Analysis and finite element approximation of an optimal control problem for the Oseen viscoelastic fluid flow

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Abstract

In this article we study a boundary control problem for an Oseen-type model of viscoelastic fluid flow. The existence of a unique optimal solution is proved and an optimality system is derived by the first-order necessary condition. We investigate finite element approximations to a solution of the optimality system, and a solution algorithm for the system based on the gradient method.

Keywords: Optimal control; Viscoelastic fluid; Finite elements; Gradient method

1. Introduction

1.1. Model equations

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ ($d = 2$ or $3$) with the Lipschitz continuous boundary $\Gamma$. Consider the Johnson–Segalman problem

\begin{align*}
\sigma + \lambda (u \cdot \nabla)\sigma + \lambda g_a(\sigma, \nabla u) - 2\alpha D(u) &= 0 \quad \text{in } \Omega, \\
-\nabla \cdot \sigma - 2(1-\alpha)\nabla \cdot D(u) + \nabla p &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega,
\end{align*}

where $\sigma$ is the stress tensor, $\lambda$ and $\alpha$ are parameters, $g_a(\sigma, \nabla u)$ is the strain-energy density function, and $D(u)$ is the deviatoric part of the strain tensor.

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where $\sigma$ denotes the polymeric stress tensor, $u$ the velocity vector, $p$ the pressure of fluid, and $\lambda$ is the Weissenberg number defined as the product of the relaxation time and a characteristic strain rate.

Assume that $p$ has zero mean value over $\Omega$. In (1.1) and (1.2), $D(u) := (\nabla u + \nabla u^T)/2$ is the rate of the strain tensor, $\alpha$ a number such that $0 < \alpha < 1$ which may be considered as the fraction of viscoelastic viscosity, and $f$ the body force. In (1.1), $g_a(\sigma, \nabla u)$ is defined by

$$g_a(\sigma, \nabla u) := \frac{1-a}{2} (\sigma \nabla u + \nabla u^T \sigma) - \frac{1+a}{2} (\nabla u \sigma + \sigma \nabla u^T)$$

for $a \in [-1, 1]$.

We use the Sobolev spaces $W^{m,p}(D)$ with norms $\|\cdot\|_{m,p,D}$ if $p < \infty$, $\|\cdot\|_{m,\infty,D}$ if $p = \infty$. We denote the Sobolev space $W^{m,2}$ by $H^m$ with the norm $\|\cdot\|_m$. The corresponding space of vector-valued or tensor-valued functions is denoted by $H^m$. If $D = \Omega$, $D$ is omitted, i.e., $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\Omega}$ and $\|\cdot\| = \|\cdot\|_{\Omega}$.

Existence of a solution to the problem (1.1)–(1.3) with the homogeneous boundary condition has been very active in the last few decades (see [11,12,15,16] and references therein). A numerical study on optimal control of viscoelastic fluid flow is found in [14], where the authors consider flow of a non-isothermal viscoelastic fluid governed by the linearized Phan-Thien–Tanner model.

One of difficulties in simulating viscoelastic flows arises from the hyperbolic nature of the constitutive equation for which one needs to use a stabilization technique such as the streamline upwinding Petrov–Galerkin (SUPG) method and the discontinuous Galerkin method. Mathematical studies of the discontinuous Galerkin method for steady state viscoelastic fluid flows are found in [2,17]. The SUPG method was investigated in [19], where the existence of a solution to the discrete problem was shown by a fixed point theory and an finite element error estimate was derived. The model equations (1.5)–(1.7) were studied in [6] and [8]. In [8] the SUPG method was examined for a continuous weak problem as well as for a discrete problem of the Oseen viscoelastic equations (1.5)–(1.7). There, it was shown that the continuous and discrete weak solutions exist under a small data assumption, and an a priori error estimate was derived.

This work is motivated by the lack of theoretical analysis on optimal control problems for non-Newtonian fluid flow, while the study of optimal control problems for Newtonian fluids has been very active in the last few decades (see [11,12,15,16] and references therein). A numerical study on optimal control of viscoelastic fluid flow is found in [14], where the authors consider flow of a non-isothermal viscoelastic fluid governed by the linearized Phan-Thien–Tanner model.
through the four-to-one contraction domain. In the article an optimality system is derived using
the Lagrange multipliers rule for vortex minimization and the system is solved by the finite
difference method.

This paper is organized as follows. In the rest of this section we define a weak formulation
of the model equations based on the streamline upwind Petrov–Galerkin (SUPG) method and
present a Lemma for the existence and uniqueness of a weak solution. In Section 2 we state the
optimal control problem, investigate the existence of a unique optimal solution, and derive an op-
timality system from which the optimal solution is determined. In Section 3 an error estimate is
derived for a finite element approximate solution to the optimality system. A computational algo-
rithm based on the gradient method is presented and investigated in Section 4. Some concluding
remarks and future work are given in Section 5.

1.2. Weak formulation of the model equations

Throughout this paper, \( C \) denotes a positive constant whose meaning and value changes with
context.

Define the function spaces for the velocity \( \mathbf{u} \), the pressure \( p \) and the stress \( \sigma \), respectively:

\[
X := \{ \mathbf{v} \in H^1(\Omega): \mathbf{v} = 0 \text{ on } \Gamma_1 \},
\]

\[
S := L^2_0(\Omega) = \{ q \in L^2(\Omega): \int_\Omega q \, d\Omega = 0 \},
\]

\[
\Sigma := \{ \mathbf{\tau} \in L^2(\Omega): \tau_{ij} = \tau_{ji}, (\mathbf{b} \cdot \nabla)\mathbf{\tau} \in L^2(\Omega) \}.
\]

Note that the velocity and pressure spaces, \( X \) and \( S \), satisfy the inf-sup condition

\[
\inf_{q \in S} \sup_{\mathbf{v} \in X} \frac{(q, \nabla \cdot \mathbf{v})}{\|\mathbf{v}\|_1 \|q\|_0} \geq C. \tag{1.10}
\]

We also define the norm \( \| \cdot \|_b \) as

\[
\|\mathbf{\tau}\|_b := (\|\mathbf{\tau}\|_0^2 + \lambda^2 (\mathbf{b} \cdot \nabla)\mathbf{\tau}\|_0^2)^{1/2}.
\]

Note that \( \Sigma \) is a Hilbert space with associated inner product

\[
(\sigma, \mathbf{\tau})_b = (\sigma, \mathbf{\tau}) + \lambda^2 ((\mathbf{b} \cdot \nabla)\sigma, (\mathbf{b} \cdot \nabla)\mathbf{\tau}).
\]

We introduce the weak divergence free space

\[
\mathbf{V} = \{ \mathbf{v} \in X: \int_\Omega q \, \text{div} \, \mathbf{v} \, d\Omega = 0 \quad \forall q \in L^2_0(\Omega) \}.
\]

In deriving a weak problem of (1.5)–(1.9), we use the SUPG stabilization technique to take care
of the hyperbolic character of the constitutive equation. The corresponding weak formulation is
then given by

\[
(\sigma, \mathbf{\tau} + \lambda \delta (\mathbf{b} \cdot \nabla)\mathbf{\tau}) + \lambda ((\mathbf{b} \cdot \nabla)\sigma, \mathbf{\tau} + \lambda \delta (\mathbf{b} \cdot \nabla)\mathbf{\tau}) + \lambda (g_{\alpha}(\sigma, \nabla \mathbf{b}), \mathbf{\tau} + \lambda \delta (\mathbf{b} \cdot \nabla)\mathbf{\tau}) - 2\alpha (D(\mathbf{u}), \mathbf{\tau} + \lambda \delta (\mathbf{b} \cdot \nabla)\mathbf{\tau}) = 0 \quad \forall \mathbf{\tau} \in \Sigma, \tag{1.11}
\]

\[
(\sigma, D(\mathbf{v})) + 2(1 - \alpha)(D(\mathbf{u}), D(\mathbf{v})) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{v})_{\Gamma_2} \quad \forall \mathbf{v} \in \mathbf{X}, \tag{1.12}
\]

\[
(q, \nabla \cdot \mathbf{u}) = 0 \quad \forall q \in L^2_0(\Omega), \tag{1.13}
\]
where $\delta > 0$ is a constant. Using the weak divergence free space $V$, the weak formulation (1.11)–(1.13) is equivalent to

$$
(\sigma, \tau + \lambda\delta (b \cdot \nabla)\tau) + \lambda((b \cdot \nabla)\sigma, \tau + \lambda\delta (b \cdot \nabla)\tau) + \lambda(\sigma, \nabla b, \tau + \lambda\delta (b \cdot \nabla)\tau)
- 2\alpha(D(u), \tau + \lambda\delta (b \cdot \nabla)\tau) = 0 \quad \forall \tau \in \Sigma,
$$

(1.14)

$$
(\sigma, D(v)) + 2(1 - \alpha)(D(u), D(v)) = (f, v) + (g, v)_{\Gamma_2} \quad \forall v \in V.
$$

(1.15)

Throughout this paper we use the bilinear form $A$ defined on $(V \times \Sigma) \times (V \times \Sigma)$ by

$$
A((u, \sigma), (v, \tau)) := (\sigma, \tau + \lambda\delta (b \cdot \nabla)\tau) + \lambda((b \cdot \nabla)\sigma, \tau + \lambda\delta (b \cdot \nabla)\tau)
+ \lambda(\sigma, \nabla b, \tau + \lambda\delta (b \cdot \nabla)\tau) - 2\alpha(D(u), \tau + \lambda\delta (b \cdot \nabla)\tau)
+ 2\alpha(\sigma, D(v)) + 4\alpha(1 - \alpha)(D(u), D(v)),
$$

(1.16)

and using the bilinear form $A$, (1.14)–(1.15) can equivalently be written as

$$
A((\sigma, u), (\tau, v)) = 2\alpha[(f, v) + (g, v)_{\Gamma_2}] \quad \forall (v, \tau) \in V \times \Sigma.
$$

(1.17)

Note that as $b = 0$ on $\Gamma$ and $\nabla \cdot b = 0$, using integration by parts, we have

$$
(b \cdot \nabla)\sigma, \tau = -((b \cdot \nabla)\tau, \sigma)
$$

(1.18)

and, therefore

$$
(b \cdot \nabla)\sigma, \sigma = 0.
$$

(1.19)

It is shown in [8] that $A(\cdot, \cdot)$ is coercive and bounded if the parameters $\lambda$, $\delta$ and the bound $M$ are small enough so that

$$
1 - 2\lambda Md - \varepsilon_1\delta\lambda Md > 0,
$$

(1.20)

$$
1 - \frac{\lambda Md}{\varepsilon_1} - \varepsilon_2 > 0,
$$

(1.21)

$$
4\alpha(1 - \alpha) - \frac{\alpha^2\delta}{\varepsilon_2} > 0,
$$

(1.22)

for $\varepsilon_1, \varepsilon_2 > 0$. It has been also prove in [8] that (1.20)–(1.22) implies

$$
0 < \delta < \frac{4(1 - \alpha)}{\alpha},
$$

(1.23)

and

$$
0 < \lambda Md < \frac{(1 - \alpha)\delta}{4\alpha} - 1 + \sqrt{(\frac{(1 - \alpha)\delta}{4\alpha} - 1)^2 + \delta(\frac{(1 - \alpha)\delta}{4\alpha} - 1)}
$$

(1.24)

Therefore, we have the following result.

**Lemma 1.1.** Given $f \in H^{-1}(\Omega)$, $g \in L^2(\Gamma_2)$, and $0 < \alpha < 1$, $\lambda$, $M$, $\delta$ satisfying (1.23)–(1.24), there exists a unique solution $(u, p, \sigma) \in X \times S \times \Sigma$ satisfying (1.11)–(1.13). In addition, we have the estimate

$$
\|u\|_1 + \|p\|_0 + \|\sigma\|_0 + 2\lambda\sqrt{\delta}(b \cdot \nabla)\sigma \leq C(\|f\|_{-1} + \|g\|_{0, \Gamma_2}).
$$

(1.25)

**Proof.** See [8]. □
2. The optimal control problem

2.1. Formulation of the optimal control problem

The Johnson–Segalman problem (1.1)–(1.3) is a model of complex fluid flow of molten polymer. The geometry of four-to-one contraction commonly occurs in the forming ‘die’ for polymer fibers and films. Due to the sudden reduction in width, in the corner region a vortex appears. In this region while the flow recirculates, it has the potential to degrade, which produces an inferior product at extrusion. Hence, we would like to be able to control some parameter(s) of the flow to reduce this vortex. The shape of typical four-to-one contraction domain is presented in Fig. 1.

A measure of the vortex is the curl of the velocity field:

\[ \int_{\Omega} (\nabla \times \mathbf{u})^2 \, d\Omega. \]

Hence, we will consider minimizing the penalized functional

\[ J(u, p, \sigma, g) := \frac{1}{2} \int_{\Omega} (\nabla \times \mathbf{u})^2 \, d\Omega + \frac{\epsilon}{2} \int_{\Gamma_2} g^2 \, d\Gamma_2, \quad (2.1) \]

where \( g \) is the Neumann boundary control and \( \epsilon \) is a penalty parameter. We formulate the optimal control problem in the following terms:

\[ (P) \quad \text{Find } u, p, \sigma \text{ and } g \text{ such that the functional (2.1) is minimized subject to (1.11)–(1.13).} \]

We now define the admissibility set \( \mathcal{U}_{ad} \) as follows:

\[ \mathcal{U}_{ad} := \{ (u, p, \sigma, g) \in X \times S \times \Sigma \times \mathcal{L}^2(\Gamma_2) : J(u, p, \sigma, g) < \infty \text{ and } (1.11)–(1.13) \text{ is satisfied} \}. \quad (2.2) \]

Then \((\hat{u}, \hat{p}, \hat{\sigma}, \hat{g})\) is called an optimal solution if there exists an \( \epsilon > 0 \) such that \( J(\hat{u}, \hat{p}, \hat{\sigma}, \hat{g}) \leq J(u, p, \sigma, g) \) for all \((u, p, \sigma, g) \in \mathcal{U}_{ad} \) satisfying \( \|u - \hat{u}\|_1 + \|p - \hat{p}\|_0 + \|\sigma - \hat{\sigma}\|_b + \)
\[ \| g - \hat{g} \|_{0, \Gamma} \leq \varepsilon. \]

The optimal control problem (P) can now be formulated as a constrained minimization problem in a Hilbert space:

\[ \min_{(u, p, \sigma, g) \in \mathcal{U}_{ad}} J(u, p, \sigma, g). \tag{2.3} \]

### 2.2. Existence of an optimal control solution

The existence of an optimal solution of (2.3) is easily proven using standard arguments in the following theorem.

**Theorem 2.1.** Given \( f \in H^{-1}(\Omega) \), there exists a solution \( (u, p, \sigma, g) \in X \times S \times \Sigma \times L^2(\Gamma_2) \) of the optimal control problem (2.3).

**Proof.** We first note that the admissible set \( \mathcal{U}_{ad} \) is clearly not empty, e.g., \( (u, p, \sigma, 0) \in \mathcal{U}_{ad} \). Let \( g_n \) be a minimizing sequence for the optimal control problem and set \( u_n = u(g_n), p_n = p(g_n), \sigma_n = \sigma(g_n) \). Then \( (u_n, p_n, \sigma_n, g_n) \in \mathcal{U}_{ad} \) for all \( n \) and satisfies

\[ \lim_{n \to \infty} J(u_n, p_n, \sigma_n, g_n) = \inf_{(u, p, \sigma, g) \in \mathcal{U}_{ad}} J(u, p, \sigma, g). \]

The sequence \( g_n \) is uniformly bounded in \( L^2(\Gamma_2) \) from (2.2) and the corresponding \( (u_n, p_n, \sigma_n) \) is uniformly bounded in \( X \times S \times \Sigma \) from Lemma 1.1. We may then extract subsequences, still denoted by \( (u_n, p_n, \sigma_n, g_n) \), such that

- \( g_n \rightharpoonup \hat{g} \) in \( L^2(\Gamma_2) \),
- \( p_n \rightharpoonup \tilde{p} \) in \( L^2(\Omega) \),
- \( u_n \rightharpoonup \tilde{u} \) in \( H^1(\Omega) \),
- \( \sigma_n \rightharpoonup \tilde{\sigma} \) in \( L^2(\Omega) \),

for some \( (\tilde{u}, \tilde{p}, \tilde{\sigma}, \hat{g}) \in X \times S \times \Sigma \times L^2(\Gamma_2) \). By the process of passing to the limit, we have that \( (\tilde{u}, \tilde{p}, \tilde{\sigma}, \hat{g}) \) satisfies (1.11)–(1.13). Now, by the weak lower semi-continuity of \( J(\cdot, \cdot, \cdot, \cdot) \), we conclude that \( (\tilde{u}, \tilde{p}, \tilde{\sigma}, \hat{g}) \) is an optimal solution, i.e.,

\[ \inf_{(u, p, \sigma, g) \in \mathcal{U}_{ad}} J(u, p, \sigma, g) = \lim_{n \to \infty} J(u_n, p_n, \sigma_n, g_n) = J(\tilde{u}, \tilde{p}, \tilde{\sigma}, \hat{g}). \]

Thus, we have shown that an optimal solution belonging to \( \mathcal{U}_{ad} \) exists. Finally, the uniqueness of the optimal solution follows from the convexity of the functional and the linearity of the constraint equations. \( \square \)

### 2.3. First-order necessary condition

The first-order optimality condition associated with problem (P) can be derived by Gâteaux derivative. We shall show that the optimal solution must satisfy the first-order necessary condition.

**Theorem 2.2.** Let \( f \in H^{-1}(\Omega) \) and \( g \in L^2(\Gamma_2) \). The mapping \( g \to (u(g), p(g), \sigma(g)) \), defined as the solution of (1.11)–(1.13), has a Gâteaux derivative \( (d(u(g), p(g), \sigma(g))/dg) \cdot h \) in every direction \( h \) in \( L^2(\Gamma_2) \). Furthermore, \( (\bar{w}, \bar{\xi}, \bar{\eta}) := (d(u(g), p(g), \sigma(g))/dg) \cdot h \) is the solution of the problem.
In deriving the above relation, we used (1.18),

\[
\begin{align*}
(\bar{\eta} + \lambda (\bar{b} \cdot \nabla) \bar{\eta}, \tau + \lambda \delta (\bar{b} \cdot \nabla) \tau) + \lambda \left( g_a (\bar{\eta}, \nabla \bar{b}), \tau + \lambda \delta (\bar{b} \cdot \nabla) \tau \right) \\
- 2 \alpha (D(\bar{w}), \tau + \lambda \delta (\bar{b} \cdot \nabla) \tau) = 0 \quad \forall \tau \in \Sigma,
\end{align*}
\]

(2.4)

\[
(\bar{\eta}, D(v)) + 2 (1 - \alpha) (D(\bar{w}), D(v)) - (\bar{\xi}, \nabla \cdot v) = (h, v)_{I_2} \quad \forall v \in \mathbf{X},
\]

(2.5)

\[
(q, \nabla \cdot \bar{w}) = 0 \quad \forall q \in L_0^2(\Omega).
\]

(2.6)

Finally, \((\bar{w}, \bar{r}, \bar{\eta}) \in \mathbf{X} \times \mathbf{S} \times \Sigma \).

**Proof.** It is immediate from the linearity of (1.11)–(1.13). \(\square\)

**Theorem 2.3.** If \((u, p, \sigma, g)\) is an optimal solution for \((P)\), then the equality

\[
\int_{\Gamma_2} g \cdot h \, d\Gamma = \frac{1}{\epsilon} \int_{\Gamma_2} w \cdot h \, d\Gamma
\]

(2.7)

holds, where \(w\) is the adjoint state that is the solution of the adjoint problem

\[
-2 \alpha (\eta + \lambda \delta (\bar{b} \cdot \nabla) \eta, D(v)) + 2 (1 - \alpha) (D(w), D(v)) - (\bar{\xi}, \nabla \cdot v) \quad \forall v \in \mathbf{X},
\]

(2.8)

\[
(\eta - \lambda (\bar{b} \cdot \nabla) \eta, \tau - \lambda \delta (\bar{b} \cdot \nabla) \tau) + \lambda \left( \tilde{g}_a (\eta + \lambda \delta (\bar{b} \cdot \nabla) \eta, \nabla \bar{b}), \tau \right) \]

\[
+ (D(w), \tau) = 0 \quad \forall \tau \in \Sigma,
\]

(2.9)

\[
(\nabla \cdot w, q) = 0 \quad \forall q \in S.
\]

(2.10)

In (2.8) \(\tilde{g}_a (\eta, \nabla \bar{b})\) is defined by

\[
\tilde{g}_a (\eta, \nabla \bar{b}) := \frac{1}{2} \left( \eta \nabla \bar{b}^T + \nabla \eta \right) - \frac{1}{2} \left( \nabla \eta \bar{b}^T + \eta \nabla \bar{b} \right).
\]

(2.11)

**Proof.** Let \((u, p, \sigma, g)\) be an optimal solution. The Gâteaux derivative of the functional \(J (g)\) in the direction of \(h\) is defined by

\[
\frac{d J (u(g), p(g), \sigma (g), g)}{dg} \cdot h = \int_{\Omega} \left( \nabla \times u(g) \right) \left( \nabla \times \frac{d u(g)}{dg} \right) \cdot h \, d\Omega + \epsilon \int_{\Gamma_2} g \cdot h \, d\Gamma
\]

\[
= \int_{\Omega} \left( \nabla \times u(g) \right) \left( \nabla \times \tilde{w}(h) \right) \, d\Omega + \epsilon \int_{\Gamma_2} g \cdot h \, d\Gamma.
\]

(2.12)

Setting \((v, q, \tau) = (w, \bar{\xi}, \eta)\) in (2.4)–(2.6) and \((v, q, \tau) = (\bar{w}, \bar{\xi}, \bar{\eta})\) in (2.8)–(2.10), we have that

\[
\int_{\Omega} \left( \nabla \times u(g) \right) \left( \nabla \times \tilde{w}(h) \right) \, d\Omega = - \int_{\Gamma_2} w \cdot h \, d\Gamma.
\]

In deriving the above relation, we used (1.18),

\[
\begin{align*}
(g_a (\tau, \nabla \bar{b}), \eta) &= \frac{1 - a}{2} \left[ (\tau \nabla \bar{b}, \eta) + (\nabla \bar{b}^T \tau, \eta) \right] - \frac{1 + a}{2} \left[ (\nabla \bar{b} \tau, \eta) + (\tau \nabla \bar{b}^T, \eta) \right] \\
&= \frac{1 - a}{2} \left[ (\eta \nabla \bar{b}^T, \tau) + (\nabla \eta \bar{b}, \tau) \right] - \frac{1 + a}{2} \left[ (\nabla \bar{b} \tau, \eta) + (\eta \nabla \bar{b}, \tau) \right] \\
&= (\tilde{g}_a (\eta, \nabla \bar{b}), \tau)
\end{align*}
\]
and, similarly

\[(g_a(\mathbf{r}, \nabla \mathbf{b}), (\mathbf{b} \cdot \nabla)\eta) = (\tilde{g}_a((\mathbf{b} \cdot \nabla)\eta, \nabla \mathbf{b}), \mathbf{r}).\]

Now, from the definition of the optimal control problem, if \((\mathbf{u}, p, \sigma, \mathbf{g})\) is an optimal solution and the Gâteaux derivative of the cost functional exists, the latter must be zero for all directions \(h \in L^2(\Gamma_2)\). Thus, (2.7) holds. \(\square\)

### 2.4. The optimality system

Collecting the results in Section 2.3, we obtain the optimality system

\[
\begin{aligned}
&\left\{ \begin{aligned}
&\left(\sigma + \lambda(\mathbf{b} \cdot \nabla)\sigma, \mathbf{r} + \lambda\delta(\mathbf{b} \cdot \nabla)\mathbf{r} \right) + \lambda(g_a(\sigma, \nabla \mathbf{b}), \mathbf{r} + \lambda\delta(\mathbf{b} \cdot \nabla)\mathbf{r}) \\
&\quad - 2\alpha(D(\mathbf{u}), \mathbf{r} + \lambda\delta(\mathbf{b} \cdot \nabla)\mathbf{r}) = 0 \quad \forall \mathbf{r} \in \mathbf{\Sigma}, \\
&\left(\eta - \lambda(\mathbf{b} \cdot \nabla)\eta, \mathbf{r} - \lambda\delta(\mathbf{b} \cdot \nabla)\mathbf{r} \right) + \lambda(\tilde{g}_a((\mathbf{b} \cdot \nabla)\eta, \nabla \mathbf{b}), \mathbf{r}) \\
&\quad + (D(\mathbf{w}), \mathbf{r}) = 0 \quad \forall \mathbf{r} \in \mathbf{\Sigma}, \\
&-2\alpha(\eta + \lambda\delta(\mathbf{b} \cdot \nabla)\eta, D(\mathbf{v})) + 2(1 - \alpha)(D(\mathbf{w}), D(\mathbf{v})) = -(\xi, \nabla \cdot \mathbf{v}) \\
&\quad - (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}, \\
&\left(\mathbf{g}, \mathbf{h}\right)_{\Gamma_2} = \frac{1}{\epsilon}(\mathbf{w}, \mathbf{h})_{\Gamma_2} \quad \forall \mathbf{h} \in L^2(\Gamma_2).
\end{aligned} \right.
\]

(\text{OS})

The optimality condition (the last equation in (OS)) can be substituted into the state equations and thus, we have

\[
\begin{aligned}
&\left\{ \begin{aligned}
&\left(\sigma + \lambda(\mathbf{b} \cdot \nabla)\sigma, \mathbf{r} + \lambda\delta(\mathbf{b} \cdot \nabla)\mathbf{r} \right) + \lambda(g_a(\sigma, \nabla \mathbf{b}), \mathbf{r} + \lambda\delta(\mathbf{b} \cdot \nabla)\mathbf{r}) \\
&\quad - 2\alpha(D(\mathbf{u}), \mathbf{r} + \lambda\delta(\mathbf{b} \cdot \nabla)\mathbf{r}) = 0 \quad \forall \mathbf{r} \in \mathbf{\Sigma}, \\
&\left(\eta - \lambda(\mathbf{b} \cdot \nabla)\eta, \mathbf{r} - \lambda\delta(\mathbf{b} \cdot \nabla)\mathbf{r} \right) + \lambda(\tilde{g}_a((\mathbf{b} \cdot \nabla)\eta, \nabla \mathbf{b}), \mathbf{r}) \\
&\quad + (D(\mathbf{w}), \mathbf{r}) = 0 \quad \forall \mathbf{r} \in \mathbf{\Sigma}, \\
&-2\alpha(\eta + \lambda\delta(\mathbf{b} \cdot \nabla)\eta, D(\mathbf{v})) + 2(1 - \alpha)(D(\mathbf{w}), D(\mathbf{v})) = -(\xi, \nabla \cdot \mathbf{v}) \\
&\quad - (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}, \\
&\left(\nabla \cdot \mathbf{w}, q\right) = 0 \quad \forall q \in \mathbf{S}.
\end{aligned} \right.
\]

(\text{OS}’)

Given \(\mathbf{f} \in H^{-1}(\Omega)\), the regularity of the solutions to problem (OS) are the following:

\[\begin{aligned}
&\left(\mathbf{u}, p, \sigma, \mathbf{w}, r, \eta, \mathbf{g}\right) \in \mathbf{X} \times \mathbf{S} \times \mathbf{\Sigma} \times \mathbf{X} \times \mathbf{S} \times L^2(\Gamma_2).
\end{aligned}\]

(2.13)

To study the adjoint system in (OS), define the bilinear form \(\tilde{A}\) defined on \((\mathbf{V} \times \mathbf{\Sigma}) \times (\mathbf{V} \times \mathbf{\Sigma})\) by

\[
\tilde{A}((\eta, \mathbf{w}), (\mathbf{r}, \mathbf{v})) := -2\alpha(\eta + \lambda\delta(\mathbf{b} \cdot \nabla)\eta, D(\mathbf{v})) + 2(1 - \alpha)(D(\mathbf{w}), D(\mathbf{v})) \\
+ 2\alpha(\eta - \lambda(\mathbf{b} \cdot \nabla)\eta, \mathbf{r} - \lambda\delta(\mathbf{b} \cdot \nabla)\mathbf{r}) \\
+ 2\alpha\lambda(\tilde{g}_a((\mathbf{b} \cdot \nabla)\eta, \nabla \mathbf{b}), \mathbf{r}) + 2\alpha(D(\mathbf{w}), \mathbf{r}).
\]

(2.14)

Using the adjoint bilinear form \(\tilde{A}\) defined by (2.14), the adjoint equations in (OS) can equivalently be written as

\[
\tilde{A}((\eta, \mathbf{w}), (\mathbf{v}, \mathbf{r})) = -(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) \quad \forall (\mathbf{v}, \mathbf{r}) \in \mathbf{V} \times \mathbf{\Sigma}.
\]

(2.15)
We can show by the similar way in [8] that $\tilde{A}$ is coercive and continuous if the parameters $\lambda$, $\alpha$, $\delta$ and the bound $M$ are small enough so that

$$1 - 2\lambda Md - \epsilon_{1} \delta \lambda Md > 0, \quad (2.16)$$

$$2 \alpha - 2 \alpha^2 \frac{\lambda Md}{\epsilon_{1}} - \epsilon_{2} > 0, \quad (2.17)$$

$$2(1 - \alpha) - \frac{\alpha^2 \delta}{\epsilon_{2}} > 0, \quad (2.18)$$

for $\epsilon_{1}, \epsilon_{2} > 0$. Therefore, by the Lax–Migram theorem and the inf-sup condition (1.10), the adjoint system (2.15) admits a unique solution $(w, \xi, \eta) \in X \times S \times \Sigma$ if (2.16)–(2.18) is satisfied. And, furthermore, we have the estimate

$$\|w\|_1 + \|\xi\|_0 + \|\eta\|_0 \leq C \|u\|_1 \leq C \quad (2.19)$$

by (1.25).

3. Finite element approximation

3.1. Finite element spaces

Suppose $T_h$ is a triangulation of $\Omega$ such that $\overline{\Omega} = \bigcup K$: $K \in T_h$. Assume that there exist positive constants $c_1, c_2$ such that

$$c_1 \rho_K \leq h_K \leq c_2 \rho_K,$$

where $h_K$ is the diameter of $K$, $\rho_K$ is the diameter of the greatest ball included in $K$, and $h = \max_{K \in T_h} h_K$.

Let $P_k(K)$ denote the space of polynomials of degree less than or equal to $k$ on $K \in T_h$. We define finite element spaces for an approximation of $(u, p, \sigma)$:

$$X^h := \{ v \in X \cap (C^0(\overline{\Omega}))^d : v|_K \in P_2(K)^d \ \forall K \in T_h \},$$

$$S^h := \{ q \in S \cap C^0(\overline{\Omega}) : q|_K \in P_1(K) \ \forall K \in T_h \},$$

$$\Sigma^h := \{ \tau \in \Sigma \cap (C^0(\overline{\Omega}))^{d \times d} : \tau|_K \in P_1(K)^{d \times d} \ \forall K \in T_h \},$$

$$V^h := \{ v \in X^h : (q, \nabla \cdot v) = 0 \ \forall q \in S^h \}.$$

The finite element spaces defined above satisfy the standard approximation properties (see [3] or [10]), i.e., there exist an integer $k$ and a constant $C$ such that

$$\inf_{v^h \in X^h} \| v - v^h \|_1 \leq C h^{2} \| v \|_3 \ \forall v \in H^3(\Omega), \quad (3.1)$$

$$\inf_{q^h \in S^h} \| q - q^h \|_0 \leq C h^{2} \| q \|_2 \ \forall q \in H^2(\Omega), \quad (3.2)$$

and

$$\inf_{\tau^h \in \Sigma^h} \| \tau - \tau^h \|_0 \leq C h^{2} \| \tau \|_2 \ \forall \tau \in H^2(\Omega). \quad (3.3)$$

It is also well known that the Taylor–Hood pair $(X^h, S^h)$ satisfies the inf-sup (or LBB) condition

$$\inf_{\tau^h \in \Sigma^h} \sup_{0 \neq q^h \in S^h} \frac{(q^h, \nabla \cdot v^h)}{\| v^h \|_1 \| q^h \|_0} \geq C, \quad (3.4)$$
where \( C \) is a positive constant independent of \( h \).

The finite element approximation of (1.11)–(1.13) is then as follows: find \( u^h \in X^h, \ p^h \in S^h, \ \sigma^h \in \Sigma^h \) such that

\[
\begin{align*}
(\sigma^h + \lambda(b \cdot \nabla)\sigma + \lambda g_\sigma(\sigma^h, \nabla b) - 2\alpha D(u^h), \tau^h + \lambda \delta b \cdot \nabla \tau^h) &= 0 \quad \forall \tau^h \in \Sigma^h, \\
(\sigma^h, d(v^h)) + 2(1 - \alpha)(d(u^h), d(v^h)) - (p^h, \nabla \cdot v^h) &= (f, v^h) \quad \forall v^h \in X^h, \\
(q^h, \nabla \cdot u^h) &= 0 \quad \forall q^h \in S^h.
\end{align*}
\]

(3.5) (3.6) (3.7)

Notice that, in view of (3.4), (3.5)–(3.7) is equivalent to: find \( u^h \in V^h \) and \( \sigma^h \in \Sigma^h \) such that

\[
\begin{align*}
(\sigma^h + \lambda(b \cdot \nabla)\sigma + \lambda g_\sigma(\sigma^h, \nabla b) - 2\alpha D(u^h), \tau^h + \lambda \delta b \cdot \nabla \tau^h) &= 0 \quad \forall \tau^h \in \Sigma^h, \\
(\sigma^h, d(v^h)) + 2(1 - \alpha)(d(u^h), d(v^h)) &= (f, v^h) \quad \forall v^h \in V^h.
\end{align*}
\]

(3.8) (3.9)

In [8] we proved the following a priori error estimate.

**Lemma 3.1.** For \( (u^h, p^h, \tau^h) \) satisfying (3.5)–(3.7), and \( (u, p, \sigma) \in H^3(\Omega) \times H^2(\Omega) \times H^2(\Omega) \) satisfying (1.11)–(1.13), we have the error estimate

\[
\begin{align*}
\|u - u^h\|_1 + \|\sigma - \sigma^h\|_0 + \sqrt{\delta \lambda} \|b \cdot \nabla (\sigma - \sigma^h)\|_0 + \|p - p^h\|_0 &
\leq C[h^2\|u\|_3 + h\|\sigma\|_2 + h^2\|p\|_2].
\end{align*}
\]

(3.10)

**3.2. Quotation of some results concerning the approximation of a class of nonlinear problems**

Here for the sake of completeness, we will state the relevant results specialized to our needs. The nonlinear problems considered in [4] and [10] are of the type

\[
F(\lambda, \psi) := \psi + AG(\lambda, \psi) = 0,
\]

(3.11)

where \( A \in L(Y; X) \), \( G \) is a \( C^2 \) mapping from \( \Lambda \times X \) into \( Y \), \( X \) and \( Y \) are Banach spaces, and \( \Lambda \) is a compact interval of \( \mathbb{R} \). We say that \( \{ (\lambda, \psi(\lambda)): \lambda \in \Lambda \} \) is a branch of solutions of (3.11) if \( \lambda \rightarrow \psi(\lambda) \) is a continuous function from \( \Lambda \) into \( X \) such that \( F(\lambda, \psi(\lambda)) = 0 \). The branch is called a nonsingular branch if we also have that \( D_\psi F(\lambda, \psi(\lambda)) \) is an isomorphism from \( X \) into \( X \) for all \( \lambda \in \Lambda \). Here, \( D_\psi \) denotes the Fréchet derivative with respect to \( \psi \). Approximations are defined by introducing a subspace \( X^h \subset X \) and an approximating operator \( A^h \in L(Y; X^h) \). Then we seek \( \psi^h \in X^h \) such that

\[
F^h(\lambda, \psi^h) := \psi^h + A^h G(\lambda, \psi^h) = 0.
\]

(3.12)

We will assume that there exists another Banach space \( Z \), contained in \( Y \), with continuous imbedding such that

\[
D_\psi G(\lambda, \psi) \in L(X; Z), \quad \forall \lambda \in \Lambda, \forall \psi \in X.
\]

(3.13)

Concerning the operator \( A^h \), we assume the approximation properties

\[
\lim_{h \to 0} \| (A^h - A)y \|_X = 0 \quad \forall y \in Y,
\]

(3.14)

and

\[
\lim_{h \to 0} \| A^h - A \|_{L(Z; X)} = 0.
\]

(3.15)

Note that (3.13) and (3.15) imply that the operator \( D_\psi G(\lambda, \psi) \in L(X, X) \) is compact. Moreover, (3.15) follows from (3.14) whenever the imbedding \( Z \subset Y \) is compact.

Now we can state the first result of [4] and [10] that used in the sequel.
Theorem 3.2. Let $X$ and $Y$ be Banach spaces and $A$ a compact subset of $\mathbb{R}$. Assume that $G$ is a $C^2$ mapping from $\Lambda \times X$ into $Y$ and that $D^2 G$ is bounded on all sets of $\Lambda \times X$. ($D^2 G$ represents second Fréchet derivative of $G$). Assume that (3.13)–(3.15) hold and $\{(\lambda, \psi(\lambda)) : \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (3.11). Then, there exists a neighborhood $\mathcal{O}$ of the origin in $X$ and for $h \leq h_0$ small enough, a unique $C^2$ function $\lambda \in \Lambda \to \psi^h(\lambda) \in X^h$ such that $\{(\lambda, \psi^h(\lambda)) : \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (3.12) and $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ for all $\lambda$. Moreover, there exists a constant $C > 0$, independent of $h$ and $\lambda$, such that

$$\|\psi^h(\lambda) - \psi(\lambda)\|_X \leq C \left\| (A^h - A) G(\lambda, \psi(\lambda)) \right\|_X, \quad \forall \lambda \in \Lambda. \tag{3.16}$$

For the second result, we have to introduce two other Banach spaces $H$ and $W$, such that $W \subset X \subset H$, with continuous imbeddings and assume that

$$\forall w \in W \text{ the operator } D\psi G(\lambda, w) \text{ may be extended as a linear operator of } \mathcal{L}(H; Y), \tag{3.17}$$

and the mapping $w \to D\psi G(\lambda, w)$ is continuous from $W$ onto $\mathcal{L}(H; Y)$.

We also suppose that

$$\lim_{h \to 0} \|A^h - A\|_{\mathcal{L}(Y; H)} = 0. \tag{3.18}$$

Then we may state the following additional result.

Theorem 3.3. Assume the hypotheses of Theorem 3.2, and also assume that (3.17) and (3.18) hold. Assume in addition that

for each $\lambda \in \Lambda$, $\psi^h(\lambda) \in W$ and the function $\lambda \to \psi(\lambda)$ is continuous from $\Lambda$ into $W$. \tag{3.19}

and

for each $\lambda \in \Lambda$, $D\psi F(\lambda, \psi(\lambda))$ is an isomorphism of $H$. \tag{3.20}

Then, for $h \leq h_1$ sufficiently small, there exists a constant $C$, independent of $h$ and $\lambda$, such that

$$\|\psi^h(\lambda) - \psi(\lambda)\|_H \leq C \left\| (A^h - A) G(\lambda, \psi(\lambda)) \right\|_H + \|\psi^h(\lambda) - \psi(\lambda)\|_{X^h}, \quad \forall \lambda \in \Lambda. \tag{3.21}$$

3.3. Error estimate for the optimality system

The finite element approximation of a solution of the optimality system (OS') is defined as follows. Seek $(u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h) \in X^h \times S^h \times \Sigma^h \times X^h \times S^h \times \Sigma^h$ such that

$$\begin{cases}
\left(\sigma^h + \lambda (b \cdot \nabla)\sigma^h, \tau^h + \lambda \delta (b \cdot \nabla)\tau^h\right) + \lambda \left(g_\alpha (\sigma^h, \nabla b), \tau^h + \lambda \delta (b \cdot \nabla)\tau^h\right) - 2\alpha (D(u^h), \tau^h + \lambda \delta (b \cdot \nabla)\tau^h) = 0 & \forall \tau^h \in \Sigma, \\
\left(\sigma^h, D(w^h)\right) + 2(1 - \alpha) (D(u^h), D(w^h)) - (p^h, \nabla \cdot v^h) = (f, v^h) + \frac{1}{\varepsilon} (v^h, v^h)_{L^2} & \forall v^h \in X^h, \\
\left(q^h, \nabla \cdot u^h\right) = 0 & \forall q^h \in S^h(\Omega), \\
\left(\eta^h - \lambda (b \cdot \nabla)\eta^h, \tau^h + \lambda \delta (b \cdot \nabla)\tau^h\right) + \lambda \left(g_\alpha (\eta^h + \lambda \delta (b \cdot \nabla)\eta^h, \nabla b), \tau^h\right) + (D(w^h), \tau^h) = 0 & \forall \tau^h \in \Sigma^h, \\
-2\alpha (\eta^h + \lambda \delta (b \cdot \nabla)\eta^h, D(w^h)) + 2(1 - \alpha) (D(w^h), D(v^h)) = -\left(\nabla \chi, \chi \cdot v^h\right) & \forall v^h \in X^h, \\
\left(\nabla \cdot w^h, q^h\right) = 0 & \forall q^h \in S^h.
\end{cases} \tag{DOS'}$$
We obtain an error estimate for the finite element approximation using the Brezzi–Rappaz–Raviart Theory [4]. Let

\[ W := X \times S \times \Sigma \times X \times S \times \Sigma, \]
\[ Y := X^{-1} \times X^{-1}, \]
\[ Z := L^2(\Omega) \times L^2(\Omega), \]

and

\[ W^h := X^h \times S^h \times \Sigma^h \times X^h \times S^h \times \Sigma^h. \]

Define \( T \in \mathcal{L}(Y, W) \) as \( T(\tilde{f}, \tilde{h}) = (u, p, \sigma, w, \xi, \eta) \) such that

\[
\begin{align*}
(\sigma + \lambda (b \cdot \nabla)\sigma, \tau + \lambda \delta (b \cdot \nabla)\tau) + \lambda (g_a(\sigma, \nabla b), \tau + \lambda \delta (b \cdot \nabla)\tau) & - 2\alpha (D(u), \tau + \lambda \delta (b \cdot \nabla)\tau) = 0 \quad \forall \tau \in \Sigma, \\
(\sigma, D(v)) + 2(1 - \alpha)(D(u), D(v)) - (p, \nabla \cdot v) &= (\tilde{f}, v) \quad \forall v \in X, \\
(q, \nabla \cdot u) &= 0 \quad \forall q \in S, \\
(\eta - \lambda (b \cdot \nabla)\eta, \tau - \lambda \delta (b \cdot \nabla)\tau) + \lambda (\bar{g}_a(\eta + \lambda \delta (b \cdot \nabla)\eta, \nabla b), \tau) & + 2\alpha(\eta + \lambda \delta (b \cdot \nabla)\eta, D(v)) + 2(1 - \alpha)(D(w), D(v)) - (\xi, \nabla \cdot v) \\
= (\tilde{h}, v) & \quad \forall v \in X, \\
(\nabla \cdot w, q) &= 0 \quad \forall q \in S.
\end{align*}
\]

The discrete operator \( T^h \in \mathcal{L}(Y, W^h) \) is also defined similarly: for \((u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h) \in W^h \) and \((\tilde{f}, \tilde{h}) \in Y \)

\[
T^h(\tilde{f}, \tilde{h}) = (u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h)
\]

if and only if the above system is satisfied with \((u, p, \sigma, w, \xi, \eta) \) replaced by \((u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h) \) for all \((v^h, q^h, \tau^h) \in X^h \times S^h \times \Sigma^h \). Note that the adjoint problem (3.25)–(3.27) is equivalent to

\[
\tilde{A}(\tilde{(w), (v), (\tau)}) = (\tilde{h}, v) \quad \forall (v, \tau) \in V \times \Sigma,
\]

where \( \tilde{A} \) is coercive and continuous under the conditions (2.16)–(2.18). Hence, if \((w, \xi, \eta) \in H^3(\Omega) \times H^2(\Omega) \times H^2(\Omega) \) satisfies (3.25)–(3.27) and \((w^h, \xi^h, \eta^h) \in X^h \times S^h \times S^h \) is a numerical approximation of the solution, the following error estimate holds:

\[
\| w - w^h \|_1 + \| \eta - \eta^h \|_0 + \sqrt{\delta \lambda} \| b \cdot \nabla (\eta - \eta^h) \|_0 + \| \xi - \xi^h \|_0 \leq C \left[ h^2 \| w \|_3 + h \| \eta \|_2 + h^2 \| \xi \|_2 \right].
\]

See [8] for details. Now, by (3.10) and (3.28), we have

\[
\| (u, p, \sigma, w, \xi, \eta) - (u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h) \|_W \leq C \left[ h^2 \| u \|_3 + \| w \|_3 \right] + h \left[ \| \sigma \|_2 + \| \eta \|_2 \right] + h^2 \left[ \| p \|_2 + \| \xi \|_2 \right].
\]

Let \( \Lambda \) be a positive interval containing \( 1/\epsilon \) and define the operator \( G \) from \( \Lambda \times W \) to \( Y \) as follows. For \((\tilde{f}, \tilde{h}) \in Y \) and \((1/\epsilon, (u, p, \sigma, w, \xi, \eta)) \in \Lambda \times W \),

\[
G(\epsilon, (u, p, \sigma, w, \xi, \eta)) = (\tilde{f}, \tilde{h})
\]
if and only if

\[
\begin{align*}
(\sigma + \lambda(b \cdot \nabla)\sigma, \tau + \lambda\delta(b \cdot \nabla)\tau) + \lambda(g_a(\sigma, \nabla b), \tau + \lambda\delta(b \cdot \nabla)\tau) \\
- 2\alpha(D(u), \tau + \lambda\delta(b \cdot \nabla)\tau) = 0 \quad \forall \tau \in \Sigma,
\end{align*}
\]

\[
(\tilde{f}, v) = -(f, v) - \frac{1}{\epsilon}(w, v)_{L^2} \quad \forall v \in X,
\]

\[
(q, \nabla \cdot u) = 0 \quad \forall q \in S.
\]

Then, by the definition of the operators \( T, T^h \) and \( G \), the optimality system (OS') can be expressed as

\[
(u, p, \sigma, w, \xi, \eta) + TG(\epsilon, (u, p, \sigma, w, \xi, \eta)) = 0.
\]

And similarly, the discrete optimality system (DOS') is equivalent to

\[
(u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h) + T^h G(\epsilon, (u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h)) = 0.
\]

Note that the operator \( G \) is linear and of class \( C^\infty \). Since \( A \) is compact, \( D^2 G \) is bounded on all bounded sets of \( A \times W \) by (1.25) and (2.19). The derivative of \( G \), \( D\varphi G \), is defined by

\[
D\varphi G(\epsilon, (u, p, \sigma, w, \xi, \eta))(\tilde{u}, \tilde{p}, \tilde{\sigma}, \tilde{w}, \tilde{\xi}, \tilde{\eta}) = (\tilde{f}, \tilde{h})
\]

for \((\tilde{u}, \tilde{p}, \tilde{\sigma}, \tilde{w}, \tilde{\xi}, \tilde{\eta}) \in W \) if and only if

\[
(\tilde{\sigma} + \lambda(b \cdot \nabla)\tilde{\sigma}, \tau + \lambda\delta(b \cdot \nabla)\tau) + \lambda(g_a(\tilde{\sigma}, \nabla b), \tau + \lambda\delta(b \cdot \nabla)\tau) \\
- 2\alpha(D(\tilde{u}), \tau + \lambda\delta(b \cdot \nabla)\tau) = 0 \quad \forall \tau \in \Sigma,
\]

\[
(\tilde{f}, v) = -\frac{1}{\epsilon}(\tilde{w}, v)_{L^2} \quad \forall v \in X,
\]

\[
(q, \nabla \cdot \tilde{u}) = 0 \quad \forall q \in S(\Omega),
\]

\[
(\tilde{\eta} - \lambda(b \cdot \nabla)\tilde{\eta}, \tau - \lambda\delta(b \cdot \nabla)\tau) + \lambda(g_a(\tilde{\eta}, \nabla b), \tau + \lambda\delta(b \cdot \nabla)\tau) \\
+ (D(\tilde{w}), \tau) = 0 \quad \forall \tau \in \Sigma,
\]

\[
(\tilde{h}, v) = (\nabla \times \tilde{u}, \nabla \times v) \quad \forall v \in X,
\]

\[
(\nabla \cdot \tilde{w}, q) = 0 \quad \forall q \in S.
\]

Thus, \( D\varphi G \in \mathcal{L}(W, Y) \). Furthermore, \( D\varphi G \in \mathcal{L}(W, Z) \) and \( Z \subset Y \) is compact.

We have verified the hypotheses of the Brezzi–Rappaz–Raviart Theory, and now may have the following error estimate.

**Theorem 3.4.** Let \((u, p, \sigma, w, \xi, \eta)\), \((u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h)\) be the solution of (OS') and (DOS'), respectively. If \((u, p, \sigma, w, \xi, \eta) \in H^3(\Omega) \times H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega) \times H^2(\Omega) \times H^2(\Omega)\) and (1.20)–(1.22), (2.16)–(2.18) are satisfied,
\[ \|u - u^h\|_1 + \|\tau - \tau^h\|_0 + \sqrt{\delta\lambda}\|b \cdot \nabla(\tau - \tau^h)\|_0 + \|p - p^h\|_0 + \|w - w^h\|_1 + \|\eta - \eta^h\|_0 + \|\xi - \xi^h\|_0 \leq C[h^2\|u\|_3 + h\|\sigma\|_2 + h^2\|p\|_2 + h^2\|w\|_3 + h\|\eta\|_2 + h^2\|\xi\|_2]. \]

4. A projected gradient method

To obtain an approximate solution of the optimal control problem, one may solve the discretized optimality system (DOS) which consists of the following: the discrete state equations

- \( (\sigma^h + \lambda(b \cdot \nabla)\sigma^h, \tau^h + \lambda\delta(b \cdot \nabla)\tau^h) + \lambda\left(\bar{g}_d(\sigma^h, \nabla b), \tau^h + \lambda\delta(b \cdot \nabla)\tau^h\right) - 2\alpha(D(u^h), \tau^h + \lambda\delta(b \cdot \nabla)\tau^h) = 0 \quad \forall \tau^h \in \Sigma^h, \)
- \( (g, D(v^h)) + 2(1 - \alpha)(D(u^h), D(v^h)) - (p, \nabla \cdot v^h) = (f, v^h) + (g^h, v^h) \in \Gamma_2 \quad (4.1) \)
- \( q^h \cdot \nabla \cdot u^h = 0 \quad \forall q^h \in S^h, \)

the discrete adjoint equations

- \( (\eta^h - \lambda(b \cdot \nabla)\eta^h, \tau^h + \lambda\delta(b \cdot \nabla)\tau^h) + \lambda\left(\bar{g}_d(\eta^h + \lambda\delta(b \cdot \nabla)\eta^h, \nabla b), \tau^h\right) + (D(w^h), \tau^h) = 0 \quad \forall \tau \in \Sigma, \)
- \( -2\alpha(\eta^h + \lambda\delta(b \cdot \nabla)\eta^h, D(w^h)) + 2(1 - \alpha)(D(w^h), D(v^h)) - (\xi^h, \nabla \cdot v^h) \in \Sigma_\delta \quad (4.2) \)
- \( \nabla \cdot w^h, q^h = 0 \quad \forall q^h \in S^h, \)

and the discrete optimality condition

- \( (g^h, h^h)_{\Gamma_2} = \frac{1}{\varepsilon}(w^h, h^h)_{\Gamma_2} \quad \forall h^h \in L^2(\Gamma_2). \quad (4.3) \)

In practice, the size of the above system is huge. A gradient method is considered to overcome the computational difficulties. At each iteration the method requires the sequential solution of (4.1) and (4.2). The gradient method for minimizing a functional \( T(g^h) \) may be described as follows:

**Algorithm 4.1 (Gradient algorithm).**

1. set \( k = 0 \) and choose \( g^h(0) \),
2. set \( \delta g^h := -R\frac{dT(g^h(k))}{dg} \) and compute
   \[ \rho(k) = \arg\min_{\rho > 0} T\left(g^h(k) + \rho\delta g^h\right). \]
3. set \( g^h(k + 1) = g^h(k) + \rho(k)\delta g^h \),
4. set \( k = k + 1 \) and goto 2,

where the functional \( T \) is defined by

- \( T(g) := J(u(g), p(g), \sigma(g), g) \).

The convergence property of the above algorithm is given in the following result. The convergence of the Conjugate Gradient Algorithm can be proved in the similar fashion.
Theorem 4.1. Let \((u^h(k), p^h, \sigma^h(k), w^h(k), \xi^h(k), \eta^h(k), g^h(k))\) be the solution of the gradient algorithm and \((u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h, g^h)\) the solution of (4.1)–(4.3). Then if \(\rho(k) > 0\) is sufficiently small, \(g^h(k) \to g^h\) and thus, \((u^h(k), p^h, \sigma^h(k), w^h(k), \xi^h(k), \eta^h(k)) \to (u^h, p^h, \sigma^h, w^h, \xi^h, \eta^h)\) as \(k \to \infty\).

Proof. Let \(\rho(k)\) be the bilinear form associated with the second Fréchet derivatives of \(T(\cdot)\). Suppose that \(\rho(k)\) is chosen so that

\[
0 < \rho_* \leq \rho(k) \leq \rho^* < \frac{2m}{M^2} \quad \text{for all } k
\]

for some positive numbers \(\rho_*\) and \(\rho^*\). Then the iterates of the algorithm

\[
g(k + 1) = g(k) - \rho(k)R \frac{dT(g(k))}{dg}, \quad k = 1, 2, \ldots,
\]

converge to \(g\) for any initial iterate \(g(0) \in B\). In (4.6), \(R\) is the Riesz map from \(X^{-1}\) to \(X\) and \(\rho(k)\) is a sequence of positive step lengths.

Now, for each \(g^h \in X (:= L^2(\Gamma_2))\), the second Fréchet derivative \(T''(g^h) \cdot (\delta g^h_1, \delta g^h_2)\) may be computed by

\[
T''(g^h) \cdot (\delta g^h_1, \delta g^h_2) = \epsilon (\delta g^h_1, \delta g^h_2)_{\Gamma_2} + (\nabla \times \tilde{w}^h_1, \nabla \times \tilde{w}^h_2),
\]

where \(\tilde{w}^h_1\) and \(\tilde{w}^h_2\) are solutions of

\[
\begin{cases}
(\tilde{g}^h + \lambda (b \cdot \nabla \eta^h_1), \tau + \lambda \delta (b \cdot \nabla \eta^h_1) + \lambda (\gamma_a (\tilde{g}^h, \nabla b), \tau + \lambda \delta (b \cdot \nabla \eta^h_1) \tau = 0 \quad \forall \tau \in \Sigma^h, \\
-2 \lambda (D(\tilde{w}^h_1), \tau + \lambda \delta (b \cdot \nabla \eta^h_1) \tau = 0 \quad \forall \tau \in \Sigma^h,
\end{cases}
\]

for \(i = 1, 2\), respectively. We will show that (4.4) is satisfied for some \(M\) and \(m\). From the estimate (1.25), we have

\[
\|w^h_i\|_1 \leq C_i \|\delta g^h_i\|_{0, \Gamma_2}
\]

for \(i = 1, 2\). Then, using (4.9) and from (4.7), we have that there is some constant \(C\) depending on \(C_1\) and \(C_2\) such that

\[
|T''(g^h) \cdot (\delta g^h_1, \delta g^h_2)| \leq (\epsilon + C) \|\delta g^h_1\|_{0, \Gamma_2} \|\delta g^h_2\|_{0, \Gamma_2}
\]

and

\[
|T''(g^h) \cdot (\delta g^h_1, \delta g^h_1)| \geq \epsilon \|\delta g^h_1\|^2_{0, \Gamma_2} + \|\nabla \times w^h_1\|_0^2 \geq \epsilon \|\delta g^h_1\|^2_{0, \Gamma_2}.
\]

Remark 4.2. In Algorithm 4.1, \(R\) is the identity map from \(L^2(\Gamma_2)\) to \(L^2(\Gamma_2)\) since \(X = L^2(\Gamma_2)\). To decide \(\rho(k)\), one has to solve an optimization problem in Algorithm 4.1,

\[
\rho(k) = \arg \min_{\rho > 0} T(g^h(k) + \rho \delta g^h).
\]
For each direction $\delta g^h \in L^2(\Gamma_2)$, consider inserting
\[ u(g^h(k) + \rho \delta g^h) = u(g^h(k)) + \rho \xi \] (4.13)
into the cost functional $T$, where $\xi$ is the solution to (1.11)–(1.13) with $f = 0$ and $g = \delta g^h$.

Then one needs to solve the following minimization problem with respect to $\rho$
\[ \rho(k) = \arg \min_{\rho > 0} \left( \frac{1}{2} \int_\Omega (\nabla \times u(g^h(k) + \rho \delta g^h))^2 \, d\Omega + \frac{\epsilon}{2} \int_{\Gamma_2} (g^h(k) + \rho \delta g^h)^2 \, d\Gamma_2 \right), \] (4.14)
which is a quadratic function with respect to $\rho$ and has the unique solution
\[ \rho(k) = \frac{-(\nabla \times u(g^h(k)), \nabla \times \xi - (g^h(k), \delta g^h(k)))_{\Gamma_2}}{\epsilon ||\delta g^h(k)||_{L_2}^2 + ||\nabla \times \xi||^2_{L_2}}. \]

5. Concluding remarks and future work

We investigated an optimal control technique to minimize vortices by a Newmann type boundary control in the setting of the Oseen viscoelastic fluid flow problem. The existence of a unique optimal solution was shown and the coupled optimality system consisting of the state and adjoint equations was derived. We also studied a computational algorithm as well as a finite element error estimate for the system.

As the next step, some numerical experiments will be performed to verify the theoretical results presented in this paper. We will consider both the linearized Oseen problem (1.5)–(1.7) and the nonlinear problem (1.1)–(1.3) for experiments and make some comparisons. We will also extend this work to the power law constitutive model equation
\[ \sigma - \nu_0 |D(u)|^{r-2}D(u) = 0, \] (5.1)
where $1 < r < 2$. The power law model has been used to model the viscosity of many polymeric solutions and melts over a considerable range of shear rates [13]. The quasi-Newtonian Stokes flow equations, (1.2)–(1.3) and (5.1), were studied for existence results and finite element error estimates in appropriately chosen Sobolev spaces [1,7,9]. The results presented in the articles will be used for study on an optimal control problem associated with the power law model equations.

References

[17] K. Najib, D. Sadri, On a decoupled algorithm for solving a finite element problem for the approximation of vis-