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Article *in* Optics and Spectroscopy · May 2018

DOI: 10.1134/S0030400X18050077

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ISSN 0030-400X, Optics and Spectroscopy, 2018, Vol. 124, No. 5, pp. 720–725. © Pleiades Publishing, Ltd., 2018.
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On Quantitative Determination of the Degree of Independence of Qubit Transformation by a Quantum Gate or Channel[1]

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Received December 22, 2017

Abstract—A multiqubit channel (quantum gate) is considered. A procedure of calculating the distance from the quantum-gate matrix to the subspace of matrices, which are tensor products of the transformation matrices of qubit subsystems, is proposed. The value of this distance indicates the degree of independence of transformation of these qubit subsystems. The proposed approach is considered as applied to waveguide implementation of quantum bits.

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INTRODUCTION

Entanglement is a crucial property of a quantum system, which allows one to implement quantum algorithms [1]. It is very important to know whether individual qubits are transformed independently during performed multiqubit operations. A sufficient condition for this independence is the possibility of representation of the quantum gate (channel) matrix in the form of a tensor product of the transformation matrices of individual qubits. Otherwise, the question of the measure of the degree of independence of qubit transformation, which would make it possible to compare channels from this point of view, must be solved. This degree is proposed to be the distance from the matrix of this gate to the subspace of matrices, which are tensor products. The problem under consideration is very important for describing multiqubit quantum channels, where the matrix of qubit transformation by the channel plays a role of a quantum-gate matrix. The interest in this problem has recently renewed because of the studies on development of quantum channels through free space (see, for example, recent experiments on entanglement teleportation [2]). If there is no independence of qubit transformation, the degree of entanglement may change during the transmission, which would violate the correct operation of the channel. This problem is also important when analyzing the stability of quantum algorithms (e.g., [3–5]). It should be noted that the degree of independence of qubit transformation indicates a property of the channel that somewhat differs from the generally used accuracy of entanglement reproduction [6, 7]. This can be illustrated, in particular, by the simple example of the operator SWAP, which does not change entanglement but does not transform qubits independently.

The above-mentioned parameter and procedure for its determination proposed in this paper may be useful for other problems. If the matrix under consideration is a density matrix of a multiqubit state, its distance from the subspace of density matrices, which are density matrices of individual qubits, is the degree of entanglement of this multiqubit state [8–10]. The described distance also occurs in many other problems related to quantum transformation of information, such as quantum processes [11], quantum cryptography [12], quantum phase transitions [13], quantum speed limit [14], quantum channel capacitance [15], quantum entanglement and coherence [16–18], and recording to quantum memory [19]. Mathematical aspects of this problem were described in [20, 21]. The distance under consideration is related to the known Eckart–Young–Mirsky theorem [22] describing the approximation of matrices by lower-rank matrices. In particular, we propose a modification of this theorem and a different procedure for calculating the distance. We use the singular-value decomposition [23, 24]. It is of interest that similar problem may arise in classical computer science (e.g., for code comparison [25]).

THE MAIN RESULT

Let us introduce vectorization operator vec, which converts matrix M into vector vec$M$ according to the following rule:

$$\operatorname{vec} M = (m_{1\cdot}, \ldots, m_{p\cdot})',$$

[1] X International Conference of Young Scientists and Specialists "Optics—2017," October 16–20, 2017, St. Petersburg, Russia.

where vectors $m_1, ..., m_p$ are columns of matrix M and a prime indicates transposition operation with complex conjugation.

We consider two matrices $B_{m \times m}$ and $C_{n \times n}$ and their tensor (Kronecker) product $R_{mn\times mn} = B \otimes C$. We introduce matrix $\tilde{R}$, which can be obtained from R as follows:

$$\tilde{R}_{m^2\times n^2} = \operatorname{vec} B(\operatorname{vec} C)'.$$

It is clear that matrix $\tilde{R}$ consists of the same elements as $R_{mn\times mn} = B \otimes C$ but arranged in a different order. To perform this operation for arbitrary matrix $A_{mn \times mn}$, we divide it into blocks $m \times n$ in size and obtain $\tilde{A}_{m^2\times n^2}$ applying the same procedure as for matrix R. The main mathematical result will be formulated as a mathematical theorem.

Theorem 1. For arbitrary matrix $A_{mn \times mn}$, norm $\|A - B \otimes C\|$ is minimum if matrices $B_{m \times m}$ and $C_{n \times n}$ are such that $\operatorname{vec} B(\operatorname{vec} C)' = kbc'$. Here, $k = \sigma_1$ is the maximum number among singular numbers of matrix $\tilde{A}$ ($\tilde{A}_{m^2\times n^2}$) and $b = u_1$ and $c = v_1$ are, respectively, the right- and left-hand singular vectors of matrix A corresponding to singular number σ_1.

Remark. Concerning the measure of the degree of independence of qubit transformation, one should use the normalized distance: $\|A\|^{-1}\|A - B \otimes C\|$. For unitary matrices $p \times p$, $\|A\| = \sqrt{p}$.

DISCUSSION OF THE RESULTS

As the first example, we consider operator CNOT in the standard basis. We will find matrices B and C 2×2 in size such that their tensor product is closest to matrix U_{CNOT}:

$$U_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Accordingly,

$$\tilde{U}_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

To calculate $\|\tilde{U}_{\text{CNOT}} - \sigma_1 u_1 v_1'\|$, we construct a singular decomposition based on the known theorem (note that the singular-value decomposition is related to the Schmidt decomposition, which is widely used in quantum theory of information).

Theorem 2. For any matrix $A_{m \times n}$, there are unitary matrices U and V such that $A = U\lambda V'$, where $\lambda_{m \times n}$ are matrices with nonzero elements in diagonal $\lambda_{jj} = \sigma_j$ with $\{\sigma_i\}_{i=1}^{s}$ being the singular numbers of matrix $A_{m \times n}$. Here, s is the rank of A and $U_{m \times m}$ and $V_{n \times n}$ are two unitary matrices formed by the left-hand ($\{u_i\}_{i=1}^{m}$) and right-hand ($\{v_j\}_{j=1}^{n}$) singular vectors. There is the following singular decomposition of the matrix:

$$A = \sum_{i=1}^{s} \sigma_i u_i v_i.$$

In the case of CNOT, we have $\sigma_1 = \sigma_2 = \sqrt{2}$, $\sigma_3 = \sigma_4 = 0$, and $\|\tilde{U}_{\text{CNOT}} - \sigma_1 u_1 v_1\| = \sqrt{2}$.

Let us consider two-qubit operator U_{SWAP}, which is characterized by the fact that it does not induce entanglement. However, it does not transform qubits independently. Its matrix is

$$U_{\text{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case, we have $\tilde{U}_{\text{SWAP}} = U_{\text{SWAP}}$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$, and $\|\tilde{U}_{\text{SWAP}} - \sigma_1 u_1 v_1'\| = \sqrt{3}$.

Let us consider the density matrix for the standard two-qubit pure state:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle,$$

where $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. The criterion of nonentanglement in this case is $\delta = ad - bc = 0$. The $|ad - bc|$ value can be considered as the degree of entanglement. Density matrix S is $S = |\psi\rangle\langle\psi|$. According to our procedure, we obtain

$$\tilde{S}^*\tilde{S} = \begin{bmatrix} A\begin{bmatrix} A & \bar{B} \\ B & C \end{bmatrix} & B\begin{bmatrix} A & \bar{B} \\ B & C \end{bmatrix} \\ \bar{B}\begin{bmatrix} A & \bar{B} \\ B & C \end{bmatrix} & C\begin{bmatrix} A & \bar{B} \\ B & C \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A & B \\ \bar{B} & C \end{bmatrix} \otimes \begin{bmatrix} A & \bar{B} \\ B & C \end{bmatrix},$$

where $A = |a|^2 + |b|^2$, $C = |c|^2 + |d|^2$, $B = a\bar{c} + b\bar{d}$, and $\bar{B} = c\bar{a} + d\bar{b}$. Correspondingly, eigenvalues σ_i of a 4×4 matrix can be obtained as products of eigenvalue λ_j of 2×2 matrices:

$$\begin{aligned} \lambda_{1,2} &= 2^{-1} \pm \sqrt{4^{-1} - AC + |B|^2} \\ &= 2^{-1} \pm \sqrt{4^{-1} - |ad - bc|^2}. \end{aligned}$$

Accordingly, we find distance ρ from the subspace of nonentangled states:

$$\rho = 1 - \frac{|\sigma_1|^4 C_1 + 2\delta^4 - \delta^2(\sigma_1^2 + \overline{\sigma_1^2})C_2}{C_2(|\sigma_1|^4 + \delta) - 2\delta^2(\sigma_1^2 + \overline{\sigma_1^2})},$$

$$C_1 = A^4 + C^4 + 4|B|^2(AC + C^2 + A^2) + 2|B|^4,$$

$$C_2 = A^2 + C^2 + 2|B|^2.$$

It is reasonable that we have $\rho = 0$ for $\delta = 0$ and vice versa; i.e., there is a relationship with the generally used degree of entanglement.

Let us consider our approach as applied to the problem of waveguide implementation of qubits (e.g., [26]). Qubit is considered as an electronic state (wave) in two weakly coupled (via a small hole) quantum waveguides. The positions of electron in the first and second waveguides correspond to "0" and "1," respectively. There is a superposition of the states because of the waveguide coupling. We will consider a system of two qubits, i.e., four waveguides (Fig. 1). In this case, the gate is simulated by coupling window z_{23} (the gates may differ depending on the parameters of this coupling). To describe the system, we use the model of zero-width slits, in which small holes are replaced with point holes according to a special procedure by analogy with the zero-range potential model [27–29]. This approach is based on the theory of self-conjugate expansions of symmetric operators [30]. Generally, the scheme is as follows. The initial self-conjugate operator is the orthogonal sum of Neumann's Laplacians for individual waveguides. We narrow the operator to a set of functions which turn to zero at chosen boundary points. The obtained operator is symmetric and has finite indices of error. The error elements are the Green's functions for waveguides with sources at the points chosen by us. The model operator is obtained as a self-conjugate expansion of this symmetric operator. We choose the most natural expansion, the domain of definition of which contains functions having a continuous regular part and singularities with different signs (at two sides of the hole). Within the model, we will find analytically the qubit transformation matrix.

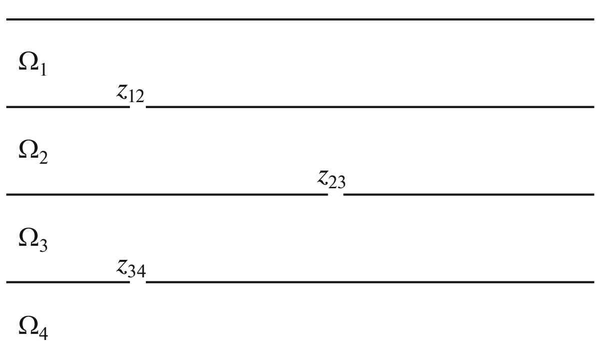

Fig. 1. Geometry of the system for waveguide implementation of two qubits: Ω_i is the quantum waveguide and z_{ij} is the coupling hole for waveguides Ω_i and Ω_j.

Let us consider the single-mode regime of the waveguides. The solution to the scattering model problem has the form

$$\begin{cases} \psi_1(z,k) + a_{12}G_1(z,z_{12},k), \quad z \in \Omega_1, \\ \psi_2(z,k) - a_{12}G_2(z,z_{12},k) + a_{23}G_2(z,z_{23},k), \\ z \in \Omega_2, \\ \psi_3(z,k) + a_{34}G_3(z,z_{34},k) - a_{23}G_3(z,z_{23},k), \\ z \in \Omega_3, \\ \psi_4(z,k) - a_{34}G_3(z,z_{34},k), \quad z \in \Omega_4. \end{cases} \tag{1}$$

Here, ψ_i is the arriving wave in Ω_i and $z = \begin{pmatrix} x \\ y \end{pmatrix}$. The waveguides are assumed to be identical. Correspondingly, Green's functions G_i are also identical. The matching conditions at the coupling points yield

$$\begin{cases} \psi_1(z_{12},k) + a_{12}g = \psi_2(z_{12},k) + a_{23}G - a_{12}g, \\ \psi_2(z_{23},k) - a_{12}G + a_{23}g \\ = \psi_3(z_{23},k) - a_{34}G - a_{23}g, \\ \psi_3(z_{34},k) + a_{34}g - a_{23} = \psi_4(z_{34},k) - a_{34}g. \end{cases} \tag{2}$$

Here,

$$G = G_i(z_{23}, z_{34}, k),$$
$$g = \lim_{x \to x_{j,j+1}} (G_i(z, z_{j,j+1}, k) - G_i(z, z_{j,j+1}, k_0)),$$

and k_0 is the model parameter characterizing the coupling force through the hole, $k_0^2 < 0$. Because of the symmetry, g and G are independent of i and j. Solutions to system (2), $a_{j,j+1}$, are substituted into (1). Taking into account the known expression of the Green's function for a waveguide, we obtain escaping waves ψ_i^{out}. They are related to arriving waves ψ_i by transformation matrix T:

$$\begin{bmatrix} \psi_1^{\text{out}} \\ \psi_2^{\text{out}} \\ \psi_3^{\text{out}} \\ \psi_4^{\text{out}} \end{bmatrix} = T \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}.$$

Thus, we find the transformation matrix. Its elements are as follows:

$$t_{11} = 1 + c_1c_2(2k)^{-1}, \quad t_{12} = -c_1c_2(2k)^{-1},$$

$$t_{13} = c_1(2k)^{-1}, \quad t_{14} = -c_1(2k)^{-1},$$

$$t_{21} = \left(-c_1c_2 + \frac{\exp^{-ikL}(1 + 2gc_1c_2)}{G}\right)(2k)^{-1},$$

$$t_{22} = 1 - \left(-c_1c_2 + \frac{\exp^{-ikL}(1 + 2gc_1c_2)}{G}\right)(2k)^{-1},$$

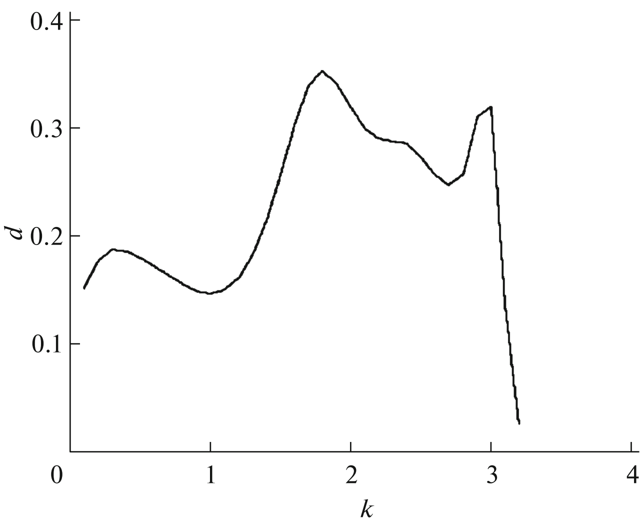

Fig. 2. Distance d from the quantum-gate matrix to the subspace of tensor products as a function of wavenumber k (dimensionless units): $k_0 = i$, $L = 1$, $x_{12} = x_{34} = 0$, and $x_{23} = 2$.

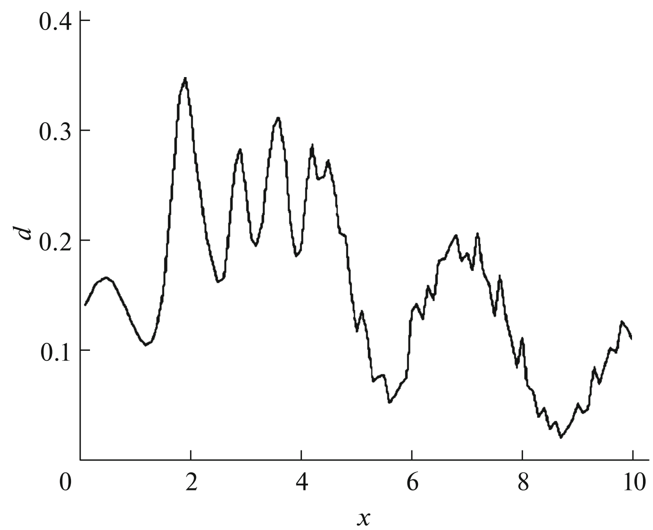

Fig. 3. Distance d from the quantum-gate matrix to the subspace of tensor products as a function of distance x between the coupling holes (dimensionless units): $k_0 = i$, $L = 1$, $x_{12} = x_{34} = 0$, $x_{23} = x$, and $k = 2$.

$$t_{23} = c_1\left(-1 + \frac{2g}{G}\exp^{-ikL}\right)(2k)^{-1},$$

$$t_{24} = -c_1\left(-1 + \frac{2g}{G}\exp^{-ikL}\right)(2k)^{-1},$$

$$t_{31} = \left(-\frac{1 + 2gc_1c_2}{G}\exp^{-ikL} + \frac{2g}{G} + c_1c_2c_4\right)(2k)^{-1},$$

$$t_{32} = \left(\frac{1 + 2gc_1c_2}{G}\exp^{-ikL} - \frac{2g}{G^2} - \frac{\exp^{ikL}}{G} - c_1c_2c_4\right)(2k)^{-1},$$

$$t_{33} = 1 - \left(\frac{2gc_1\exp^{-ikL} + \exp^{ikL}}{G} - c_1c_4\right)(2k)^{-1},$$

$$t_{34} = \left(\frac{2gc_1\exp^{-ikL} + \exp^{ikL}}{G} - c_1c_4\right)(2k)^{-1},$$

$$t_{41} = -\left(\frac{2g}{G} + c_1c_2c_4\right)(2k)^{-1},$$

$$t_{42} = -\left(\frac{\exp^{ikL}}{G} - \frac{2g}{G^2} - c_1c_2c_4\right)(2k)^{-1},$$

$$t_{43} = -\left(-\frac{\exp^{ikL}}{G} + c_1c_4\right)(2k)^{-1},$$

$$t_{44} = 1 + c_1c_4(2k)^{-1}.$$

Here, L is the distance between z_{12} and z_{23} in the longitudinal direction; i.e., $x_{23} - x_{12}$,

$$c_1 = \frac{G^2}{4g(G^2 - 2g^2)}, \quad c_2 = \frac{4g^2 - G}{G},$$

$$c_3 = \frac{G^2 - 2g^2}{G^2}, \quad c_4 = \frac{4g^2 - G^2}{G^2}.$$

Matrix T is the transformation matrix (quantum gate matrix) in our approach. Normalized distance d from the subspace of tensor products is determined numerically. In the system under consideration, the situation with implemented operation depends on wavenumber k. This dependence is shown in Fig. 2. The plot is in dimensionless units. All waveguides have an identical width, $L = 1$.

The dependence of the distance from the quantum gate matrix to the subspace of tensor products on coupling-hole position $x_{23} = x$ is shown in Fig. 3. The pronounced oscillations indicate the possibility of controlling the degree of independence of qubit transformation by changing parameters of the system.

THEOREM PROOF

We will construct matrices R, $\widetilde{R}$, and $\widetilde{A}$ in accordance with the above-described procedure. Let $\mathrm{vec}B(\mathrm{vec}C)' = kbc'$, where $\|b_{m^2\times 1}\| = \|c_{n^2\times 1}\| = 1$ and k is

the normalizing factor. We consider decomposition in singular numbers for matrix $\tilde{A}$:

$$\tilde{A} = \sum_{i=1}^{s} \sigma_i u_i v_i,$$

where $s = \operatorname{rank} \tilde{A}$, $\sigma_1 \geq \sigma_2 \geq .... \geq \sigma_s > 0$ are the singular numbers of the matrix arranged in descending order, u_i are orthonormal vectors of size m^2 ($\|u_i\| = 1$), and v_i are orthonormal vectors of size n^2 ($\|v_i\| = 1$). Then,

$$\rho = \|\tilde{A} - \operatorname{vec} B(\operatorname{vec} C)'\| = \left\| \sum_{i=1}^{s} \sigma_i u_i v_i - kbc' \right\|.$$

We consider the Frobenius norm for matrix M:

$$\|M\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2},$$

which possesses the following known property:

$$\|M\|_F^2 = \operatorname{tr}(MM') = \operatorname{tr}(M'M),$$

where $\operatorname{tr}(M)$ is the trace of matrix M and M' is the transposed matrix M. We consider

$$\rho^2 = \left\| \sum_{i=1}^{s} \sigma_i u_i v_i' - kbc' \right\|^2$$

$$= \operatorname{tr}\left(\left(\sum_{i=1}^{s} \sigma_i u_i v_i' - kbc' \right) \left(\sum_{i=1}^{s} \sigma_i u_i v_i' - kbc' \right) \right)$$

$$= \operatorname{tr}\left(\sum_{i=1}^{s} \sum_{j=1}^{s} \sigma_i \sigma_j v_i u_i' u_j v_{j'} - kcb' \sum_{i=1}^{s} \sigma_i u_i v_i' \right.$$

$$\left. - \left(\sum_{i=1}^{s} \sigma_i v_i u_i' \right) kbc' + k^2 cb'bc' \right)$$

$$= \operatorname{tr}\left(\sum_{i=1}^{s} \sigma_i^2 - kcb' \sum_{i=1}^{s} \sigma_i u_i v_i' \right.$$

$$\left. - \left(\sum_{i=1}^{s} \sigma_i v_i u_i' \right) kbc' + k^2 \right).$$

Taking into account the properties of the matrix trace, we find that

$$\rho^2 = \sum_{i=1}^{s} \operatorname{tr}(\sigma_i^2) - k \sum_{i=1}^{s} \sigma_i \operatorname{tr}(cb' v_i u_i')$$

$$- k \sum_{i=1}^{s} \sigma_i \operatorname{tr}(v_i u_i' bc') + \operatorname{tr}(k^2)$$

$$= \sum_{i=1}^{s} \sigma_i^2 - k \sum_{i=1}^{s} \sigma_i \operatorname{tr}(cb' v_i u_i')$$

$$- k \sum_{i=1}^{s} \sigma_i \operatorname{tr}(v_i u_i' bc') + k^2.$$

Note that $\operatorname{tr}(v_i u_i' bc') = \operatorname{tr}(v_i u_i' bc') = \operatorname{tr}((bc')(v_i u_i')) = \operatorname{tr}(cb' v_i u_i')$. We have

$$\rho^2 = \sum_{i=1}^{s} \sigma_i^2 - 2k \sum_{i=1}^{s} \sigma_i \operatorname{tr}(cb' v_i u_i') + k^2.$$

Let $c = [c_l]_{l=1}^{n^2}$, $b = [b_j]_{j=1}^{m^2}$, $u_i = [u_j^i]_{j=1}^{m^2}$, and $v_i = [v_l^i]_{l=1}^{n^2}$.

Then,

$$cb' u_i v_i' = \left[c_l v_t^i \sum_{j=1}^{m^2} b_j u_j^i \right]_{l,t=1}^{n^2}.$$

We arrive at the expression

$$\rho^2 = \sum_{i=1}^{s} \sigma_i^2 - 2k \sum_{i=1}^{s} \left(\sigma_i \sum_{l=1}^{n^2} (v_l^i c_l) \sum_{j=1}^{m^2} (b_j u_j^i) \right) + k^2 \tag{3}$$

$$= \sum_{i=1}^{s} \sigma_i^2 - 2k \sum_{i=1}^{s} (\sigma_i v_i' cb' u_i) + k^2.$$

We consider this expression as a function of k. It has a minimum at

$$k = \sum_{i=1}^{s} (\sigma_i v_i' cb' u_i).$$

Substituting this value into (3), we obtain

$$\sum_{i=1}^{s} \sigma_i^2 - \left(\sum_{i=1}^{s} (\sigma_i v_i' cb' u_i) \right)^2.$$

This expression is minimal if $\sum_{i=1}^{s} \sigma_i |v_i' cb' u_i|$ is maximal. Because of the order of singular numbers, we have

$$\sum_{i=1}^{s} \sigma_i |v_i' cb' u_i| \leq \sum_{i=1}^{s} \sigma_1 |v_i' cb' u_i|.$$

Due to the Hölder's inequality, we obtain

$$|v_i' c|^2 \leq \|v_i'\|^2 \|c\|^2 = 1, \quad |b' u_i|^2 \leq \|b'\|^2 \|u_i\|^2 = 1.$$

Therefore, $|v_i' cb' u_i| = 1$ for $c = v_1$ and $b = u_1$. Since σ_1 is the maximum singular number, $\sum_{i=1}^{s} \sigma_i |v_i' cb' u_i|$ is maximum at $k = \sigma_1$, $c = v_1$, and $b = \sigma_1$.

Thus, for specified matrix $A_{mn \times mn}$, $\|A - B \otimes C\|$ is minimum if matrices $B_{m \otimes m}$ and $C_{n \otimes n}$ are chosen so that $\operatorname{vec} B(\operatorname{vec} C)' = \sigma_1 u_1 v_1'$.

This norm is as follows:

$$\|A - B \otimes C\| = \|\tilde{A} - \operatorname{vec} B(\operatorname{vec} C)'\|$$

$$= \|\tilde{A} - \sigma_1 u_1 v_1'\|.$$

This norm yields the distance from matrix $A_{mn \times mn}$ to the subspace of matrices that are tensor products of matrices $m \times m$ and $n \times n$ in size.

ACKNOWLEDGMENTS

This study was supported in part by the Government of the Russian Federation (grant no. 074-U01) and Russian Science Foundation (grant no. 16-11-10330).

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Translated by A. Sin'kov