Micro-motion selective-actuation $XYZ$ flexure parallel mechanism: design and modeling

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Abstract—This paper presents the design of a selective-actuation flexure parallel mechanism that can provide three de-coupled micro-motions along the $X$-, $Y$- and $Z$-axes. This mechanism can be used as an ultra-precision positioning system. The modeling of this flexure parallel mechanism is then established based on a pseudo-rigid-body model with consideration of deformation of the flexure member. The factor of deformation allows us to formulate the accurate kinematics analysis of the flexure mechanism. Via this modeling, dimension and free shape of the mechanism are determined based on the criteria of isotropic resolution transmission scale. An experiment was set up to verify the modeling and the design. The experiment shows the advantage of the proposed model versus the model currently used. The similarity of the resolutions obtained from the experiment and predicted by the optimal design shows the validity of the resolution evaluation and the optimal design.

Keywords: Flexure; micro-motion; selective actuation; pseudo-rigid body.

1. INTRODUCTION

A flexure mechanism which has a moving platform connected to the base by at least two kinematic chains with flexure joints is called a flexure parallel mechanism (FPM). The degree-of-freedom (DOF) of the flexure joint is obtained from the deflection of flexure members and not from the sliding or rolling contacts as the DOF of traditional joint is. Therefore, a FPM possesses high stiffness, high natural frequencies and there is no error accumulation, no backlash, no friction, vacuum compatibility and no need of lubrication. FPM provides high accurate motions

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within small range and can be used as an ultra-precision manipulation system in fields like optics, precision machine tools and micro-component fabrication.

Recently, several FPM-based precision positioning stages have been developed [1–13]. Beside the topology synthesis of compliant mechanisms [12, 13], most of the flexure mechanisms are not designed based on the precision criterion but on the criteria of large range [1–4, 8, 12] and compliance/stiffness [5–8]. Only when the mechanism control is implemented, the precision of the flexure mechanism is taken into consideration. Some measures are proposed to manage the error distribution [1], to predict the error and modify the control [2, 3] or to use the precise measurement instruments to aid the control system [6, 7]. These post-design measures are not sufficient for obtaining a system with high accuracy. There is a need for pre-design measures which can guarantee the desired resolution right at the design stage. Another problem is how to model the flexure mechanism. Recent studies [1–4, 6–8] simplify the modeling of a flexure mechanism by using a pseudo-rigid-body (PRB) model, as proposed by Howell [14]. This method does not lead to accurate modeling because the deformation of the flexure members is not studied when establishing the model. Recently, to obtain a more precise PRB model, Yi et al. proposed a base-fixed flexure hinge as a two-link chain consisting of a revolving joint with torsion spring and a prismatic joint [5]. However, the elongation and the bending of the flexure hinge are actually tied together. Therefore, the replacement of a flexure hinge by a revolving joint and a prismatic joint is not sufficient and makes the modeling more complicated for a spatial flexure mechanism with more joints and links. Another disadvantage is that the deflection of passive flexure joints is still not considered. An additional problem that needs to be considered is the selective actuation (SA) of the mechanism. Usually, the cost of a precise actuator is very high. All the DOF motions of the mechanism are not always used for a positioning task. Therefore, the mechanism needs to be designed so that the actuators can be moved to obtain the desired number of DOF without affecting on the precision of the mechanism. An additional benefit of SA is that the actuators can be used for multiple tasks. As a precision positioning system cannot be re-configured often, selective actuation can be considered as the re-configurability in a precision mechanism. In addition, the motion of the end-effector relies on the actuation of all actuators. Usually, the motions are de-coupled by using a control algorithm. Selective actuation rejects fully or eliminates partially the dependence of the end-effector motion on all the actuation of actuators and, therefore, aids the decoupling control. This allows the users to make adjustments to accurately achieve one desired position.

This paper is organized as follows. First, the paper presents the design of a micro-motion selective-actuation $XYZ$ flexure parallel mechanism (SA $XYZ$ FPM) that can perform three decoupled translation motions. Secondly, a more accurate model has been realized by taking the effects of flexure deformations into account. The error of PRB model caused by deformation of flexure members, are estimated and compensated. The paper continues with the formulation of the resolution
transmission scale that represents the effect of the mechanism configuration on the resolution amplification. That scale is used to compute the average resolution of the end-effector. The global value and the uniformity of the defined scale are employed as the criterion and the constraint in an optimal computation of the dimension and free shape of the FPM. Finally, experiments are set up to verify the advantage of the stated modeling versus the PRB model and the selective-actuation characteristic. An experiment is also carried out to determine the resolution of the designed FPM equipped a feedback control system.

2. CONCEPTUAL DESIGN OF A SA XYZ FPM

Several multi-axis-positioning devices are developed based on FPM [1, 2, 4–16]. However, the coupling of motions and parasitic motions limit the accuracy of the devices. It is stated in the Introduction that a SA mechanism overcomes the above disadvantages.

A SA mechanism is defined as a mechanism whose number of DOF is equal to the number of actuators. The actuators can be removed so as to inhibit corresponding DOF, whereas other DOF motions are not affected. Each actuator provides independent DOF motion without parasitic motions.

Serially stacked multi-DOF stage mechanisms obtain the characteristic of selective actuation easily but accumulate errors. To reduce accumulated error, selective actuation is used for parallel stage design. The configuration of a SA XYZ FPM is built when observing a 3-DOF $XY\theta_Z$ mechanism with three revolving joint (R–R–R) limbs (Fig. 1). If we let the outside part free and fix the center part to a linear actuator working in the $Z$-direction (i.e., the center part can be moved in the $Z$-direction), the outside part now obtains four DOF ($X$, $Y$, $Z$ and $\theta_Z$). Three planar

![Figure 1. Planar 3-DOF mechanism with R–R–R limbs.](image-url)
3-DOF mechanisms are fixed together in one frame at the outside parts so that the three mechanism planes are perpendicular to each other (see Fig. 2). If we let each center part of the frame move along the corresponding direction perpendicular to the side containing the center part, each outside part individually can perform four DOF motions. Correspondingly, the DOF motions of the X-side, Y-side and Z-side are \((Y, Z, \theta_X, X)\), \((Z, X, \theta_Y, Y)\) and \((X, Y, \theta_Z, Z)\). Actually, the outside part of the frame possesses only three translation DOF, that is, the intersection of three above DOF set:

\[
(Y, Z, \theta_X, X) \cap (Z, X, \theta_Y, Y) \cap (X, Y, \theta_Z, Z) = (X, Y, Z).
\]

The designed micro-motion SA XYZ FPM consists of a passive 3-dimensional (3D) flexure frame fixed to three flexure prismatic joints compounded by a double-flexure linear slide (Figs 2 and 3). The dimensions of the double linear spring are determined to guarantee very high torsion and bending stiffness and to allow the linear motion. The passive 3D flexure frame has three mutually perpendicular sides. Each side is a 3-flexure-limb group that consists of a center part connected to the edge of the passive 3D frame by three symmetrically located limbs. The limb has a V-shape with three hinges to create an R–R–R kinematics chain. Each flexure prismatic joint is actuated by a linear actuator. Three linear actuators provide the platform three decoupled translation motions, following the \(X\)-, \(Y\)- and \(Z\)-axis, respectively. The number of DOF can be from 1 to 3, depending on the number of linear actuators mounted on the mechanism. The passive 3D frame is fabricated in one piece (monolithic structure) in order to avoid the error of perpendicular caused by the assembling work. In practice, there are very small deformations in the non-working axes of the
flexure element. These deformations can cause error of straightness. To overcome this problem, each side (3-flexure-limb group) of the 3D frame is designed as a symmetric group of three legs. The dimensions of the flexure elements are also determined so that the stiffness in the non-working axis is maximized. Those design points guarantee the very small and acceptable error of straightness in the expected working range. The mechanism can perform 1–3 positioning tasks and possesses advantages such as high stiffness, high natural frequencies, selective actuation, no parasitic motion, simple design and minimal number of components.

3. MODELING

Recently, the PRB model has been used as an analytical model to investigate flexure mechanisms. Because some flexure characteristics are not taken into consideration, the PRB model is suitable only for force analysis. This section presents a PRB model with consideration of deformation of flexure members (PRB-D model) that can be used to express precisely the motion of flexure mechanisms.

3.1. PRB-D model

Suppose the desired position of the end-effector and the computed active joint variables (actuated joint variables) are specified by vector $X$ and vector $x$, respectively. Forward kinematics based on the PRB model leads to $X$ as a function of $x$ as given below:

$$X = H(x). \quad (1)$$
However, actual motion of the end-effector is assumed to be the combined effect of the deformation $E_i$ and PRB modeling given as below:

$$X = H(x) + E_i.$$  \hfill (2)

Assuming that the flexure mechanism complies with the linear deformation and superposition principles, the combined effect of deformation $E_i$ is the accumulated deflection of all flexure members, so

$$E_i = \sum_{i=1}^{n} E_i,$$  \hfill (3)

where $n$ is the number of flexure members existing in the mechanism and $E_i$ is the individual deformation of flexure member $i$, that can be determined as

$$E_i = G_i \varepsilon_i,$$  \hfill (4)

where $\varepsilon_i$ is the deflection of flexure member $i$ and $G_i$ is the transformation matrix of the deformation from the flexure member $i$ to the end-effector.

The deflection of flexure member $i$ is caused by reaction forces or moments $F_i$ applied to that member given as below:

$$\varepsilon_i = K_i^{-1} G_{fi} F_i,$$  \hfill (5)

where $K_i$ is the stiffness matrix of that flexure member and matrix $G_{fi}$ is used to transform the force $F_i$ from the base frame to the local frame of the flexure member. The reaction $F_i$ is determined in the force analysis of the flexure mechanism based on the PRB model.

The total effect of deformation of flexure members on the end-effector can be expressed as

$$E_i = \sum_{i=1}^{n} G_i K_i^{-1} G_{fi} F_i.$$  \hfill (6)

Substituting equation (6) into equation (2), we have the PRB-D model of the flexure mechanism given as

$$X = H(x) + \sum_{i=1}^{n} G_i K_i^{-1} G_{fi} F_i.$$  \hfill (7)

The PRB-D model of a flexure mechanism can be summarized in a scheme as shown in Fig. 4. The PRB-D model needs extensive computations because it concerns all the flexure members. Fortunately, the flexure mechanism used for ultra-precision manipulation must have a simple structure with few flexure members.
3.2. Application to the designed XYZ FPM

3.2.1. PRB model. The PRB model of the FPM is obtained by replacing the flexure hinges of the passive 3D frame by the revolving joints and the double linear slides by the prismatic joints (see Fig. 5a). Each 3-flexure-limb group of the PRB model is illustrated in Fig. 5b as a 3-DOF planar parallel mechanism.

Denote $X = (X, Y, Z)$ and $x = (x, y, z)$ as the displacement vector of end-effector and the vector of the active joint variable, we have

$$\begin{bmatrix} ẋ \\ ẏ \\ ẑ \end{bmatrix}^T = J\begin{bmatrix} ẍ \\ ẏ \\ ẑ \end{bmatrix}^T,$$

(8)

where the Jacobian matrix $J$ is a unity matrix. To determine all joint variables (passive and active), we use the notation $\eta_{ij}$, which indicates the quantity $\eta$ of limb $j$ of the 3-flexure limb group $i$; for example, $\theta_{120}$ represents the initial angle between the $X$-axis and link $A_{12}B_{12}$ of limb 2 of 3-flexure limb group 1 and

$i = 1 : u, v$ and $w$ represent $x, y$ and $z$;

$i = 2 : u, v$ and $w$ represent $y, z$ and $x$; and

$i = 3 : u, v$ and $w$ represent $z, x$ and $y$, respectively.

Consider limb $j$ of the 3-flexure limb group $i$ of the PRB model (Fig. 6), the vector loop equation of the limb is

$$\vec{O}_iC_{ij} = \vec{O}_iA_{ij} + \vec{A}_{ij}B_{ij} + \vec{B}_{ij}C_{ij}.$$
Figure 5. Pseudo-rigid-body model. (a) Mechanism, (b) one 3-flexure limb group.

Denote $B_{ij0}$ and $C_{ij0}$ as the initial positions of joints $B_{ij}$ and $C_{ij}$ on limb $j$ of side $i$; the equation can be changed to

$$ A_{ij} \overrightarrow{B_{ij0}} + B_{ij0} \overrightarrow{C_{ij0}} + \overrightarrow{u} + \overrightarrow{v} = A_{ij} \overrightarrow{B_{ij}} + B_{ij} \overrightarrow{C_{ij}}. $$
Figure 6. Vector close loop of limb $j$ on 3-flexure limb group $i$.

With projection of the vector equation on the two axes $u$ and $v$, we have

\[ a \cos \theta_{ij0} + b \cos \psi_{ij0} + u = a \cos \theta_{ij} + b \cos \psi_{ij} \]
\[ a \sin \theta_{ij0} + b \sin \psi_{ij0} + v = a \sin \theta_{ij} + b \sin \psi_{ij}, \]

(9)

where $a, b$ are the lengths of links $A_{ij}B_{ij}, B_{ij}C_{ij}$, and $\theta_{ij0}, \psi_{ij0}, \theta_{ij}, \psi_{ij}$ are the initial and current position angles of links $A_{ij}B_{ij}, B_{ij}C_{ij}$.

Define two intermediate variables $c$ and $d$ as

\[ c = a \cos \theta_{ij0} + b \cos \psi_{ij0} + u \quad \text{and} \quad d = a \sin \theta_{ij0} + b \sin \psi_{ij0} + v. \]

(10)

Solving equation (9), we obtain passive joint variables as follows:

\[ \theta_{ij} = \gamma + \cos^{-1}\left(\frac{c^2 + d^2 + a^2 - b^2}{2a\sqrt{c^2 + d^2}}\right), \]
\[ \psi_{ij} = 2\pi + \gamma - \cos^{-1}\left(\frac{c^2 + d^2 + b^2 - a^2}{2b\sqrt{c^2 + d^2}}\right), \]

(11)

where $\gamma$ is defined by

\[ \cos \gamma = \frac{c}{\sqrt{c^2 + d^2}} \quad \text{and} \quad \sin \gamma = \frac{d}{\sqrt{c^2 + d^2}}. \]

Differentiating equation (9) and solving the equation, we have the passive joint displacement rates as follows

\[ \dot{\theta}_{ij} = \frac{\cos \psi_{ij}}{a \sin(\psi_{ij} - \theta_{ij})} \dot{u} + \frac{\sin \psi_{ij}}{a \sin(\psi_{ij} - \theta_{ij})} \dot{v} \]
\[ \dot{\psi}_{ij} = \frac{-\cos \theta_{ij}}{b \sin(\psi_{ij} - \theta_{ij})} \dot{u} + \frac{-\sin \theta_{ij}}{b \sin(\psi_{ij} - \theta_{ij})} \dot{v}. \]

(12)
3.2.2. PRB-D model. The PRB-D model of the designed FPM is:

\[
(X, Y, Z)^T = (x, y, z)^T + \sum_{i=1}^{n} G_i K_i^{-1} G_f F_i.
\]  

(13)

The terms of equation (13) are determined as follows:

The stiffness matrix \( K_i \) of a flexure hinge \( i \) is given by

\[
K_i = \begin{bmatrix}
K_{x-Fx} & 0 & 0 & 0 & 0 & 0 & K_{\alpha x-Mx} \\
0 & K_{y-Fy} & 0 & 0 & 0 & K_{\alpha y-Fy} & 0 \\
0 & 0 & K_{z-Fz} & 0 & 0 & K_{\alpha z-Fy} & 0 \\
0 & 0 & 0 & K_{\alpha z-Mz} & 0 & K_{\alpha x-Mx} & 0 \\
0 & 0 & K_{z-My} & 0 & K_{\alpha y-My} & 0 & 0 \\
0 & 0 & K_{y-Mz} & 0 & 0 & K_{\alpha y-Mz} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & K_{\alpha z-Mz}
\end{bmatrix},
\]

(14)

where the terms of the stiffness matrix are cited from Ref. [17] and can be seen in the Appendix.

Consider limb \( j \) of three 3-flexure limb group \( i \) of the frame (Fig. 7), the effect \( E_{ijw} \) of deformation of flexure hinges of the limb on the end-effector motion in the

![Figure 7. Configuration of one limb.](image)

![Figure 8. Force transformation.](image)
Micro-motion selective-actuation XYZ flexure parallel mechanism

The $w$-direction ($w$ follows the notation as given in the previous section) is

$$E_{ijw} = a \cos(\psi_{ij} - \theta_{ij})\varepsilon_{\alpha A_{ij}} + a \sin(\psi_{ij} - \theta_{ij})\varepsilon_{\alpha x A_{ij}} + \varepsilon_{z A_{ij}} + \varepsilon_{z B_{ij}} + b\varepsilon_{\alpha y B_{ij}} + \varepsilon_{z C_{ij}},$$

where $\varepsilon_{z A_{ij}}$, $\varepsilon_{z B_{ij}}$, and $\varepsilon_{z C_{ij}}$ are, respectively, the linear deformation of hinges $A_{ij}$, $B_{ij}$, and $C_{ij}$ along the $z$-direction of the local frame fixed to the corresponding flexure hinges (see Figs 7 and 8), $\varepsilon_{\alpha x A_{ij}}$ and $\varepsilon_{\alpha y A_{ij}}$ are, respectively, the rotation deformation of hinges $A_{ij}$ around the $x$- and $y$-axis of the local frame, and $\varepsilon_{\alpha y B_{ij}}$ is the rotation deformation of hinges $B_{ij}$ around the $y$-axis of the local frame.

Matrix $G_i$, expressing the flexure–hinge deformation contributed to the overall effect on the end-effector motion, can be obtained based on equation (15) as follows:

**Flexure hinge $A_{ij}$ on limb $j$ of 3-flexure limb group $i$:**

$$G_{A_{1j}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & a \cos(\psi_{1j} - \theta_{1j}) & 0 \\ 0 & 0 & 0 & 0 & a \sin(\psi_{1j} - \theta_{1j}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{A_{2j}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & a \cos(\psi_{2j} - \theta_{2j}) & 0 \\ 0 & 0 & 0 & 0 & a \sin(\psi_{2j} - \theta_{2j}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{A_{3j}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & a \cos(\psi_{3j} - \theta_{3j}) & 0 \\ 0 & 0 & 0 & 0 & a \sin(\psi_{3j} - \theta_{3j}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Flexure hinge $B_{ij}$ on limb $j$ of 3-flexure limb group $i$:**

$$G_{B_{1j}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & b & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix},$$

$$G_{B_{2j}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & b & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix},$$

$$G_{B_{3j}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & b & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}.$$

**Flexure hinge $C_{ij}$ on limb $j$ of 3-flexure limb group $i$:**

$$G_{C_{1j}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix},$$

$$G_{C_{2j}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}.$$
\[ G_{C3j} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}. \] (18)

Matrices \( G_{fi} \), transforming the reaction forces from the base frame to the flexure–hinge frame (local frame fixed to flexure hinge), are determined based on Fig. 8, relying only on the 3-flexure limb group and expressed as below:

3-flexure limb group 1

\[ G_{f1j} = \begin{bmatrix}
\cos \psi_{1j} & \sin \psi_{1j} & 0 & 0 & 0 & 0 \\
-\sin \psi_{1j} & \cos \psi_{1j} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \psi_{1j} & \sin \psi_{1j} & 0 \\
0 & 0 & 0 & -\sin \psi_{1j} & \cos \psi_{1j} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \]

3-flexure limb group 2

\[ G_{f2j} = \begin{bmatrix}
\cos \psi_{2j} & 0 & \sin \psi_{2j} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-\sin \psi_{2j} & 0 & \cos \psi_{2j} & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \psi_{2j} & \sin \psi_{2j} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\sin \psi_{2j} & 0 & \cos \psi_{2j}
\end{bmatrix}. \] (19)

3-flexure limb group 3

\[ G_{f3j} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos \psi_{3j} & \sin \psi_{3j} & 0 & 0 & 0 \\
0 & -\sin \psi_{3j} & \cos \psi_{3j} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \psi_{3j} & \sin \psi_{3j} \\
0 & 0 & 0 & 0 & -\sin \psi_{3j} & \cos \psi_{3j}
\end{bmatrix}. \]

To determine the reaction forces \( F_i \), first the actuation forces \( F_x, F_y \) and \( F_z \) are determined by using the virtual work method as follows:

\[ \delta W = F^T dx + P^T dX + \delta V = 0, \] (20)

where \( \delta W \) is the virtual work; \( F = (F_x, F_y, F_z)^T \) is actuation force; \( P = (P_X, P_Y, P_Z) \) is the load actuating at end-effector and \( \delta V \) is the differentiation of the potential energy accumulated at the flexure joints.
The potential energy is computed as
\[
V = \frac{1}{2} \sum_{i=1}^{3} \left\{ \sum_{j=1}^{3} \left[ K_{A_{ij}} (\theta_{ij} - \theta_{ij0})^2 + K_{C_{ij}} (\psi_{ij} - \psi_{ij0})^2 + K_{B_{ij}} (\psi_{ij} - \theta_{ij} - \psi_{ij0} + \theta_{ij0})^2 \right] + K_{S_{i}} (w - w_{0})^2 \right\},
\] (21)
where \( K_{A_{ij}}, K_{B_{ij}}, K_{C_{ij}} \) are, respectively, the torsional stiffness \( K_{\alpha-z-M_z} \) of the flexure hinges \( A_{ij}, B_{ij}, C_{ij} \), and \( K_{S_{i}} (i = 1, 2, 3) \) is the linear stiffness of the flexure prismatic joints of sides 1, 2 and 3 (see Appendix).

Solving equation (21), we obtain the actuation forces and reduce the number of variables in the computation of reactions. We assume that the center and the outside parts of the 3-flexure-limb groups of the passive 3D frame are absolutely rigid and the loads are distributed symmetrically on the limbs. The model of force analysis can be simplified to the 3-limb model as shown in Fig. 9.

Denote the 54 reactions at the flexure joints as
\[
\text{Joint } A_i: \ R_{A_{i}}^x, \ R_{A_{i}}^y, \ R_{A_{i}}^z, \ M_{A_{i}}^x, \ M_{A_{i}}^y, \ M_{A_{i}}^z \ (i = 1, 2, 3), \\
\text{Joint } B_i: \ R_{B_{i}}^x, \ R_{B_{i}}^y, \ R_{B_{i}}^z, \ M_{B_{i}}^x, \ M_{B_{i}}^y, \ M_{B_{i}}^z \ (i = 1, 2, 3), \\
\text{Joint } C_i: \ R_{C_{i}}^x, \ R_{C_{i}}^y, \ R_{C_{i}}^z, \ M_{C_{i}}^x, \ M_{C_{i}}^y, \ M_{C_{i}}^z \ (i = 1, 2, 3),
\]
where the reaction moments in the plane of limbs are determined as follows:
\[
M_{A_{i}}^x = K_{A_{i1}} (\theta_{11} - \theta_{110}); \ M_{C_{i}}^z = K_{C_{i1}} (\psi_{11} - \psi_{110}) \\
M_{B_{i}}^z = K_{B_{i1}} (\psi_{11} - \theta_{11} - \psi_{110} + \theta_{110})
\]
\[ M^{x}_{A2} = K_{A21} (\theta_{21} - \theta_{210}); \quad M^{x}_{C2} = K_{C21} (\psi_{21} - \psi_{210}) \]  
\[ M^{x}_{B2} = K_{B21} (\psi_{21} - \theta_{21} - \psi_{210} + \theta_{210}) \]  
\[ M^{y}_{A3} = K_{A31} (\theta_{31} - \theta_{310}); \quad M^{y}_{C3} = K_{C31} (\psi_{31} - \psi_{310}) \]  
\[ M^{y}_{B3} = K_{B31} (\psi_{31} - \theta_{31} - \psi_{310} + \theta_{310}) \]  

and the reactions at the input joints \( A_i \) are as follows:
\[ R^x_{A1} = F_z; \quad R^x_{A2} = F_x; \quad R^y_{A3} = F_y. \]  

Therefore, the remaining number of variables is 42. The reaction can be obtained by establishing and solving the Newton–Euler equations for 7 links with 42 equations.

4. RESOLUTION TRANSMISSION SCALE AND OPTIMAL DESIGN

4.1. Resolution transmission scale

Consider a non-redundant FPM with \( n \) degree-of-freedoms and denote \( dq = (dq_1, dq_2, \ldots, dq_n) \) as the vector of the infinitesimal displacements of \( n \) actuators and \( dX = (dX_1, dX_2, \ldots, dX_n) \) as the vector of the infinitesimal motions (displacements and/or orientations) of the end-effector. Let \( J \) (\( n \times n \) matrix) be the coefficient matrix of resolution transmission from actuators to the end-effector (Jacobian matrix). The amplification effect of infinitesimal actuator displacements on the infinitesimal motion of the end-effector can be described by
\[ dX = J dq. \]  

To examine the resolution of the FPM, we observe the available motions of the end-effector when one of the actuators carries out the smallest controllable motion that equal to the resolution \( \delta \) of the actuator:
\[ dq^2 = dq^T dq = dq_1^2 + dq_2^2 + \cdots + dq_n^2 = \delta^2. \]  

Assuming that the FPM operates in free singular workspace, equation (24) and equation (25) lead to:
\[ dX^T (JJ^T)^{-1} dX = \delta^2. \]  

Thus,
\[ dX^T (JJ^T)^{-1} dX = dX_1^2/\sigma_1^2 + dX_2^2/\sigma_2^2 + \cdots + dX_n^2/\sigma_n^2 = \delta^2, \]  

where \( \sigma_1, \sigma_2, \ldots, \sigma_n \) are the singular values of Jacobian \( J \) \cite{18} and
\[ JJ^T = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2). \]  

Equation (27) shows that the smallest motions (resolution) that the end-effector can perform along the principal axes are \( \sigma_1 \delta, \sigma_2 \delta, \ldots, \sigma_n \delta \), and the singular values \{\( \sigma_i \}\} are the coefficients of the resolution transmission from actuator to the end-effector.
The resolution of end-effector varies following the motion direction. Therefore, we use a concept called ‘resolution transmission scale’ to represent the average resolution of the end-effector at one particular point.

4.1.1. Resolution transmission scale. The resolution transmission (RT) scale is defined as the geometric mean of the resolution scale along all directions \( \sigma_i \).

\[
R = \sqrt[n]{\sigma_1 \sigma_2 \ldots \sigma_n}. \tag{29}
\]

Equations (28) and (29) show that the resolution transmission scale is computed as follows:

\[
R = \left(\det(JJ^T)\right)^{\frac{1}{2n}}. \tag{30}
\]

Since \( J \) is configuration dependent, the RT scale is a local performance measure that will be valid at a certain pose only. We use the average RT scale over the workspace:

\[
R_{\text{global}} = \frac{\int_W R \, dw}{\int_W dw}, \tag{31}
\]

where \( W \) is the workspace and \( dw \) is an infinitesimal area in which the RT scale is considered to be constant.

The resolution of an ultra-precision manipulation system must be uniform over the entire workspace. Therefore, to have a full assessment of resolution, we utilize an additional indicator, the uniformity \( U \), as parameter for variety evaluation given as below:

\[
U = \frac{R_{\text{min}}}{R_{\text{max}}}, \tag{32}
\]

where \( R_{\text{min}} \) and \( R_{\text{max}} \) are the minimum and the maximum of the RT scale over the entire workspace \( W \).

Both indices, the global RT scale \( R \) and the uniformity \( U \), require integration over the entire workspace \( W \) that is bounded by three workspaces:

\[
W = W_1 \cap W_2 \cap W_3, \tag{33}
\]

where \( W_1 \) is the workspace determined by the architecture reachability:

\[
W_1 = \{P(x, y, z)|\det(J) \neq 0\}; \tag{34}
\]

\( W_2 \) is the workspace determined by the actuator range:

\[
W_2 = \{P(x, y, z)|\forall i = 1, 2, \ldots, n: q_i \leq S\}, \tag{35}
\]

here the actuated joint variables \( \{q_i\} \) result from the inverse kinematics and \( S \) is the actuator range, and \( W_3 \) is the workspace based on the limited motion of flexure members:

\[
W_3 = \{P(x, y, z)|S_{\text{max}} < S_y\}, \tag{36}
\]
in the flexure members and the yield stress of the material of the FPM, respectively.

The maximal stress does not exist at the same position within the structure of the FPM when the end-effector moves within the workspace. Therefore, it is difficult to describe the closed-form boundary equation of workspace $W_3$. Consequently, the integration over the workspace based on closed-form description of the boundary cannot be guaranteed. Therefore, a numerical approach is proposed here to determine the workspace and the global indices (Fig. 10). The features of the method are the partition of a covering volume $A$ into small sampling pieces \{$v_i$\}.
Table 1.
Results of optimization

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Aluminum alloy with yield stress $S_y = 500$ MPa and Young’s modulus $E = 71$ GPa</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a = 35$ mm, $h = 60$ mm</td>
</tr>
<tr>
<td></td>
<td>$K_{\theta} = K_{\psi} = 1240$ N-mm/rad (7 N-mm/°), $K_S = 0.25$ N/mm</td>
</tr>
<tr>
<td></td>
<td>$U_{\min} = 0.6$</td>
</tr>
<tr>
<td>Outputs</td>
<td>Dimension: $k_{ab} = b/a = 1.14$</td>
</tr>
<tr>
<td></td>
<td>Free shape/initial posture: $\theta_{ij0} = 80^\circ$, $\psi_{ij0} = 165^\circ$</td>
</tr>
<tr>
<td></td>
<td>Global RT scale: $R_{\text{global}} = 6.1$ and uniformity $U = 0.61$</td>
</tr>
</tbody>
</table>

and the piece-by-piece test in order to know whether the piece $v_i$ belongs to the workspace. The dimension of the samples is determined based on the required accuracy. The samples that belong to the workspace are added to workspace variable $W$ of the algorithm and the local RT scales at those samples are added to the summing variable $\text{Sum}R$. Samples are compared to identify the maximal and the minimal values of RT scale. After all of the sampling pieces are examined, the workspace is obtained and stored in variable $W$. The summing variable $\text{Sum}R$ is divided by the number of samples identified within the workspace in order to obtain the global RT scale $R_{\text{global}}$. The uniformity $U$ is determined based on the minimal and the maximal values of the local RT scales.

### 4.2. Optimal design

In general, the global scale of RT needs to be minimized to obtain the highest resolution. However, the Jacobian $J$ is a coefficient matrix of the force transmission from actuators to the end-effector. The decrease of the RT scale leads to the increase of the actuation load requirement and causes difficulty in the actuator selection because the precise actuator usually possesses small load capability. To satisfy both requirements, i.e., high resolution and small actuation load, the global RT scale $R_{\text{global}}$ needs to be as close to 1 as possible. Moreover, to limit the variety of the RT scale in the workspace, the uniformity must be kept as close to unity as possible. The optimal design of the FPM can be formulated as:

\[
\text{Minimize: } |M_{\text{global}} - 1|, \quad (37)
\]

Subject to: $U \geq U_{\min}$, \quad (38)

where $U_{\min}$ is the allowed lower limit of uniformity.

Optimization variables: The dimension of the PRB and PRB-D models and the initial posture/free shape of the FPM.

The conjugate direction method is used to reduce computation resource and to speed up the searching process. The result of the optimization is shown in Table 1.
5. EXPERIMENTS

Theoretically, the workspace of the \(XYZ\) stage is determined by equation (33) and yield stress \(S_y = 500\) MPa. Practically, the picomotors with the actuation force limit of 22 N are used to actuate the stage. Therefore, only 20\% of the max strain (maximum strain caused when \(S_y\) is used as the limit of strain in equation (36)) is accepted and the developed \(XYZ\) stage has workspace of 800 \(\mu\)m \(\times\) 800 \(\mu\)m \(\times\) 800 \(\mu\)m. An experiment is set up to verify the selective-actuation characteristic and the PRB-D model (see Fig. 11). The motion of the end-effector is measured by a probe with a resolution of 0.1 \(\mu\)m, an increase of measuring force of 0.04 N/mm and measurement range from \(-0.37\) mm to \(+1.6\) mm. The stiffness of measuring action is very small (0.04 N/mm), the deformation of probe’s stand is negligible and the measuring result can be considered the motion of end-effector. When the actuator in the \(X\)-direction is working, the unexpected motions \(dZ\) and \(dY\) of the end-effector along the \(Z\)- and \(Y\)-axis is measured. The unexpected motions are less than 7\% (see Fig. 12). To verify the advantage of the PRB-D model versus the PRB model, the solutions of the kinematics based on the PRB-D and PRB models are applied to control the FPM. The PRB model’s error increases very early even when the end-effector displacement is under 5 \(\mu\)m, whereas the errors of PRB-D model can be negligible. The error produced by the PRB-D model is only 1/3 of that by the PRB model (Fig. 12). It is possible to study the tendency of the error of PRB-D model in Fig. 13 and provide compensation for the control of the system.

![Figure 11. Selective-actuation XYZ flexure parallel mechanism and experimental setup.](image-url)
Figure 12. Unexpected motion $dZ$ versus the actuated motion $X$.

Figure 13. Comparison between pseudo-rigid-body model and pseudo rigid-body model with consideration of deformation.

A feedback system is established to control the developed FPM (see Fig. 14). The motion of the end-effector of FPM is measured by the optical sensors (LC-2430, Keyence) with a resolution of 0.02 $\mu$m and measurement range of 1 mm,
and the data obtained are fed back to the controlling program. We let the FPM provide a series of small motions. The range of the motions is gradually reduced and the response of end-effector position is recorded to determine the smallest controllable motion of the FPM. The responses of the end-effector position to the
position commands with ranges of 300, 20, 0.2 and 0.1 µm are illustrated in Fig. 15. Resolution of the developed FPM equipped with a feedback system is determined experimentally as 0.2 µm, whereas the resolution as predicted by the design is \( \delta R_{\text{global}} \approx 0.18 \) µm (resolution of the used actuator \( \delta \approx 0.03 \) µm and global RT scale \( R_{\text{global}} \approx 6.1 \)).

6. CONCLUSIONS

The design of a translational FPM with three de-coupled motions is presented. The selective-actuation configuration allows the designed mechanism to be used in a multi-positioning task with different requirements for the number of DOF. Moreover, the expensive precision actuator can be removed and reused for another task. The selective-actuation configuration is also useful for control aspect. It allows the user to adjust the mechanism easily and rapidly to obtain the precise desired position. A design of the micro-motion mechanism is needed to consider the factors of resolution. Therefore, the resolution transmission scale is defined to evaluate the resolution criteria of the mechanism. The defined scale is utilized as the criteria of the optimal design for dimension and free shape of the flexure parallel mechanism. The PRB-D model based on the PRB model currently used and the addition of the deformation of flexure members is specified to obtain the precise analytical model for the position control and the formulation of the optimal design. The experiment confirmed the selective-actuation characteristic of the developed FPM and the validity of the PRB-D model in precision positioning. Also, the experimental result proves that the desired resolution of the FPM can be achieved by the optimization based on the global value and the uniformity of the resolution transmission scale.

Acknowledgements

This project is financially sponsored by Ministry of Education, Singapore under ARP RG 06/02 and Innovation in Manufacturing Science and Technology (IMST) program of Singapore-MIT Alliance. We would like to thank Prof. David Butler for his highly valuable comments and suggestions.

REFERENCES

16. Laboratoire de Systemes Robotiques EPFL-LSRO, website http://lsro.epfl.ch

**APPENDIX**

The inverse of stiffness of flexure hinges

\[
\begin{align*}
\frac{1}{K_{x-Fx}} &= \frac{\pi (r/t)^{1/2} - 2.57}{Ee}, \\
\frac{1}{K_{y-Fy}} &= \frac{9\pi r^{5/2}}{2Eet^{5/2}} + \frac{\pi (r/t)^{1/2} - 2.57}{Ge}, \\
\frac{1}{K_{z-Fz}} &= \frac{12\pi r^2}{Ee^3} \left[ (r/t)^{1/2} - 1/4 \right] + \frac{\pi (r/t)^{1/2} - 2.57}{Ge}, \\
\frac{1}{K_{ax-Mx}} &= \frac{9\pi r^{1/2}}{8Get^{5/2}}, \\
\frac{1}{K_{ay-My}} &= \frac{12}{Ee^3} \left[ (r/t)^{1/2} - 2.57 \right], \\
\frac{1}{K_{az-Mz}} &= \frac{9\pi r^{1/2}}{2Eet^{5/2}}, \\
\frac{1}{K_{y-Mz}} &= \frac{1}{K_{az-Fy}} = \frac{9\pi r^{3/2}}{2Eet^{5/2}}.
\end{align*}
\]
Figure 16. Flexure hinge dimension.

\[
\frac{1}{K_{z-M_y}} = \frac{1}{K_{\alpha_y-F_z}} = \frac{12r}{Ee^3} \left[ \pi (r/t)^{1/2} - 2.57 \right],
\]

where \( e, t \) and \( r \) are, respectively, the thickness of the link, the thickness and the radius of the hinge described in Fig. 16; \( E \) and \( G \) are the Young’s modulus and the shear modulus of the used material.

Stiffness of a flexure prismatic joint double:

\[
K_{S_i} = \frac{8K_{a_z-M_z}}{L^2},
\]

(A2)

where \( L \) is the length of the links in the double linear spring used as a flexure prismatic joint.