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Effects of heat and mass transfer on nonlinear MHD boundary layer flow over a shrinking sheet in the presence of suction *

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Abstract This work is concerned with Magnetohydrodynamic viscous flow due to a shrinking sheet in the presence of suction. The cases of two dimensional and axisymmetric shrinking are discussed. The governing boundary layer equations are written into a dimensionless form by similarity transformations. The transformed coupled nonlinear ordinary differential equations are numerically solved by using an advanced numeric technique. Favorability comparisons with previously published work are presented. Numerical results for the dimensionless velocity, temperature and concentration profiles as well as for the skin friction, heat and mass transfer and deposition rate are obtained and displayed graphically for pertinent parameters to show interesting aspects of the solution.

Key words shrinking sheet, suction at the surface, Runge Kutta Gill method, magnetic effect

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Introduction

Shrink packaging has been well established for over three decades. Due to its practical and cost-related advantages, it has found its use in many industries at various stages of the packaging process. Shrink wrapping, bundle wrapping, etc, is a process during which a product item or a group of items are wrapped in a loose sleeve or envelope of plastic film, which, upon application of heat, shrinks and tightly conforms to the shape of the enclosed contents. The key element of this process is the shrink film. Shrink film can be made from a variety of materials, each having different strengths, shrink characteristics transparency, and luster. Poly vinyl chloride (PVC), one of the first materials used, is now being phased out due to its toxic properties, although it is still used abundantly in underdeveloped countries. PVC shrink film is transparent, has good intensity, excellent seal and is moisture-proof, making it a kind of ideal packaging material.

Once a product is wrapped, the plastic film around it has to be shrunk to complete the process. Shrinking of the film can be achieved through a number of methods which, depending on application, may range from hand-held gas or electric hot air blowers, to fully automated room-size ovens. However, the best results are obtained through the use of a shrink tunnel. The shrink tunnel is simply a recirculated hot air chamber sitting on top of a conveyor which carries the product through the chamber. The essential requirements for a good shrink tunnel

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are a constant yet adjustable air temperature and conveyor speed, and a symmetrical air circulation pattern. As the pre-wrapped product is transported through the hot chamber, the film temperature rises and the film shrinks to conform to the shape of the enclosed product. At that stage, the film is not strong enough to compress the contents. When the pack emerges from the hot chamber, the film cools and shrinks a little more, tightening the pack into a stable, easily handled unit. The degree of final compression is determined mainly by the thickness and type of film used.

In many transport processes in nature and in industrial applications, heat and mass transfer with magnetic strength are a consequence of buoyancy effects caused by diffusion of heat and chemical species. The study of such processes is useful for improving a number of chemical technologies, such as polymer production and food processing. In nature, the presence of pure air or water is impossible. Some foreign mass may be present either naturally or mixed with the air or water. The present trend in the field of chemical reaction analysis is to give a mathematical model for the system to predict the reactor performance. A large amount of research work has been reported in this field. In particular, the study of heat and mass transfer with chemical reaction is of considerable importance in the chemical and hydrometallurgical industries. Chemical reaction effects on heat and mass transfer laminar boundary layer flow have been discussed by many authors[1−15] in various situations.

On the other hand, when a conductive fluid moves through a magnetic field, ionized gas is electrically conductive; the fluid may be influenced by the magnetic field. Magnetohydrodynamic (MHD) mixed convection heat transfer flow in porous and non-porous media is of considerable interest in the technical field due to its frequent occurrence in industrial technology and geothermal application, high temperature plasmas applicable to nuclear fusion energy conversion, liquid metal fluids, and (MHD) power generation systems. Laminar boundary layer flow over a wedge with suction/injection has been discussed by many authors[16−20].

One can find an abundant number of articles in the literature regarding different problems for Newtonian and non-Newtonian fluids, with and without heat transfer analysis dealing with the stretching flow problems. However, investigations regarding the flow problems due to a shrinking sheet are scarcely available in literature. To the best of our knowledge, only very few such attempts[21−24] are available in the literature. In Ref. [11] Wang presented an unsteady shrinking film solution, and in Ref. [24] Miklavcic and Wang proved the existence and uniqueness of a steady viscous hydrodynamic flow due to a shrinking sheet for a specific value of the suction parameter.

Effects of heat and mass transfer on MHD mixed convection flow have been studied by many authors in different situations. However, so far no attempt has been made to analyze the effects of heat and mass transfer on MHD boundary layer flow over a shrinking sheet in the presence of suction. It is hoped that the results obtained will not only provide useful information for applications, but also serve as a complement to previous studies.

1 Mathematical analysis

A two-dimensional, laminar, mixed convection flow of a viscous flow of a viscous incompressible fluid past a horizontal shrinking sheet in the presence of suction is considered. The physical model of the analysis is shown in Fig. 1. It is assumed that the concentration C of the diffusing species in the binary mixture is very small in comparison to the chemical species which are present. The x and y axes are taken along and perpendicular to the sheet, respectively, as shown in Fig. 1. The fluid is assumed to be Newtonian and electrically conducting, and the

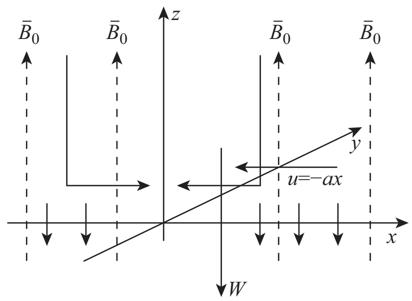


Fig. 1 Flow analysis on shrinking surface[22,29]

flow is confined to $z > 0$. A constant magnetic field of strength B_0 acts in the direction of the z axis. The induced magnetic field is negligible, which is a valid assumption on a laboratory scale. The assumption is justified when the magnetic Reynolds number is small[22]. Since no external electric field is applied and the effect of polarization of the ionized fluid is negligible, we can assume that the electric field $E = 0$. The chemical reactions do not take place in the flow and a constant suction is imposed at the horizontal surface, see Fig. 1. The boundary layer equations governing the MHD flow are as follows:
continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0; \tag{1}$$

momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{\sigma B_0^2}{\rho}u, \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - \frac{\sigma B_0^2}{\rho}v, \tag{3}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right); \tag{4}$$

energy equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right); \tag{5}$$

diffusion equation

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right), \tag{6}$$

where u, v, w are velocity components in the x, y and z directions respectively, ν is the kinematic viscosity, p is the pressure, σ is the electrical conductivity, ρ is the density of the fluid, B_0 is the magnetic induction, α is the thermal conductivity of the fluid and D is the effective diffusion coefficient. The above equations are derived by considering a zero electric field and incorporating the small magnetic Reynolds number assumption.

The boundary conditions applicable to the present flow are

$$\begin{cases} u = -U = -ax, \quad v = -a(m-1)y, \quad w = -W, \quad T = T_{\rm w}, \quad C = C_{\rm w} \quad \text{at} \quad z = 0; \\ u \to 0, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{as} \quad z \to \infty, \end{cases} \tag{7}$$

in which $a > 0$ is the shrinking constant, W is the suction velocity. $m = 1$ when the sheet shrinks in the x-direction only, and $m = 2$ when the sheet shrinks axisymmetrically[22−23]. If the disturbance level is systematically decreased[29], the three-dimensional state evolves to an almost two-dimensional recirculation. Here, the key finding is that the intensity of the flow response is proportionate to the amplitude of the inflow disturbance, meaning that the breakup of the flow in the step region is a linear (i.e., small) perturbation of the two-dimensional base flow. This is so because from the surface the fluid has no lateral velocities and the pressure is uniform.

Introducing the following similarity transformations:

$$u = axf'(\eta), \quad v = a(m-1)yf'(\eta), \quad w = -\sqrt{a\nu}mf(\eta), \tag{8}$$

where

$$\eta = \sqrt{\frac{a}{\nu}}z, \quad \theta = \frac{T - T_\infty}{T_\mathrm{w} - T_\infty}, \quad \phi = \frac{C - C_\infty}{C_\mathrm{w} - C_\infty}, \tag{9}$$

equation (1) is automatically satisfied; equation (4) becomes

$$\frac{p}{\rho} = \nu\frac{\partial w}{\partial z} - \frac{w^2}{2} + \text{constant}; \tag{10}$$

equations (2), (3), (5) and (6) reduce to ordinary differential equations:

$$f''' - M^2 f' - f'^2 + mff'' = 0, \tag{11}$$

$$\theta'' - Prf'\theta + mPrf\theta' = 0, \tag{12}$$

$$\phi'' - Scf'\phi + mScf\phi' = 0, \tag{13}$$

where the Prandtl number Pr, the Schmidt number Sc, the the magnetic parameter M^2, and the suction parameter S are defined as

$$Pr = \frac{\nu}{\alpha}, \quad Sc = \frac{\nu}{D}, \quad M^2 = \frac{\sigma B_0^2}{\rho a}, \quad S = \frac{W}{m\sqrt{a\nu}}, \tag{14}$$

with boundary conditions:

$$\begin{cases} \eta = 0 : f(0) = S, \quad f'(0) = -1, \quad \theta(0) = 1, \quad \phi(0) = 1; \\ \eta \to \infty : f'(\infty) = 0, \quad \theta(\infty) = 0, \quad \phi(\infty) = 0, \end{cases} \tag{15}$$

where S is the suction parameter if $S > 0$, and injection if $S < 0$.

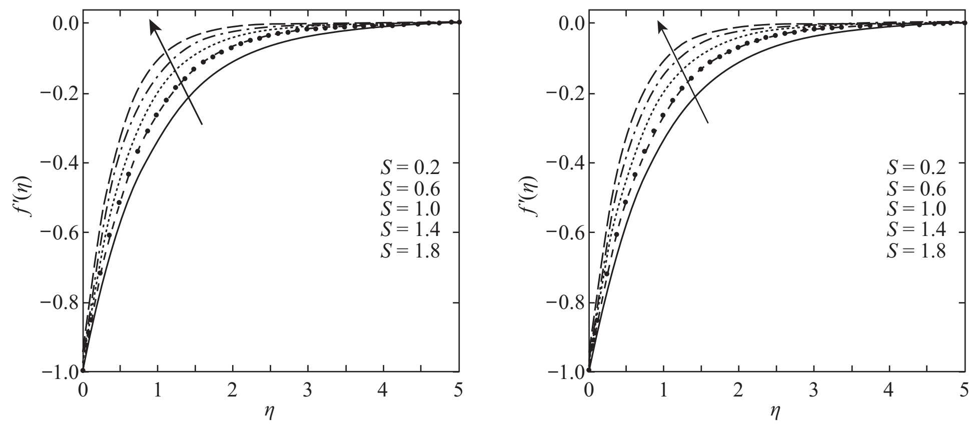
2 Numerical solution

The set of non-linear ordinary differential equations (11)–(13) with boundary conditions (15) were numerically solved using the Runge Kutta Gill algorithm[25] with systematic guessing of $f''(0), \theta'(0)$ and $\phi'(0)$ by the shooting technique until the boundary conditions at infinity $f'(0), \theta(0)$ and $\phi(0)$ decay exponentially to zero. The step size $\Delta\eta = 0.001$ is used while obtaining the numerical solution with $\eta_{\max}$, and an accuracy to the fifth decimal place is sufficient for convergence. The computations were done through a program which uses a symbolic and computational computer language Matlab. A step size of $\Delta\eta = 0.001$ was selected to be satisfactory for a convergence criterion of 10^{-7} in nearly all cases. The value of η_∞ was found for each iteration loop by the assignment statement $\eta_\infty = \eta_\infty + \Delta\eta$. The maximum value of η_∞, for each group of parameters α, γ, M^2 and m determined when the values of unknown boundary conditions at $\eta = 0$ do not change, enables successful looping with error less than 10^{-7}. The results of the effects of heat and mass transfer on a non-linear MHD boundary layer flow over a shrinking sheet in the presence of suction are discussed in detail in the following section.

3 Results and discussion

Numerical computations are carried out for $1 \leq M^2 \leq 3$, $0.32 \leq Sc \leq 0.78$, $0.71 \leq Pr \leq 5.0$ and $1 \leq m \leq 3.0$. Typical velocity, temperature and concentration profiles are shown in the following figures for $Pr = 0.71$ and various values for the governing parameters M^2, Sc and m.

In the absence of energy and diffusion equations, in order to ascertain the accuracy of our numerical results, the present study is compared with the available exact solution provided by Sajid and Hayat[23]. The velocity profiles for $M^2 = 2.0$, $m = 1.0$ are compared with the available exact solution of Sajid and Hayat[23] in Fig. 2. It is observed that the agreement with the theoretical solution of the velocity profile is excellent.



(a) Results of Sajid and Hayat[23]

(b) Results of the present work

Fig. 2 Comparison of the velocity profile with Sajid and Hayat[23], where $Pr = 0.71$, $Sc = 0.62$, $m = 1$, $M^2 = 2.0$

Figure 3 illustrates the influence of suction parameter S on the velocity, temperature and concentration profiles, respectively. The imposition of wall fluid suction for this problem has the effect of increasing the entire hydrodynamics and reduces the thermal and concentration boundary layers causing the fluid velocity to increase while decreasing its temperature and concentration. The decrease in the concentration boundary layer is caused by two effects: (i) the direct action of the suction, and (ii) the indirect action of the suction causing a thicker thermal boundary layer, which corresponds to a lower temperature gradient, and a consequent increase in the shrinking surface and a higher concentration gradient.

Figure 4 presents typical profiles of velocity, temperature and concentration for various values of the magnetic parameter, for a physical situation with a uniform chemical reaction. It is clearly shown that the velocity of the fluid increases and the temperature and the concentration of the fluid slightly decrease with the increasing strength of the magnetic field. The effects of a transverse magnetic field on an electrically conducting fluid gives rise to a resistive-type force called the Lorentz force. This force has the tendency to slow down the motion of the fluid. This result qualitatively agrees with expectations, since the magnetic field exerts a retarding force on the mixed convection flow. The application of a magnetic field moving with the free stream has a tendency to induce a motive force which decreases the motion of the fluid and increases its boundary layer.

Figure 5 indicates the effects of the Prandtl number on velocity, temperature and concentration field. In this figure, we depict the temperature distribution for $Pr \approx 1.0$ as corresponding to air and $Pr \approx 5.0$ as corresponding to water at the room temperature and one atmosphere pressure, highlighting the effect of the shrinking sheet. It is observed that the velocity of the fluid increases and the temperature decreases with an increase of the Prandtl number, whereas the concentration of the fluid is not significant with an increase of the Prandtl number. Physically, it means that the momentum boundary layer increases and the thermal boundary layer decreases with an increase in the Prandtl number.

Figure 6 presents typical profiles for velocity, temperature and concentration for various values of the shrinking sheet parameter, for a physical situation with porosity effects. It is clearly shown that the velocity of the fluid increases and the temperature and the concentration of the fluid decrease with the increase of the strength of the shrinking sheet. Due to the shrinking of the sheet, it is also seen that the changes in velocity, temperature and concentration of the fluid are very rapid, and all these physical behaviors are due to the combined effects of the strength of the shrinking of the sheet and porosity at the wall of the sheet.

Effects of the Schmidt number on velocity, temperature and concentration profiles are shown in Fig. 7. It is seen from this figure that the concentration of the fluid decreases with an increase of the Schmidt number, whereas the profiles of velocity and temperature are not significantly altered with an increase of the Schmidt number. This causes the concentration buoyancy effects to decrease, yielding a small reduction in the fluid velocity. The decrease of the concentration field due to an increase of Sc shows that it increases gradually as we replace Hydrogen ($Sc = 0.32$) by water vapour ($Sc = 0.62$) and Ammonia ($Sc = 0.78$) in the said sequence. All these

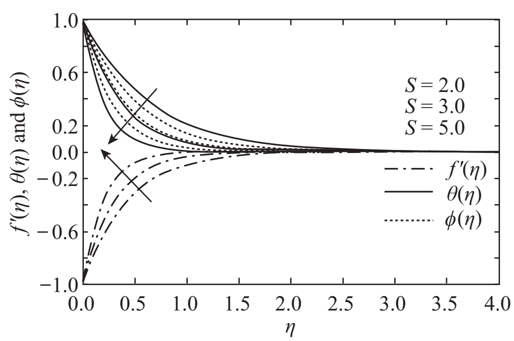


Fig. 3 Suction effect on velocity, temperature and concentration profiles where $Pr = 0.71$, $Sc = 0.62$, $m = 1$, $M^2 = 1$

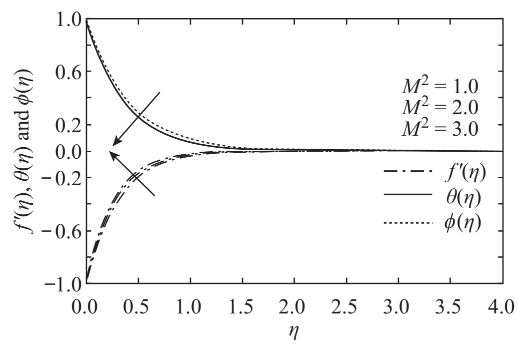


Fig. 4 Magnetic effect on velocity, temperature and concentration profiles where $Pr = 0.71$, $Sc = 0.62$, $m = 1$, $S = 3$

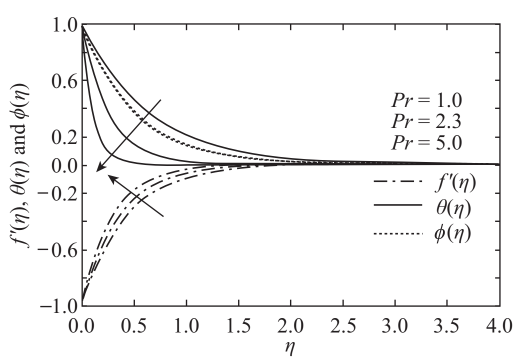


Fig. 5 Prandtl number effect on velocity, temperature and concentration profiles where $M^2 = 1$, $Sc = 0.62$, $m = 1$, $S = 3$

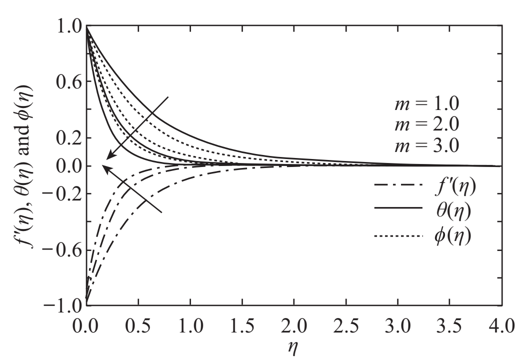


Fig. 6 Shrinking parameter effect on velocity, temperature and concentration profiles where $Pr = 0.71$, $M^2 = 1$, $Sc = 0.62$, $S = 3$

physical behaviors are due to the combined effects of the strength of magnetic effect and porosity at the wall of the sheet.

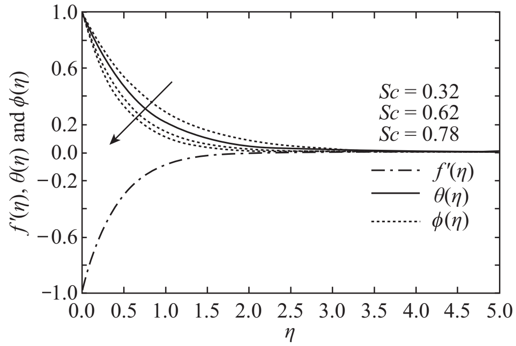


Fig. 7 Schmidt number effect on velocity, temperature and concentration profiles where $Pr = 0.71$, $M^2 = 1$, $m = 1$, $S = 3$

Table 1 presents the effects of suction, magnetic and shrinking parameters on the values of the skin-friction $f''(0)$, the rate of heat transfer $\theta'(0)$, and the rate of mass transfer $\phi'(0)$. The results show that the skin friction at the wall of the surface increases whereas the rates of heat and mass transfer decrease with the increase of suction, magnetic and shrinking parameters, respectively.

Table 1 Analysis for skin friction and rate of heat and mass transfer

$f''(0)$	$\theta'(0)$	$\phi'(0)$	Parameter	
2.414 214	−1.493 370	−3.503 611	$Pr = 1.0$	
2.816 591	−3.912 338	−6.416 284	$Pr = 2.3$	Prandtl number
3.000 001	−5.267 754	−8.106 047	$Pr = 3.0$	
2.414 214	−1.493 292	−1.917 597	$S = 2.0$	
3.302 776	−2.665 537	−2.410 283	$S = 3.0$	Suction
4.236 068	−3.749 705	−2.937 482	$S = 4.0$	
3.302 776	−2.665 537	−2.410 283	$M^2 = 1.0$	
3.561 553	−2.680 315	−2.417 000	$M^2 = 2.0$	Magnetic
3.791 288	−2.692 318	−2.422 522	$M^2 = 3.0$	
2.414 214	−1.493 292	−1.917 597	$m = 1.0$	
4.124 816	−3.608 226	−2.865 386	$m = 2.0$	Shrinking
6.001 955	−5.653 797	−3.944 462	$m = 3.0$	

4 Conclusions

In this work the problem of effects of heat and mass transfer on the non-linear MHD boundary layer flow over a shrinking sheet in the presence of suction is investigated. The effect of the strength of the magnetic field is expected to alter the momentum boundary layer significantly. It is observed that the shrinking of the sheet has a substantial effect on the flow field and, thus, on the heat and mass transfer rate from the sheet to the fluid. It is expected that this research may prove to be useful for the study of movement of oil or gas and water through the reservoir of an oil or gas field, in the migration of the underground water, and in filtration and water purification processes. It is interesting to note that the results of this investigation play a very important role in packaging units. Shrink wrapping, bundle wrapping, etc, is a process during which a product item or a group of items are wrapped in a loose sleeve or envelope of plastic film, which, upon application of heat, shrinks and tightly conforms to the shape of the enclosed contents. The key element of this process is the shrink film. A heat shrinkable fabric is impregnated with a plastic or resin material to reduce its air permeability prior to any assembly of the fabric on aircraft. In comparison to the stretching sheet problem, it is found that the results in the case of hydrodynamic flow are not stable for the shrinking surface and only the flows are meaningful in the case of a shrinking surface. The results of the problem are also of great interest in geophysics, in the study of the interaction of the geomagnetic field with the fluid in the geothermal region.

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