} = a_0 + a_1 (\sigma + j\omega) + a_2 (\sigma + j\omega)^2 + \ldots + a_n (\sigma + j\omega)^n \]

\[ + S(\sigma, \omega, \dot{\omega}) \bar{y}, \tag{23} \]

where \( S(\sigma, \omega, \dot{\omega}) \) includes all terms containing \(\dot{\omega}\). This component can be accounted for as an additional feedback – in the same way as it was done for the integrator.

If we introduce certain modified frequency response as

\[
W_i^*(\sigma, \omega, \dot{\omega}) = \frac{\bar{y}}{\bar{u}} = \frac{1}{Q(\sigma + j\omega) + S(\sigma, \omega, \dot{\omega})} = \frac{W_i(\sigma + j\omega)}{1 + W_i(\sigma + j\omega)S(\sigma, \omega, \dot{\omega})} \tag{24}
\]
we can write the dynamic harmonic balance equation as follows:

\[ N(a)W_i^*(\sigma, \omega, \dot{\omega}) = -1 \]  \hspace{1cm} (25)

Obviously, equation (25) can be split into two equations: for real and imaginary parts, or for equations of the magnitude balance and phase balance.

Equation (25) must be complemented with an equation that relates the difference of the decays at the input and the output of the linear part and the frequency rate of change – in the same way as it was done in the example of the integrator analysis. We note that the amplitude of the control (first harmonic) is as follows

\[ a_u = a \| Q(\sigma + j\omega) + S(\sigma, \omega, \dot{\omega}) \|, \]  

and, therefore, its time derivative is

\[ \dot{a}_u = \frac{d}{dt} \| Q(\sigma + j\omega) + S(\sigma, \omega, \dot{\omega}) \| + a \frac{d}{dt} \| Q(\sigma + j\omega) + S(\sigma, \omega, \dot{\omega}) \|. \]  

The decay of signal \( u(t) \) is computed as

\[ \sigma_u = \frac{\dot{a}_u}{a_u} = \frac{\dot{a}_u \| Q(\sigma + j\omega) + S(\sigma, \omega, \dot{\omega}) \| + a \frac{d}{dt} \| Q(\sigma + j\omega) + S(\sigma, \omega, \dot{\omega}) \|}{a \| Q(\sigma + j\omega) + S(\sigma, \omega, \dot{\omega}) \|} = \sigma + \frac{d}{dt} \| Q(\sigma + j\omega) + S(\sigma, \omega, \dot{\omega}) \|. \]

Because we disregard \( \dot{\sigma} \) and \( \dot{\omega} \) we can rewrite the last formula as follows:

\[ \sigma_u = \sigma + \frac{\partial}{\partial \omega} \| Q(\sigma + j\omega) + S(\sigma, \omega, \dot{\omega}) \| \frac{\dot{\omega}}{\dot{\omega}} = \sigma - \frac{\partial}{\partial \omega} \| W_i^*(\sigma, \omega, \dot{\omega}) \| \frac{\dot{\omega}}{\dot{\omega}} \]

The derivative in the last formula defines the slope of the magnitude-frequency characteristic of \( W_i^*(\sigma, \omega, \dot{\omega}) \). Considering also the formula for the decay of \( u(t) \) derived above through the DF, we can now write the condition of the balance of the decays in the closed-loop system as follows:

\[ \frac{\partial}{\partial \omega} \| W_i^*(\sigma, \omega, \dot{\omega}) \| \frac{\dot{\omega}}{\dot{\omega}} = -\frac{\partial}{\partial \omega} \| N(a) \| \sigma \]  \hspace{1cm} (26)

We can formulate the dynamic harmonic balance principle as follows. \textit{At every time during a single-frequency mode transient oscillation, the oscillation can be described as a process of variable instantaneous frequency, amplitude, decay and frequency rate of change (time derivative), which must satisfy equations (25) and (26).}
It is worth noting that \( \frac{\partial \ln |W_i'(\sigma, \omega, \dot{\omega})|}{\partial \ln \omega} = 0.05 \frac{\partial M(\omega)}{\partial \log \omega} \), where \( M(\omega) \) is the magnitude frequency response [dB] (Bode plot) for the transfer function \( W_i'(\sigma, \omega, \dot{\omega}) \). Therefore, this term gives the slope of the Bode plot. Both \( \dot{\omega} / \omega \) and \( \sigma = \dot{a} / a \) are relative rates of change of the frequency and the amplitude, respectively. Equation (26), therefore, is establishing the balance between those two rates of change. The use of the dynamic HB principle is illustrated by the following example.

Overall solution is obtained via integration of the first-order differential equation (10) with \( \sigma \) expressed through \( a \) using the three algebraic equations presented above.

V. ANALYSIS OF DECAYING OSCILLATIONS OF ROCKING BLOCK

Consider planar rocking motion of the 2-dimensional block (Fig. 1).

![Rocking block](image)

Fig. 1. Rocking block

The block of width \( c \) and height \( b \) and the base on which the block rests are assumed to be rigid. The possible considered motion includes only rocking (we assume that sliding and liftoff do not take place). However, we consider that friction is present in the form of a viscous friction, so that the resulting motion is alternating rocking around pivots A and B decaying due to the energy dissipation. Rocking
around pivot point A can be described by the following equation:

\[ mR^2 \ddot{\gamma}_A + \eta \dot{\gamma}_A - mg \sin \gamma_A = 0, \]

where \(|\gamma_A| < \arcsin \frac{b}{2R}\), \(\gamma_A\) is the angle between the line connecting pivot A and the center of mass of the block and the vertical line. And rocking around pivot B is given by:

\[ mR^2 \ddot{\gamma}_B + \eta \dot{\gamma}_B - mg \sin \gamma_B = 0, \]

where \(|\gamma_B| < \arcsin \frac{b}{2R}\), \(\gamma_B\) is the angle between the line connecting pivot B and the center of mass of the block and the vertical line. Introduce the variable \(x\) that describes the angular position of the block in both motions, which is the angle between the vertical axis of the block and the true vertical line:

\[ x = \arcsin \frac{b}{2R} - \gamma_B = \gamma_A - \arcsin \frac{b}{2R} \]

Now the motions are described as follows. Rocking around pivot A is given by

\[ mR^2 \ddot{x} + \eta \dot{x} - mg \sqrt{1 - \frac{b^2}{4R^2}} \sin x = mg \frac{b}{2R} \cos x, \] \text{if } x < 0, \tag{27} \]

And rocking around pivot B is given by

\[ mR^2 \ddot{x} + \eta \dot{x} - mg \sqrt{1 - \frac{b^2}{4R^2}} \sin x = -mg \frac{b}{2R} \cos x, \] \text{if } x > 0 \tag{28} \]

Attributing the right-hand side of (27) and (28) to control, we can rewrite (27) and (28) as an equation of a system with a discontinuous control:

\[ mR^2 \ddot{x} + \eta \dot{x} - mg \sqrt{1 - \frac{b^2}{4R^2}} \sin x = u \] \tag{29} \]

\[ u = \begin{cases} 
mg \frac{b}{2R} \cos x & \text{if } x < 0 \\
-mg \frac{b}{2R} \cos x & \text{if } x > 0
\end{cases} \]

One can see that the control can be presented as
\[ u = -mg \frac{b}{2R} \sgn x \cdot \cos x \]  

(30)

We now carry out analysis of motions in the system using the dynamic harmonic balance principle introduced above. We note that the system has two nonlinearities: \( f_1(x) = \sin x \), and \( f_2(x) = \sgn x \cdot \cos x \). Both nonlinearities are odd-symmetric. The second nonlinearity is depicted in Fig.2.

![Second nonlinearity](image)

Fig. 2. Second nonlinearity

Find the describing functions for these nonlinearities. For the first nonlinearity the DF is given by [1]:

\[
N_1(a) = 2J_1(a)/a,
\]

(31)

where \( J_1(a) \) is the Bessel function of the first order. We shall now derive the DF for the second nonlinearity. According to the DF definition:

\[
N_2(a) = \frac{4}{\pi a} \int_0^{\pi/2} f_2(a \sin \psi) \sin \psi \, d\psi = \frac{4}{\pi a} \int_0^{\pi/2} \cos(a \sin \psi) \sin \psi \, d\psi
\]

Considering the fact that we are mostly interested in the value of the DF at small values of the amplitude \( a \), we will use the expansion of the function \( A(a) = a N_2(a) \) that represents the first harmonic of the output signal of the nonlinearity into the Taylor series at \( a=0 \) with the account of only first three terms in the series:
\[ A(0) = \frac{4}{\pi}, \]
\[
\frac{d^2 A(a)}{da^2}
\bigg|_{a=0} = -\frac{8}{3\pi}
\]

Therefore \( A(a) \approx \frac{4}{\pi} - \frac{8}{3\pi}a^2 \) and the whole DF for the second nonlinearity can be written as
\[
N_2(a) \approx \frac{4}{\pi a} - \frac{8a}{3\pi}
\]  \hspace{1cm} (32)

We can now carry out analysis of the motions in the system using the formulated dynamic harmonic balance principle. The system can be presented as the block diagram (Fig. 3), with the transfer function of the linear part given as follows:
\[
W_l(s) = \frac{1}{mR^2s^3 + \eta}j
\]

Fig. 3. Block diagram of rocking block dynamics

The modified frequency response can be written for \( W_l(s) \) as follows:
\[
W_l^*(\sigma, \omega, \dot{\omega}) = \frac{1}{mR^2(\sigma + j\omega)^2 + \eta(\sigma + j\omega) + j\dot{\omega}}
\]  \hspace{1cm} (33)

The nonlinearity in the Lure representation of the system can be written as \( f(x) = f_2(x) - f_1(x) \).
Therefore, the DF of the combined nonlinearity is

\[ N(a) = mg \frac{b}{2R} N_2(a) - mg \sqrt{1 - \frac{b^2}{4R^2} N_1(a)} = mg \frac{b}{2R} \left( \frac{4}{\pi a} - \frac{8a}{3\pi} \right) - mg \sqrt{1 - \frac{b^2}{4R^2} \frac{2J_1(a)}{a}} \]  \tag{34}

Complex equation (25) can be rewritten as two equations: for the real and imaginary parts as follows:

\[ 2mR^2 \sigma \omega + \eta \omega + \dot{\omega} = 0 \]  \tag{35}

\[ mR^2 (\sigma^2 - \omega^2) + \eta \sigma = -N(a) \]  \tag{36}

And the third algebraic equation is obtained per (26), considering that

\[ \frac{\partial \ln W^*_r(\sigma, \omega, \dot{\omega})}{\partial \omega} = \frac{2\omega mR^2}{mR^2 (\sigma^2 - \omega^2) + \eta \sigma}, \quad \frac{dJ_1(a)}{da} = \frac{J_0(a) - J_2(a)}{2}, \] with \( J_0(a) \) and \( J_2(a) \) being the Bessel functions of zero and second order \[19\], respectively, and

\[ \frac{\partial \ln N}{\partial \ln a} = \frac{1}{N(a)} \left[ -mg \frac{b}{2R} \left( \frac{4}{\pi a} + \frac{8a}{3\pi} \right) + mg \sqrt{1 - \frac{b^2}{4R^2} \left( \frac{2J_1(a)}{a} - J_0(a) + J_2(a) \right)} \right] \] as follows:

\[ \sigma = \frac{2\omega mR^2}{mR^2 (\sigma^2 - \omega^2) + \eta \sigma} \dot{\omega} \]

\[ = \left\{ N^{-1}(a) \left[ -mg \frac{b}{2R} \left( \frac{4}{\pi a} + \frac{8a}{3\pi} \right) + mg \sqrt{1 - \frac{b^2}{4R^2} \left( \frac{2J_1(a)}{a} - J_0(a) + J_2(a) \right)} \right] + 1 \right\} \sigma \]  \tag{37}

Expressing \( \dot{\omega} \) from (35) and substituting into (37), we obtain two equations with three unknown variables. We also denote \( g_1(a) = -N(a) = -mg \frac{b}{2R} \left( \frac{4}{\pi a} - \frac{8a}{3\pi} \right) + mg \sqrt{1 - \frac{b^2}{4R^2} \frac{2J_1(a)}{a}} \) and

\[ g_2(a) = N^{-1}(a) \left[ -mg \frac{b}{2R} \left( \frac{4}{\pi a} + \frac{8a}{3\pi} \right) + mg \sqrt{1 - \frac{b^2}{4R^2} \left( \frac{2J_1(a)}{a} - J_0(a) + J_2(a) \right)} \right] + 1 \]

\[ = N^{-1}(a) mg \sqrt{1 - \frac{b^2}{4R^2} \left( J_2(a) - J_0(a) \right)} \]

With this notation the set of two equations can be written as
\[
\begin{aligned}
&\begin{cases}
    mR^2(\sigma^2 - \omega^2) + \eta \sigma = g_1(a) \\
    \sigma + \frac{2\omega^2 mR^2 (2mR^2 \sigma + \eta)}{mR^2 (\sigma^2 - \omega^2) + \eta \sigma} = g_2(a) \sigma
\end{cases}
\end{aligned}
\]

(38)

We shall solve (38) as a set of equations for \(\sigma\) and \(\omega\), considering amplitude \(a\) a given parameter. With this approach, if we express \(\omega\) from the first equation of (38) and substitute in the second equation, we arrive at one algebraic equation with one unknown variable \(\sigma\).

\[
\frac{2mR^2 \left(2mR^2 \sigma + \eta\right) \left(\sigma^2 - g_1(a) - \eta \sigma\right)}{\sigma g_1(a)} = g_2(a) - 1,
\]

which can be transformed into third-order algebraic equation for \(\sigma\):

\[
c_0 + c_1 \sigma + c_2 \sigma^2 + c_3 \sigma^3 = 0,
\]

(39)

where \(c_0 = -2g_1(a)\eta\), \(c_1 = -4mR^2 g_1(a) + 2\eta^2 - g_1(a)(g_2(a) - 1)\), \(c_2 = 6mR^2 \eta\), \(c_3 = 4m^2 R^4\).

Solution of the cubic equation (39) is well known (Cardano’s formula and other methods [19]). In the considered example we can use the Cardano’s formula, which allows one to analytically find the real root of the equation (39). This is possible due to the fact that \(\sigma\) is a real value. With \(a\) and \(\sigma\) available, we can numerically integrate the differential equation (10). After that \(\omega\) is computed through the formula that was used for the substitution (based on (36)). Note: equations (35) - (37) are solved at every step of the integration of (10).

The transient processes of the rocking motion for the block parameters \(R=1\), \(b=0.2\), \(m=1\), \(\eta = 0.4\) and initial amplitude \(a(0) = 0.05\) are presented in Fig. 4 for the presented dynamic HB-based solution and simulations based on the original differential equations.
Fig. 4. Rocking block motions (dynamic harmonic balance-based analysis and simulations)

One can see that the proposed approach provides a good estimate of the transient dynamics of the rocking block. In fact there is only some phase mismatch accumulated at the beginning of the transient; later this phase shift does not change.

VI. CONCLUSION

The dynamic harmonic balance condition is formulated, and a frequency-domain approach to analysis of transient oscillatory processes is developed. The previously developed approach that was quasi-static with respect to frequency, i.e. did not account for the rate of change of the frequency, is improved providing now equations of the full dynamic harmonic balance. The proposed method is applied to analysis of the decaying motions of increasing frequency of the block alternately rocking about its two pivots. The proposed approach may find numerous applications in various areas of engineering such as in solving the problems that involve estimation of the dynamics of oscillations near the periodic solution, converging or diverging oscillations, etc.

REFERENCES


