Generalized integral transforms and convolution products on function space

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In this paper, we use a Gaussian process to define a generalized integral transform (GIT) and a generalized convolution product (GCP) of functionals defined on a function space. We establish the existence and some properties for the GIT, the GCP and the inverse integral transform. Finally, we prove a Fubini theorem for the GIT and the GCP.

Keywords: generalized integral transform; generalized convolution product; inverse integral transform; Gaussian process

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1. Introduction and definitions

Let $C_0[0, T]$ denote one-parameter Wiener space; that is the space of continuous real-valued functions $x$ on $[0, T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0, T]$, and let $m$ denote the Wiener measure. $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional $F$ by

$$\int_{C_0[0,T]} F(x) \, dm(x).$$

A subset $B$ of $C_0[0, T]$ is said to be scale-invariant measurable provided $\rho B$ is $\mathcal{M}$-measurable for all $\rho \geq 0$, and a scale-invariant measurable set $N$ is said to be a scale-invariant null set provided $m(\rho N) = 0$ for all $\rho \geq 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) [10].

Let $K = K_0[0, T]$ be the space of all complex-valued continuous functions defined on $[0, T]$ which vanish at $t = 0$. Cameron and Martin [1] introduced a transform of functionals defined on
which is called the Fourier–Wiener transform

\[ \mathcal{F}_{1,i}(F)(y) = \int_{C_0[0,T]} F(x + iy) \, dm(x), \quad y \in K. \]

Later it was modified in [2] which is called the modified Fourier–Wiener transform

\[ \mathcal{F}_{\sqrt{2},i}(F)(y) = \int_{C_0[0,T]} F(\sqrt{2}x + iy) \, dm(x), \quad y \in K. \]

Lee [13] introduced the integral transform \( \mathcal{F}_{\gamma,\beta} \) in his unifying paper,

\[ \mathcal{F}_{\gamma,\beta}(F)(y) = \int_{C_0[0,T]} F(\gamma x + \beta y) \, dm(x), \quad y \in K. \]

For certain values of the parameters \( \gamma \) and \( \beta \) and for certain classes of functionals, the Fourier–Wiener transform [1], the modified Fourier–Wiener transform [2], the Fourier–Feynman transform and the Gauss transform are special cases of Lee’s integral transform \( \mathcal{F}_{\gamma,\beta} \). These transforms play an important role in studying stochastic processes and functional integrals on infinite dimensional spaces.

In [6,12], the authors studied some integral transforms and established several basic formulas involving these integral transforms, convolution products and inverse integral transforms (IITs) of functionals in \( L^2(C_0[0,T]) \). In [5], the authors gave a necessary and sufficient condition that a functional \( F \) in \( L^2(C_{a,b}[0,T]) \) has a generalized integral transform (GIT) also belonging to \( L^2(C_{a,b}[0,T]) \), where \( C_{a,b}[0,T] \) is a more general function space than the Wiener space \( C_0[0,T] \). In [8,9], the authors introduced the concepts of the generalized Feynman integral and the generalized Fourier–Feynman transform via a Gaussian process, and that was further developed in [4,14].

The main purpose of this paper is fourfold. We first use the concept of the special Gaussian process introduced in [8] and used in [4,9,14] to define a GIT and a generalized convolution product of functionals defined on \( K \). Secondly and thirdly, we define a class \( S_\alpha \) of functionals which contains many functionals of interest in the Feynman integration theory and quantum mechanics, and then establish the existence and basic formulas for the GIT, the GCP and the IIT of functionals in \( S_\alpha \). Finally, we establish a Fubini theorem for GITs and GCPs of functionals in \( S_\alpha \).

First we recall the Gaussian process introduced in [8] and used in [4,9,14]. Let \( h \) be a non-zero element of \( L^2[0,T] \). We define the Gaussian process \( Z \) by

\[ Z(x,t) = \int_0^t h(s) \, d\tilde{x}(s), \]

where \( \int_0^t h(s) \, d\tilde{x}(s) \) denotes the Paley–Wiener–Zygmund integral. For each \( v \in L^2[0,T] \), let \( \langle v, x \rangle = \int_0^T v(t) \, d\tilde{x}(t). \) From [8], we note that

\[ \langle v, Z(x, \cdot) \rangle = \langle vh, x \rangle \]

for \( h \in L^\infty[0,T] \) and s-a.e. \( x \in C_0[0,T] \). Thus, throughout this paper, we require \( h \) to be in \( L^\infty[0,T] \) rather than simply in \( L^2[0,T] \).

**Remark 1.1** The Wiener process \( x(t) \) used in [6,7,11,12] is a stationary Gaussian process with mean 0 and variance \( t \). The Gaussian process \( Z(x, t) \) used in this paper is a non-stationary Gaussian process with mean 0 and variance \( \int_0^t h^2(s) \, ds \).
Now we are ready to define the GIT via the Gaussian process of functionals defined on $K$.

**Definition 1.2** Let $F$ be a functional defined on $K$. For each pair of non-zero complex numbers $\gamma$ and $\beta$, the GIT $\mathcal{F}_{\gamma, \beta, h}$ of $F$ is defined by

$$
\mathcal{F}_{\gamma, \beta, h}(F)(y) \equiv \int_{C_{0}[0, T]} F(\gamma Z(x, \cdot) + \beta y) \, dm(x), \quad y \in K,
$$

if it exists.

**Remark 1.3** When $h(t) \equiv 1$ on $[0, T]$, then $Z(x, t) = x(t)$ and so our integral transforms $\mathcal{F}_{\gamma, \beta, h}$ is the integral transform introduced by Lee [13] and used in [6,7,11,12]. In particular, when $\gamma = 1$ and $\beta = i$, $\mathcal{F}_{1, i, 1}$ is the Fourier–Wiener transform introduced by Cameron and Martin [1], and when $\gamma = \sqrt{2}$ and $\beta = i$, $\mathcal{F}_{\sqrt{2}, i, 1}$ is the modified Fourier–Wiener transform used by Cameron and Martin [2].

Next, we define the GCP via the Gaussian process of functionals defined on $K$.

**Definition 1.4** Let $F$ and $G$ be functionals defined on $K$. Then the GCP $(F \ast G)_\gamma$ of $F$ and $G$ is defined by

$$
(F \ast G)_\gamma(y) = \int_{C_{0}[0, T]} F\left(\frac{y + \gamma Z(x, \cdot)}{\sqrt{2}}\right) G\left(\frac{y - \gamma Z(x, \cdot)}{\sqrt{2}}\right) \, dm(x), \quad y \in K,
$$

if it exists.

**Remark 1.5** The generalized transforms and convolutions appearing in [4,8,9,14] are defined in terms of analytic Feynman integrals which are defined using values of $\gamma$ with Re$(\gamma) \geq 0$.

Now we finish this section by a simple example for GITs of functionals defined on $K$.

**Example 1.6** Let $F$ and $G$ be functionals defined by $F(x) = x(t)$ and $G(x) = \int_{0}^{T} \exp\{x(t)\} \, dt$. Then we have the following integration formulas for GITs of $F$ and $G$;

$$
\mathcal{F}_{\gamma, \beta, h} F(y) = \gamma^2 \int_{0}^{T} h^2(s) \, ds + \beta^2 y^2(t),
$$

and

$$
\mathcal{F}_{\gamma, \beta, h} G(y) = \int_{0}^{T} \exp\left\{\beta y(t) + \frac{\gamma^2}{2} \int_{0}^{t} h^2(s) \, ds\right\} \, dt
$$

for $y \in K$.

2. Existences theorems

In this section, we define a class $\mathcal{S}_{\alpha}$ of functionals which will be used in this paper. We then establish the existence for the GIT, the GCP and the IIT of functionals in $\mathcal{S}_{\alpha}$.

For each complex number $\alpha$, let $\mathcal{S}_{\alpha}$ be the class of functionals of the form

$$
F(x) = \int_{L_{2}[0, T]} \exp\{\alpha \langle v, x \rangle\} \, df(v)
$$

for s-a.e. $x \in C_{0}[0, T]$, where $f$ is an element of $M(L_{2}[0, T])$, the space of complex-valued countably additive Borel measures on $L_{2}[0, T]$. 
Remark 2.1 Note that \( M(L_2[0, T]) \) is a Banach algebra under the total variation norm where the convolution is taken as the multiplication. One can show that the correspondence \( f \rightarrow F \) is injective, carries convolution into pointwise multiplication and that for each complex number \( \alpha \), the space \( S_\alpha \) is a Banach algebra with norm
\[
\|F\| = \|f\| = \int_{L_2[0, T]} |d f(v)|.
\]
In particular, if \( \alpha = i \), then \( S_i \) is the Banach algebra \( S \) introduced by Cameron and Storvick [3].

Next, we recall an integration formula which will be used several times in this paper. For each \( \alpha \in \mathbb{C} \) and for \( v \in L_2[0, T] \),
\[
\int_{C_0[0, T]} \exp(\alpha \langle v, x \rangle) \, dm(x) = \exp\left(\frac{\alpha^2}{2} \|v\|^2 \right). \tag{2.2}
\]

Definition 2.2 Let \( \mathbb{C} \) be the class of all complex numbers. For each \( \alpha \in \mathbb{C} \), let
\[
E_\alpha \equiv \{ (\gamma, \beta) \in \mathbb{C} \times \mathbb{C} : \text{Re}(\alpha^2 \gamma^2) \leq 0 \},
\]
and
\[
A_\alpha \equiv \{ (\gamma, \beta) \in E_\alpha : \gamma^2 + \beta^2 = 1 \}.
\]

Note that if \( (\gamma, \beta) \) is an element of \( E_\alpha \) (or \( A_\alpha \)), it may not be an element of \( E_{\alpha \beta} \) (or \( A_{\alpha \beta} \)).

In our first theorem, we obtain a formula for the GIT of functionals from \( S_\alpha \) into \( S_{\alpha \beta} \).

Theorem 2.3 Let \( F \in S_\alpha \) be given by Equation (2.1). Then for all \( (\gamma, \beta) \in E_\alpha \), the GIT \( \mathcal{F}_{\gamma, \beta, h}F \) exists and is given by the formula
\[
\mathcal{F}_{\gamma, \beta, h}F(y) = \int_{L_2[0, T]} \exp\left(\alpha \beta \langle v, y \rangle + \frac{\alpha^2 \gamma^2}{2} \|vh\|^2 \right) \, df(v) \tag{2.3}
\]
for s-a.e. \( y \in C_0[0, T] \). Furthermore the GIT \( \mathcal{F}_{\gamma, \beta, h}F \), as a function of \( y \), is an element of \( S_{\alpha \beta} \).

In fact,
\[
\mathcal{F}_{\gamma, \beta, h}F(y) = \int_{L_2[0, T]} \exp(\alpha \beta \langle v, y \rangle) \, d\phi_1(v),
\]
where \( \phi_1 \) is an element of \( M(L_2[0, T]) \), defined as in the following proof.
Proof Using Equations (1.1) and (2.2) it follows that for s-a.e. $y \in C_0[0, T],$

$$\mathcal{F}_{\gamma, \beta, h} F(y) = \int_{C_0[0, T]} F(\gamma Z(x, \cdot) + \beta y) \, dm(x)$$

$$= \int_{C_0[0, T]} \int_{L_2[0, T]} \exp\{\alpha \gamma (vh, x) + \alpha \beta \langle v, y \rangle\} \, df(v) \, dm(x)$$

$$= \int_{L_2[0, T]} \exp\left\{ \alpha \beta \langle v, y \rangle + \frac{\alpha^2 \gamma^2}{2} \|vh\|_2^2 \right\} \, df(v)$$

(2.4)

and so we have established Equation (2.3). Now let $\phi_1$ be a set function defined by

$$\phi_1(E) = \int_E \exp\left\{ \frac{\alpha^2 \gamma^2}{2} \|vh\|_2^2 \right\} \, df(v)$$

for $E \in B(L_2[0, T])$. Then $\phi_1$ is an element of $M(L_2[0, T])$ since $\text{Re}(\alpha^2 \gamma^2) \leq 0$ and so the last expression in Equation (2.4) becomes

$$\int_{L_2[0, T]} \exp\{\alpha \beta \langle v, y \rangle\} \, d\phi_1(v).$$

Hence the GIT $\mathcal{F}_{\gamma, \beta, h} F$ is an element of $\mathcal{S}_{\alpha \beta}$. ■

Remark 2.4 Note that for a given non-zero complex number $\alpha = a + ib$, there are many complex numbers $\gamma$ so that $\text{Re}(\alpha^2 \gamma^2) \leq 0$. In fact, we can take $\gamma = c + id$ so that

$$\begin{cases} 
  c^2 \geq d^2 & \text{if } a = 0, b \neq 0 \\
  c^2 \leq d^2 & \text{if } a \neq 0, b = 0 \\
  (a^2 - b^2)(c^2 - d^2) \leq 4abcd & \text{otherwise}
\end{cases}$$

In the next theorem, we obtain a formula for the GCP of functionals from $\mathcal{S}_\alpha$ into $\mathcal{S}_\alpha$.

Theorem 2.5 Let $F$, $h$, and $f$ be as in Theorem 2.3. Let $G \in \mathcal{S}_\alpha$ be given by

$$G(x) = \int_{L_2[0, T]} \exp\{\alpha \langle w, x \rangle\} \, dg(w)$$

for s.a.e. $x \in C_0[0, T]$, where $g$ is an element of $M(L_2[0, T])$. Then for all $(\gamma, \beta) \in E_\alpha$, the GCP $(F * G)_\gamma$, of $F$ and $G$ exists and is given by the formula

$$(F * G)_\gamma(y) = \int_{L_2[0, T]} \int_{L_2[0, T]} \exp\left\{ \frac{\alpha}{\sqrt{2}} \langle v + w, y \rangle + \frac{\alpha^2 \gamma^2}{4} \| (v - w)h\|_2^2 \right\} \, df(v) \, dg(w)$$

(2.5)

for s.a.e. $y \in C_0[0, T]$. Furthermore, the GCP $(F * G)_\gamma$, as a function of $y$, is an element of $\mathcal{S}_\alpha$. In fact,

$$(F * G)_\gamma(y) = \int_{L_2[0, T]} \exp\{\alpha \langle k, y \rangle\} \, d\phi_2(k),$$

where $\phi_2$ is an element of $M(L_2[0, T])$, as in the following proof.
Proof Using Equations (1.2) and (2.2), it follows that for s-a.e. \( y \in C_0[0, T] \),

\[
(F \ast G)_\gamma(y) = \int_{C_0[0,T]} F\left(\frac{y + \gamma Z(x, \cdot)}{\sqrt{2}}\right) G\left(\frac{y - \gamma Z(x, \cdot)}{\sqrt{2}}\right) \, dm(x)
\]

\[
= \int_{C_0[0,T]} \int_{L_2[0,T]} \int_{L_2[0,T]} \exp\left\{ \frac{\alpha y}{\sqrt{2}} (v - w) h, x \right\} \\
+ \frac{\alpha}{\sqrt{2}} (v + w, y) \right\} \, df(v) \, dg(w) \, dm(x)
\]

\[
= \int_{L_2[0,T]} \int_{L_2[0,T]} \exp\left\{ \frac{\alpha}{\sqrt{2}} (v + w, y) + \frac{\alpha^2 \gamma^2}{4} \| (v - w) h \|_2^2 \right\} \, df(v) \, dg(w)
\]

and so we have established Equation (2.5). Now, let \( \phi \) be a set function defined by

\[
\phi(E) = \int_E \exp\left\{ -\frac{\alpha^2 \gamma^2}{4} \| (v - w) h \|_2^2 \right\} \, df(v) \, dg(w)
\]

for \( E \in B(L_2[0, T] \times L_2[0, T]) \) and \( \rho : L_2[0, T] \times L_2[0, T] \to L_2[0, T] \) be a function defined by \( \rho(v, w) = (v + w)/\sqrt{2} \). Then \( \phi_2 = \phi \circ \rho^{-1} \) is an element of \( M(L_2[0, T]) \) since \( \text{Re}(\alpha^2 \gamma^2) \leq 0 \) and so the last expression in Equation (2.6) becomes

\[
\int_{L_2[0,T]} \exp[\alpha(k, y)] \, d\phi_2(k).
\]

Hence the GCP \((F \ast G)_\gamma\) is an element of \( S_\alpha \). \( \blacksquare \)

In the next theorem, we establish the IIT of the GIT.

**Theorem 2.6** Let \( F, h \) and \( f \) be as in Theorem 2.3. Then for all \((\gamma, \beta)\) with \( \text{Re}(\alpha^2 \gamma^2) = 0 \) and \( \text{Re}(\alpha^2 \gamma^2 / \beta^2) = 0 \),

\[
\mathcal{F}_{i(\gamma/\beta), (1/\beta), h}(\mathcal{F}_{\gamma, \beta, h} F)(y) = F(y) = \mathcal{F}_{\gamma, \beta, h}(\mathcal{F}_{i(\gamma/\beta), (1/\beta), h} F)(y)
\]

for s-a.e. \( y \in C_0[0, T] \). That is to say, \( \mathcal{F}_{i(\gamma/\beta), (1/\beta), h} \) is the IIT of the GIT.

**Proof** Using Equations (2.3), (1.1) and (2.2) it follows that for s-a.e. \( y \in C_0[0, T] \),

\[
\mathcal{F}_{i(\gamma/\beta), (1/\beta), h}(\mathcal{F}_{\gamma, \beta, h} F)(y) = \int_{C_0[0,T]} \mathcal{F}_{\gamma, \beta, h} F\left( i \frac{\gamma}{\beta} Z(x, \cdot) + \frac{1}{\beta} y \right) \, dm(x)
\]

\[
= \int_{C_0[0,T]} \int_{L_2[0,T]} \exp\left\{ \alpha(v, y) + i \alpha \gamma (vh, x) \\
+ \frac{\alpha^2 \gamma^2}{2} \| vh \|_2^2 \right\} \, df(v) \, dm(x)
\]

\[
= \int_{L_2[0,T]} \exp[\alpha(v, y)] \, df(v) = F(y).
\]
Using again Equations (2.3), (1.1) and (2.2) it follows that for s-a.e. \( y \in C_0[0, T] \),

\[
\mathcal{F}_{\gamma, \beta, h}(\mathcal{F}_{(y/\beta, (1/\beta), h} F)(y) = \int_{C_0[0,T]} \mathcal{F}_{i(\gamma/\beta), (1/\beta), h} F(y Z(x, \cdot) + \beta y) \, dm(x)
\]

\[
= \int_{C_0[0,T]} \int_{L_2[0,T]} \exp \left\{ \frac{\alpha \langle v, y \rangle}{\sqrt{2}} + \frac{\alpha^2 \gamma^2}{2} \right\} d f(v) \, dm(x)
\]

\[
= \int_{L_2[0,T]} \exp \left\{ \frac{\alpha \langle v, y \rangle}{\sqrt{2}} \right\} d f(v) = F(y),
\]

which completes the proof of Theorem 2.6.

3. Basic formulas

In the first theorem of this section, we establish a basic formula that the GIT of the GCP is the product of their GITs.

**Theorem 3.1** Let \( F, G, f, g \) and \( h \) be as in Theorem 2.5. Then for all \((\gamma, \beta) \in E_{\alpha}\),

\[
\mathcal{F}_{\gamma, \beta, h}(F \ast G)_{\gamma}(y) = \mathcal{F}_{\gamma, \beta, h} F \left( \frac{y}{\sqrt{2}} \right) \mathcal{F}_{\gamma, \beta, h} G \left( \frac{y}{\sqrt{2}} \right)
\]

as elements of \( S_{\alpha \beta} \). Also, both sides of the expression in Equation (3.1) are given by the formula

\[
\int_{L_2[0,T]} \int_{L_2[0,T]} \exp \left\{ \frac{\alpha \beta}{\sqrt{2}} \langle v + w, y \rangle + \frac{\alpha^2 \gamma^2}{2} (\| v h \|^2 + \| w h \|^2) \right\} \, d f(v) \, d g(w).
\]

**Proof** Using Equations (1.1), (2.5) and (2.2) it follows that for s-a.e. \( y \in C_0[0, T] \),

\[
\mathcal{F}_{\gamma, \beta, h}(F \ast G)_{\gamma}(y) = \int_{C_0[0,T]} (F \ast G)_{\gamma}(y Z(x, \cdot) + \beta y) \, dm(x)
\]

\[
= \int_{C_0[0,T]} \int_{L_2[0,T]} \int_{L_2[0,T]} \exp \left\{ \frac{\alpha \gamma}{\sqrt{2}} (v + w) h, x \right\}
\]

\[
+ \frac{\alpha \beta}{\sqrt{2}} (v + w, y) + \frac{\alpha^2 \gamma^2}{4} (\| v - w \|^2) \right\} \, d f(v) \, d g(w) \, dm(x)
\]

\[
= \int_{L_2[0,T]} \int_{L_2[0,T]} \exp \left\{ \frac{\alpha \beta}{\sqrt{2}} \langle v + w, y \rangle
\]

\[
+ \frac{\alpha^2 \gamma^2}{2} (\| v h \|^2 + \| w h \|^2) \right\} \, d f(v) \, d g(w).
\]

The last equality in Equation (3.2) follows immediately from Equation (2.2) and the parallelogram law of the norm. On the other hand, using Equation (2.3) it follows that for s-a.e. \( y \in C_0[0, T] \),

\[
\mathcal{F}_{\gamma, \beta, h} F \left( \frac{y}{\sqrt{2}} \right) = \int_{L_2[0,T]} \exp \left\{ \frac{\alpha \beta}{\sqrt{2}} (v, y) + \frac{\alpha^2 \gamma^2}{2} \| v h \|^2 \right\} \, d f(v),
\]
and

\[ \mathcal{F}_{\gamma, \beta, h} G \left( \frac{y}{\sqrt{2}} \right) = \int_{L_2[0,1]} \exp \left\{ \frac{\alpha \beta}{\sqrt{2}} \langle w, y \rangle + \frac{\alpha^2 \gamma^2}{2} \|wh\|^2 \right\} dg(w), \]

which completes the proof of Theorem 3.1.

In the next theorem, we obtain two basic formulas for the GCPs, whose proofs follow immediately from Equations (3.1) and (2.7).

**Theorem 3.2**  Let \( F, G, f, g \) and \( h \) be as in Theorem 3.1. Then for all \((\gamma, \beta)\) with \( \text{Re}(\alpha^2 \gamma^2) = 0 \),

\[ (F \ast G)_{\gamma}(y) = \mathcal{F}_{l(\gamma/\beta), (1/\beta), h}(\mathcal{F}_{\gamma, \beta, h} F (+/\sqrt{2}) \mathcal{F}_{\gamma, \beta, h} G (+/\sqrt{2}))(y) \]  

(3.3)

as elements of \( S_{\alpha \beta} \). Also, by interchanging two pairs \((\gamma, \beta)\) and \((i(\gamma/\beta), (1/\beta))\) in Equation (3.3), for all \((i(\gamma/\beta), (1/\beta))\) with \( \text{Re}(\alpha^2 \gamma^2/\beta^2) = 0 \), we can establish a basic formula

\[ (F \ast G)_{i \gamma}(y) = \mathcal{F}_{\gamma, \beta, h}(\mathcal{F}_{i(\gamma/\beta), (1/\beta), h} F (+/\sqrt{2}) \mathcal{F}_{i(\gamma/\beta), (1/\beta), h} G (+/\sqrt{2}))(y) \]  

(3.4)

as elements of \( S_{\alpha} \).

The following formulas (3.5) through (3.8) follow from Equations (3.1) and (3.3) by letting \( F(y) = G(y) \) or by letting \( G(y) \) identically one on \( C_0[0, T] \).

**Corollary 3.3**  Let \( F, f \) and \( h \) be as in Theorem 3.1. Then for all \((\gamma, \beta)\) with \( \text{Re}(\alpha^2 \gamma^2) = 0 \),

\[ \mathcal{F}_{\gamma, \beta, h}(F \ast F)_{\gamma}(y) = \left[ \mathcal{F}_{\gamma, \beta, h} F \left( \frac{y}{\sqrt{2}} \right) \right]^2 \]  

(3.5)

as elements of \( S_{\alpha \beta} \),

\[ \mathcal{F}_{\gamma, \beta, h}(F \ast 1)_{\gamma}(y) = \mathcal{F}_{\gamma, \beta, h} F \left( \frac{y}{\sqrt{2}} \right) = \mathcal{F}_{\gamma, \beta, h}(1 \ast F)_{\gamma}(y) \]  

(3.6)

as elements of \( S_{\alpha \beta} \),

\[ (F \ast F)_{\gamma}(y) = \mathcal{F}_{i(\gamma/\beta), (1/\beta), h}(\mathcal{F}_{\gamma, \beta, h} F (+/\sqrt{2})^2)(y) \]  

(3.7)

as elements of \( S_{\alpha} \) and

\[ (F \ast 1)_{\gamma}(y) = \mathcal{F}_{i(\gamma/\beta), (1/\beta), h}(\mathcal{F}_{\gamma, \beta, h} F (+/\sqrt{2}))(y) \]  

(3.8)

as elements of \( S_{\alpha} \). Similarly by interchanging two pairs \((\gamma, \beta)\) and \((i(\gamma/\beta), (1/\beta))\) in Equations (3.5)–(3.8), we can establish other basic formulas, as in Theorem 3.2.

In the next theorem, we obtain a basic formula for the GCP of GITs.

**Theorem 3.4**  Let \( F, G, f, g \) and \( h \) be as in Theorem 3.1. Then for all \((\gamma, \beta) \in E_{\alpha} \),

\[ (\mathcal{F}_{\gamma, \beta, h} F \ast \mathcal{F}_{\gamma, \beta, h} G)_{i(\gamma/\beta)}(y) = \mathcal{F}_{\gamma, \beta, h}(F (+/\sqrt{2}) G (+/\sqrt{2}))(y) \]  

(3.9)

as elements of \( S_{\alpha \beta} \).

**Proof**  Equation (3.9) follows from Equation (3.4) with the functionals \( F \) and \( G \) being replaced by the functionals \( \mathcal{F}_{\gamma, \beta} F \) and \( \mathcal{F}_{\gamma, \beta} G \), respectively. \[ \blacksquare \]
Corollary 3.5  Let $F, G, f, g$ and $h$ be as in Theorem 3.1. Then for all $(i(\gamma/\beta), (1/\beta)) \in E_\alpha$,

$$(\mathcal{F}_{i(\gamma/\beta),(1/\beta),h} F \ast \mathcal{F}_{i(\gamma/\beta),(1/\beta),h} G)(y) = \mathcal{F}_{i(\gamma/\beta),(1/\beta),h}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2}))(y) \quad (3.10)$$

as elements of $S_{\alpha/\beta}$, and for all $(\gamma, \beta) \in E_\alpha$,

$$(\mathcal{F}_{\gamma,\beta,h} F \ast \mathcal{F}_{\gamma,\beta,h} F)(i(\gamma/\beta))(y) = \mathcal{F}_{\gamma,\beta,h}(F(\cdot/\sqrt{2}))(y) \quad (3.11)$$

as elements of $S_{\alpha\beta}$ and

$$(\mathcal{F}_{\gamma,\beta,h} F \ast 1)(i(\gamma/\beta))(y) = \mathcal{F}_{\gamma,\beta,h}(F(\cdot/\sqrt{2}))(y) \quad (3.12)$$

as elements of $S_{\alpha\beta}$.

Proof  Equation (3.10) follows immediately by interchanging two pairs $(\gamma, \beta)$ and $(i(\gamma/\beta), (1/\beta))$ in Equation (3.9), Equation (3.11) follows by letting $G(y) = F(y)$ in Equation (3.9), while Equation (3.12) follows by letting $G(y)$ identically one in Equation (3.9). □

4. Fubini theorems

In this section, we establish a Fubini theorem for GITs and GCPs of functionals in $S_\alpha$.

Remark 4.1  Note that $A_\alpha \subset E_\alpha$ for all $\alpha \in \mathbb{C}$ and so using Theorems 2.3 and 2.5, for all $(\gamma, \beta) \in A_\alpha$ and for each $F, G \in S_\alpha$, the GIT $\mathcal{F}_{\gamma,\beta,h} F$ as an element of $S_{\alpha\beta}$ and the GCP $(F \ast G)_\gamma$ as an element of $S_\alpha$ always exist. Furthermore, we observe that $(0, 1)$ and $(0, -1)$ are elements of $A_\alpha$ and so $\mathcal{F}_{0,1,h} F(y) = F(y)$ and $\mathcal{F}_{0,-1,h} F(y) = F(-y)$ for s-a.e. $y \in C_0[0, T]$.

As the first theorem in this section, we establish a Fubini theorem for GITs.

Theorem 4.2  Let $F, h$ and $f$ be as in Theorem 2.3, and let $(\gamma_1, \beta_1)$ and $(\gamma_2, \beta_2)$ be elements of $A_\alpha$ with $(\gamma_1, \beta_1) \in A_{\alpha\beta_2}$ and $(\gamma_2, \beta_2) \in A_{\alpha\beta_1}$. Then

$$(\mathcal{F}_{\gamma_2,\beta_2,h}(\mathcal{F}_{\gamma_1,\beta_1,h} F))(y) = \mathcal{F}_{\gamma_1,\beta_1,\gamma_2,\beta_2,h}(\mathcal{F}_{\gamma_1,\beta_1,h} F)(y) = \mathcal{F}_{\gamma_2,\beta_1,h}(\mathcal{F}_{\gamma_1,\gamma_2,\beta_2,h} F)(y) \quad (4.1)$$

as elements of $S_{\alpha\beta_1\beta_2}$ with $\gamma' = \sqrt{\gamma_1^2 + \beta_1^2 \gamma_2^2}$ and $\beta' = \beta_1 \beta_2$. Also, both sides of the expression in Equation (4.1) are given by the formula

$$\int_{L_2[0,T]} \exp \left\{ \alpha \beta_1 \beta_2 \langle v, y \rangle + \frac{\alpha^2}{2} (\gamma_1^2 + \beta_1^2 \gamma_2^2) \| v h \|_2^2 \right\} df(v).$$

Furthermore, $(\gamma', \beta')$ is an element of $A_\alpha$ and hence $\mathcal{F}_{\gamma',\beta',h} F$ always exists.
Proof Using Equations (1.1), (2.3) and (2.2) it follows that for s-a.e. \( y \in C_0[0, T] \),

\[
\mathcal{F}_{\gamma_2, \beta_2, h} (\mathcal{F}_{\gamma_1, \beta_1, h} F)(y) = \int_{C_0[0, T]} \int_{C_0[0, T]} F(\gamma_1 Z(z, \cdot) + \beta_1 \gamma_2 Z(x, \cdot) + \beta_1 \beta_2 y) \, dm(z) \, dm(x)
\]

\[
= \int_{L_2[0, T]} \exp \left\{ \alpha \beta_1 \beta_2 (v, y) + \frac{\alpha^2}{2} (\gamma_1^2 + \beta_1^2 \gamma_2^2) \|vh\|^2_2 \right\} \, df(v).
\]

On the other hand, using Equations (1.1), (2.3) and (2.2) it follows that for s-a.e. \( y \in C_0[0, T] \),

\[
\mathcal{F}_{\gamma_1, \beta_1, h} (\mathcal{F}_{\gamma_2, \beta_2, h} F)(y) = \int_{C_0[0, T]} \int_{C_0[0, T]} F(\gamma_1 z + \beta_1 \gamma_2 x + \beta_1 \beta_2 y) \, m(dz) \, dm(x)
\]

\[
= \int_{L_2[0, T]} \exp \left\{ \alpha \beta_1 \beta_2 (v, y) + \frac{\alpha^2}{2} (\gamma_2^2 + \beta_1^2 \gamma_1^2) \|vh\|^2_2 \right\} \, df(v).
\]

Note that \( \gamma_1^2 + \beta_1^2 \gamma_2^2 = \gamma_2^2 + \beta_2^2 \gamma_1^2 \) because \((\gamma_1, \beta_1)\) and \((\gamma_2, \beta_2)\) are elements of \( A_\alpha \) and so Equation (4.1) is established. Also note that \((\gamma')^2 + (\beta')^2 = 1\) and \(\text{Re}(\alpha^2 \gamma^2) = \text{Re}(\alpha^2 \gamma'^2) + \text{Re}(\alpha^2 \beta^2 \gamma^2) \leq 0\) because \((\gamma_1, \beta_1) \in A_\alpha\) and \((\gamma_2, \beta_2) \in A_{\alpha \beta_1}\). Hence \((\gamma', \beta')\) is an element of \( A_\alpha \) and hence \( \mathcal{F}_{\gamma', \beta', h} F \) always exists.

We obtain the following corollary by letting \( \gamma_1 = \gamma_2 = \gamma \) and \( \beta_1 = \beta_2 = \beta \), or by letting \( \gamma_1 = r, \gamma_2 = i(\gamma/\beta), \beta_1 = \beta \) and \( \beta_2 = (1/\beta) \) in Equation (4.1).

**Corollary 4.3** Let \( F, h \) and \( f \) be as in Theorem 4.2, and let \((\gamma, \beta)\) be an element of \( A_\alpha \cap A_{\alpha \beta}\). Then

\[
\mathcal{F}_{\gamma, \beta, h} (\mathcal{F}_{\gamma, \beta, h} F)(y) = \mathcal{F}_{\gamma'', \beta'', h} F(y)
\]

(4.2)

as elements of \( S_{\alpha \beta^2} \) with \( \gamma'' = \sqrt{\gamma^2 + \beta^2 \gamma^2} \) and \( \beta'' = \beta^2 \). Also, both sides of the expression in Equation (4.2) are given by the formula

\[
\int_{L_2[0, T]} \exp \left\{ \alpha \beta^2 (v, y) + \frac{\alpha^2}{2} (\gamma^2 + \beta^2 \gamma^2) \|vh\|^2_2 \right\} \, df(v).
\]

Furthermore, for all \((\gamma, \beta) \in A_\alpha\) with \(\text{Re}(\alpha^2 \gamma^2) = 0\) and \(\text{Re}(\alpha^2 \gamma^2 / \beta^2) = 0\),

\[
\mathcal{F}_{\gamma, \beta, h} (\mathcal{F}_{\gamma, \beta, h} F)(y) = \mathcal{F}_{\gamma''/(\beta''/(1/\beta), h)} (\mathcal{F}_{\gamma', \beta', h} F)(y)
\]

(4.3)

as elements of \( S_\alpha \).

In the last theorem, we establish a Fubini theorem for GCPs.

**Theorem 4.4** Let \( F, G, f, g \) and \( h \) be as in Theorem 3.1, let \((\gamma_1, \beta_1)\) and \((\gamma_2, \beta_2)\) be elements of \( A_\alpha \) and moreover \((\gamma_1, \beta_1) \in A_{\alpha \beta_2}\) and \((\gamma_2, \beta_2) \in A_{\alpha \beta_1}\). Then

\[
\mathcal{F}_{\gamma_1, \beta_1, h} (\mathcal{F}_{\gamma_2, \beta_2, h} F * \mathcal{F}_{\gamma_2, \beta_2, h} G)(y_1) = \mathcal{F}_{\gamma_2, \beta_2, h} (\mathcal{F}_{\gamma_1, \beta_1, h} F * \mathcal{F}_{\gamma_1, \beta_1, h} G)(y_2)
\]

(4.3)

as elements of \( S_{\alpha \beta_1 \beta_2} \).
Proof  Equation (4.3) follows from the calculations

\[ \mathcal{F}_{\gamma_1,\beta_1,h}(\mathcal{F}_{\gamma_2,\beta_2,h}F \ast \mathcal{F}_{\gamma_2,\beta_2,h}G)_{\gamma_1}(y) = \mathcal{F}_{\gamma_1,\beta_1,h}(\mathcal{F}_{\gamma_2,\beta_2,h}F) \left( \frac{y}{\sqrt{2}} \right) \mathcal{F}_{\gamma_1,\beta_1,h}(\mathcal{F}_{\gamma_2,\beta_2,h}G) \left( \frac{y}{\sqrt{2}} \right) \]

\[ = \mathcal{F}_{\gamma_2,\beta_2,h}(\mathcal{F}_{\gamma_1,\beta_1,h}F \ast \mathcal{F}_{\gamma_1,\beta_1,h}G)_{\gamma_2}(y) \]

\[ = \mathcal{F}_{\gamma_2,\beta_2,h}(\mathcal{F}_{\gamma_1,\beta_1,h}F \ast \mathcal{F}_{\gamma_1,\beta_1,h}G)_{\gamma_1}(y) \]

for s-a.e. \( y \in C_0[0, T] \). The first equality in (4.4) follows from Theorem 3.1 with the functionals \( F \) and \( G \) being replaced by the functionals \( \mathcal{F}_{\gamma_2,\beta_2,h}F \) and \( \mathcal{F}_{\gamma_2,\beta_2,h}G \). The second equality in (4.4) follows from Equation (4.1), while the third equality is again an application of Theorem 3.1. Furthermore, both sides of the expression in (4.3) are elements of \( S_{\alpha\beta} \).

We obtain the following corollary by letting \( G(y) = F(y) \) or by letting \( G(y) \) be identically one on \( C_0[0, T] \).

**Corollary 4.5** Let \( F, h \) and \( f \) be as in Theorem 4.2, let \( (\gamma_1, \beta_1) \) and \( (\gamma_2, \beta_2) \) be elements of \( A_\alpha \cap A_{\alpha\beta_2} \) and \( A_\alpha \cap A_{\alpha\beta_1} \) respectively. Then

\[ \mathcal{F}_{\gamma_1,\beta_1,h}(\mathcal{F}_{\gamma_2,\beta_2,h}F \ast \mathcal{F}_{\gamma_2,\beta_2,h}F)_{\gamma_1}(y) = \mathcal{F}_{\gamma_1,\beta_1,h}(\mathcal{F}_{\gamma_2,\beta_2,h}F) \left( \frac{y}{\sqrt{2}} \right) \mathcal{F}_{\gamma_1,\beta_1,h}(\mathcal{F}_{\gamma_2,\beta_2,h}F) \left( \frac{y}{\sqrt{2}} \right) \]

\[ = \mathcal{F}_{\gamma_2,\beta_2,h}(\mathcal{F}_{\gamma_1,\beta_1,h}F \ast \mathcal{F}_{\gamma_1,\beta_1,h}F)_{\gamma_2}(y), \]

and

\[ \mathcal{F}_{\gamma_1,\beta_1,h}(\mathcal{F}_{\gamma_2,\beta_2,h}F \ast 1)_{\gamma_1}(y) = \mathcal{F}_{\gamma_1,\beta_1,h}(\mathcal{F}_{\gamma_2,\beta_2,h}F) \left( \frac{y}{\sqrt{2}} \right) \]

\[ = \mathcal{F}_{\gamma_2,\beta_2,h}(\mathcal{F}_{\gamma_1,\beta_1,h}F \ast 1)_{\gamma_2}(y) \]

as elements of \( S_{\alpha\beta_1\beta_2} \).

We obtain the following corollary by letting \( \gamma_1 = \gamma_2 = \gamma \) and \( \beta_1 = \beta_2 = \beta \) in Equation (4.3).

**Corollary 4.6** Let \( F, G, h, f \) and \( g \) be as in Theorem 4.2, and let \( (\gamma, \beta) \) be an element of \( A_\alpha \cap A_{\alpha\beta} \). Then

\[ \mathcal{F}_{\gamma,\beta,h}(\mathcal{F}_{\gamma,\beta,h}F \ast \mathcal{F}_{\gamma,\beta,h}G)_{\gamma}(y) = \mathcal{F}_{\gamma,\beta,h}F \left( \frac{y}{\sqrt{2}} \right) \mathcal{F}_{\gamma,\beta,h}G \left( \frac{y}{\sqrt{2}} \right) \]

as elements of \( S_{\alpha\beta} \) with \( \gamma'' = \sqrt{\gamma^2 + \beta^2 \gamma^2} \) and \( \beta'' = \beta \).

We finish this section by stating a remark for \( n \)-dimensional versions of Theorems 4.2 and 4.4, respectively.

**Remark 4.7** Clearly there are \( n \)-dimensional versions of Theorems 4.2 and 4.4. First, in case of GITs, using mathematical induction our next formula (4.5) follows readily from Theorem 4.2.
Under some appropriate conditions,
\[ \mathcal{F}_{\gamma_1, \beta_1, h} \cdots \mathcal{F}_{\gamma_n, \beta_n, h} F(y) = \mathcal{F}_{\gamma'' n, \beta'' n, h} F(y) = \mathcal{F}_{\gamma_1, \beta_1, h} \cdots \mathcal{F}_{\gamma_n, \beta_n, h} F(y) \]  
(4.5)
as elements of \( S_{a \beta'' n} \) with \( \gamma'' = \sqrt{\sum_{k=1}^{n} (\gamma_k^2 \prod_{j=1}^{k} \beta_j^2 - 1)} \) and \( \beta'' = \beta_1 \beta_2 \cdots \beta_n \) where \( \beta_0 = 1 \). In particular,
\[ \mathcal{F}_{\gamma', \beta', h} \cdots \mathcal{F}_{\gamma', \beta', h} F(y) = \mathcal{F}_{\sqrt{1 - \beta'' n, \beta'' n, h}} F(y) \]
as elements of \( S_{a \beta'' n} \). Next, in case of GCPs, for the sake of simplicity, we will only state a three-dimensional version. Under some appropriate conditions,
\[ \mathcal{F}_{\gamma_3, \beta_3, h}(\mathcal{F}_{\gamma_2, \beta_2, h} \mathcal{F}_{\gamma_1, \beta_1, h} F \ast \mathcal{F}_{\gamma_2, \beta_2, h} \mathcal{F}_{\gamma_1, \beta_1, h} G)_{\gamma_3}(y) \]
\[ = \mathcal{F}_{\gamma_3, \beta_3, h}(\mathcal{F}_{\gamma_1, \beta_1, h} \mathcal{F}_{\gamma_2, \beta_2, h} F \ast \mathcal{F}_{\gamma_1, \beta_1, h} \mathcal{F}_{\gamma_2, \beta_2, h} G)_{\gamma_3}(y) \]
\[ = \mathcal{F}_{\gamma_2, \beta_2, h}(\mathcal{F}_{\gamma_3, \beta_3, h} \mathcal{F}_{\gamma_1, \beta_1, h} F \ast \mathcal{F}_{\gamma_3, \beta_3, h} \mathcal{F}_{\gamma_1, \beta_1, h} G)_{\gamma_2}(y) \]
\[ = \mathcal{F}_{\gamma_1, \beta_1, h}(\mathcal{F}_{\gamma_3, \beta_3, h} \mathcal{F}_{\gamma_2, \beta_2, h} F \ast \mathcal{F}_{\gamma_3, \beta_3, h} \mathcal{F}_{\gamma_2, \beta_2, h} G)_{\gamma_1}(y) \]
(4.6)
as elements of \( S_{a \beta_3 \beta_2 \beta_1} \). The conditions are neither difficult nor complicated, but rather long. So we only state the formulas without stating conditions.

5. Additional results

In this section, we give some additional results as remarks.

Remark 5.1 Let \( F \in S_a \) be given by Equation (2.1). Then for all \( (\gamma, \beta) \) and \( (i(\gamma/\beta), (1/\beta)) \) in \( A_a \) with Re(\( \alpha^2 \)) \leq 0,
\[ \int_{C_0[0, T]} \mathcal{F}_{\gamma', \beta', h} F(y) \, dm(y) = \int_{C_0[0, T]} F(Z(y, \cdot)) \, dm(y) \]
\[ = \int_{C_0[0, T]} \mathcal{F}_{i(\gamma/\beta), (1/\beta), h} F(y) \, dm(y). \]  
(5.1)

Also, three sides of the expression in Equation (5.1) are given by the formula
\[ \int_{L^1[0, T]} \exp \left\{ \frac{\alpha^2}{2} \|v h\|_2^2 \right\} \, df(v) < \infty. \]

Furthermore, for all \( (\gamma, \beta) \in A_a \) with Re(\( \alpha^2 \beta^2 \)) \leq 0,
\[ \int_{C_0[0, T]} \mathcal{F}_{\gamma, \beta, h}(F \ast G)_\gamma(y) \, dm(y) = \int_{C_0[0, T]} (F \ast G)_\gamma(Z(y, \cdot)) \, dm(y) \]
\[ = \int_{C_0[0, T]} \mathcal{F}_{\gamma, \beta, h} \left( \frac{y}{\sqrt{2}} \right) \mathcal{F}_{\gamma, \beta, h} G \left( \frac{y}{\sqrt{2}} \right) \, dm(y). \]  
(5.2)

Also, three sides of the expression in Equation (5.2) are given by the formula
\[ \int_{L^2[0, T]} \int_{L^2[0, T]} \exp \left\{ \frac{\alpha^2 \beta^2}{4} \|v + w\|_2^2 + \frac{\alpha^2 \beta^2}{2} (\|v h\|_2^2 + \|w h\|_2^2) \right\} \, df(v) \, dg(w) < \infty. \]
Remark 5.2 Let \( F \in \mathcal{S}_\alpha \) be given by Equation (2.1). Then for all \((\gamma_1, \beta_1) \in (E_\alpha / A_\alpha) \cap (E_\alpha) \cap (E_\alpha / A_\alpha) \cap (E_\alpha / A_\alpha) \cap (E_\alpha / A_\alpha) \),

\[
\mathcal{F}_{\gamma_2, \beta_2, h_2} (\mathcal{F}_{\gamma_1, \beta_1, h_1} F)(y) = \mathcal{F}_{\gamma_1, \beta_1, h_1} (\mathcal{F}_{\gamma_2, \beta_2, h_2} F)(y)
\]

(5.3)

if and only if \( \beta_1^2 = \beta_2^2 = 1 \). Also, both sides of the expression in Equation (5.3) are given by the formula

\[
\int_{L_2[0,T]} \exp \left\{ \pm \alpha \langle v, y \rangle + \frac{\alpha^2}{2} (\gamma_1^2 v h_1 \| v \|_2^2 + \gamma_2^2 v h_2 \| v \|_2^2) \right\} \, d f(v)
\]

as element of \( \mathcal{S}_{\pm \alpha} \).

Note that if \( v \in L_2[0, T] \) is a function of bounded variation, then for each \( y \in K \),

\[
|\langle v, y \rangle| = \left| v(T) y(T) - \int_0^T y(t) \, d v(t) \right|
\]

\[
\leq |v(T)||y(T)| + \left| \int_0^T y(t) \, d v(t) \right|
\]

\[
\leq \| y \|_\infty \left( |v(T)| + \left| \int_0^T d v(t) \right| \right)
\]

\[
\leq \| y \|_\infty (|v(T)| + V_0^T(v)) < \infty,
\]

where \( V_0^T(v) \) is the total variation of \( v \) on \([0, T]\).

Remark 5.3 Let \( F \in \mathcal{S}_\alpha \) be given by Equation (2.1) and let \( v \in L_2[0, T] \) be a function of bounded variation. Let \( \{(\gamma_n, \beta_n)\} \) be a sequence in \( E_\alpha \) with \( \gamma_n \to \gamma \neq 0 \) and \( \beta_n \to \beta \neq 0 \) as \( n \to \infty \) for some \((\gamma, \beta) \in E_\alpha \). Suppose that for all \( n = 1, 2, \ldots, |\alpha \beta_n| \leq M \) for some positive real number \( M \). Then using the dominated convergence theorem,

\[
\lim_{n \to \infty} \mathcal{F}_{\gamma_n, \beta_n, h} F(y) = \lim_{n \to \infty} \int_{L_2[0,T]} \exp \left\{ \alpha \beta_n \langle v, y \rangle + \frac{\alpha^2 \gamma_n^2}{2} \| v \|_2^2 \right\} \, d f(v)
\]

\[
= \int_{L_2[0,T]} \exp \left\{ \alpha \beta \langle v, y \rangle + \frac{\alpha^2 \gamma^2}{2} \| v \|_2^2 \right\} \, d f(v)
\]

\[
= \mathcal{F}_{\gamma, \beta, h} F(y).
\]

Definition 5.4 Let \( F \) be a functional defined on \( K \). Then the first variation (FV) is defined by formula

\[
\delta F(x|u) = \frac{\partial}{\partial k} F(x + ku) \bigg|_{k=0}, \quad x, u \in K,
\]

if it exists.

Remark 5.5 Let \( F \in \mathcal{S}_\alpha \) be given by Equation (2.1) and let \( v \in L_2[0, T] \) be a function of bounded variation. Let \( u(t) = \int_0^T z(s) \, ds \) for some \( z \in L_2[0, T] \). Then for all complex numbers \( \alpha \) with \( |\alpha| \leq M \) for some positive real number \( M \),

\[
\delta F(x|u) = \int_{L_2[0,T]} \alpha \langle v, u \rangle \exp \{ \alpha \langle v, x \rangle \} \, d f(v)
\]

\[
= \int_{L_2[0,T]} \exp \{ \alpha \langle v, x \rangle \} \, d \phi_3(v),
\]

where \( \phi_3 \) is an element of \( M(L_2[0, T]) \) and so \( \delta F(x|u) \) is an element of \( \mathcal{S}_\alpha \) as a function of \( x \).
In the last remark, we establish basic formulas involving FVs under some appropriate conditions.

**Remark 5.6** Let $F$ and $G$ be given by Theorem 2.5. Then

$$
\beta \mathcal{F}_{\gamma, \beta, h} \delta F(\cdot | u)(y) = \delta \mathcal{F}_{\gamma, \beta, h} F(y | u),
$$

(5.4)

$$
\delta (F * G)(y | u) = (\delta F(\cdot / \sqrt{2}) * G)(y) + (F * \delta G(\cdot | \sqrt{2}) ) (y),
$$

(5.5)

$$
\beta^2 \mathcal{F}_{\gamma, \beta, h} (\delta F(\cdot | u) * \delta G(\cdot | u))(y) = \delta \mathcal{F}_{\gamma, \beta, h} (F \ast \delta G)(y | u),
$$

(5.6)

$$
\beta \mathcal{F}_{\gamma, \beta, h} (F * G)(y | u) = \delta \mathcal{F}_{\gamma, \beta, h} (F \ast G)(y | u),
$$

(5.7)

and

$$
\beta^2 (\mathcal{F}_{\gamma, \beta, h} \delta F(\cdot | u) * \mathcal{F}_{\gamma, \beta, h} \delta G(\cdot | u))(y) = (\delta \mathcal{F}_{\gamma, \beta, h} F(\cdot | u) * \delta \mathcal{F}_{\gamma, \beta, h} G(\cdot | u))(y).
$$

(5.8)

Furthermore, by interchanging the two pairs $(\gamma, \beta)$ and $(i(\gamma / \beta), (1/\beta))$ in Equations (5.4)–(5.8), we can establish other basic formulas, as in Theorem 3.2.

**References**


