Abstract

We present a new semi-topological quantity, called the absolute hodograph winding number, that measures how close the quintic PH spline interpolating a given sequence of points is to the cubic spline interpolating the same sequence. This quantity then naturally leads into a new criterion of determining the best quintic PH spline interpolant. This seems to work favorably compared with the elastic bending energy criterion developed by Farouki [Farouki, R.T., 1996. The elastic bending energy of Pythagorean-hodograph curves. Comput. Aided Geom. Design 13 (3), 227–241]. We also present a fast method that is a modification of the method of Albrecht, Farouki, Kuspa, Manni, and Sestini [Albrecht, G., Farouki, R.T., 1996. Construction of \( C^2 \) Pythagorean-hodograph interpolating splines by the homotopy method. Adv. Comput. Math. 5 (4), 417–442; Farouki, R.T., Kuspa, B.K., Manni, C., Sestini, A., 2001. Efficient solution of the complex quadratic tridiagonal system for \( C^2 \) PH quintic splines. Numer. Algorithms 27 (1), 35–60]. While the basic scheme of our approach is essentially the same as theirs, ours differs in that the underlying space in which the Newton–Raphson method is applied is the double covering space of the hodograph space, whereas theirs is the hodograph space itself. This difference, however, seems to produce more favorable results, when viewed from the above mentioned semi-topological criterion.

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1. Introduction

Pythagorean-hodograph (PH) curve, which was first introduced in Farouki and Sakkalis (1990), Farouki (1992), offers computational advantage over ordinary polynomial curves in CAD/CAGD. For example, the speed of a PH curve is a polynomial—thus the arc length of a PH curve can be evaluated exactly by evaluating the polynomial—and the offset curves are rational curves which can also be expressed exactly in the CAD/CAGD system. There are many other attracting features the PH curves have: for example, the curvature can be expressed as a the rational function, and so on.

On the other hand, despite such attractive features, the PH curves are not easy to use in practice. One of the reasons is that the problem of constructing a PH spline curve interpolating a sequence of given points is not yet
well understood. For example, while the number of solutions of the PH spline interpolation problem proliferates exponentially as the number of points to be interpolated increases, it is not yet clear which of those solutions should be deemed best; and then even if there is one, it is not easy to find such best solution either.

In this paper, we study the quintic PH spline interpolating a sequence of given points. This problem was first studied by Albrecht and Farouki (1996). They showed that there are distinct $2^{N+k}$ ($k \in \{-1, 0, 1\}$) solutions when $N + 1$ points to be interpolated are given and developed a very nice algorithm to find all such solutions. Since there are $2^{N+k}$ solutions to the PH spline interpolation problem when there are $N + 1$ points to be interpolated, the first order of business is to decide which of these $2^{N+k}$ solutions should be regarded as the best. Farouki gave a nice and intuitive criterion using the concept of the elastic bending energy (Farouki, 1996). In most cases, it produces a nice (and correct) solution. However, as we shall see in the subsequent sections, it sometimes does not seem to produce the best solution, at least from the topological viewpoint we advocate. The reason is that a certain curve may have small elastic bending energy even if it loops.

We look at this problem from a different angle. In the previous paper, the authors together with Farouki and Moon studied the criterion of selecting the best quintic Hermite interpolant from the topological viewpoint (Choi et al., 2007). In there, they compare the PH quintic curves with the unique cubic spline to the same Hermite data and choose the PH quintic interpolant that is topologically closest to the cubic curve by introducing the topological quantity, called the hodograph winding number. The comparison of the PH quintic curve with the cubic curve makes sense because both have the same degree of freedom.

In this paper, we continue to adopt this philosophy of emphasizing the topological aspects of the problem. Therefore the natural growth of this idea is for us to compare the solutions of the PH quintic spline problem with the corresponding cubic spline. The trouble is that when we try to apply the same concept, i.e., the hodograph winding number, there appear too many solutions with the zero hodograph winding number. A visual inspection quickly depicts that many of the solution with zero hodograph winding number in fact have many loops whose windings cancel each other to produce the net zero hodograph winding number. These bad solutions should be ruled out. Our way of ruling them out is to introduce a modified quantity called the absolute hodograph winding number. As we shall see in the subsequent section, this absolute hodograph winding number turns out to be a rather powerful discriminant. In particular, we can easily construct, out of this, a selection criterion that picks out the best interpolant, which is presented in Section 4. We then perform extensive numerical experiments, which seems to confirm the usefulness and validity of our method.

While this selection criterion presented in Section 4 works quite well, it becomes unworkable when the number of points to be interpolated gets big. So one must have a different method to deal with the PH quintic spline problem in this case. It turns out that the method developed by Albrecht, Farouki, Kuspa, Manni, and Sestini comes in handy (Albrecht and Farouki, 1996; Farouki et al., 2001). They have devised a novel Newton–Raphson method (and initial point selection strategy) that seems to converge nicely in almost all cases to a solution of the PH quintic spline problem. The results are reported as excellent.

It turns out that the method of Albrecht, Farouki, Kuspa, Manni, and Sestini can be modified somewhat to produce even better results in terms of the selection criterion introduced in Section 4. In Section 5, we present this modified method. The main difference in our case is that the Newton–Raphson method and initial point selection is done in the double covering space of the hodograph space rather than the hodograph space itself as is the case with the Albrecht–Farouki–Kuspa–Manni–Sestini method. This seemingly minor change seems to produce not insignificant improvement as can be seen in the results in Section 5. The reason is that by running the algorithm in the double covering space, the ambiguities and perhaps the difficulties of choosing the branches that come with the solutions of the quadratics equations can be satisfactorily resolved. We performed extensive experiments with excellent result. In particular, it was very difficult to find the case in which our algorithm fails to produce the best solution. We believe the method we presented here can be profitably adopted in practice.

2. Preliminary

As in Choi et al. (2007), Farouki (1994), Farouki and Neff (1995), Moon et al. (2001), the complex representation will be used for planar curves. A planar PH quintic curve $\gamma(t) \in \mathbb{C}[t]$ is characterized by a quadratic curve $\chi(t) \in \mathbb{C}[t]$ through the relation,

$$\gamma'(t) = \chi(t)^2.$$
The first-order Hermite interpolation problem is to find a planar PH quintic curve \( \gamma(t) \in \mathbb{C}[t] \) such that
\[
\gamma(0) = p_0, \quad \gamma(1) = p_1, \quad \gamma'(0) = a, \quad \gamma'(1) = b
\]
for given first-order Hermite data \( p_0, p_1, a, b \in \mathbb{C} \). Such curve has the form
\[
\gamma(t) = p_0 + \int_0^t \chi(s)^2 \, ds
\]
for some quadratic complex polynomial \( \chi(t) \). Thus, this problem is reduced to finding a complex quadratic polynomial \( \chi(t) \in \mathbb{C}[t] \) such that
\[
\int_0^1 \chi(t)^2 \, dt = p_1 - p_0, \quad \chi(0)^2 = a, \quad \chi(1)^2 = b.
\]

Generally, there are four distinct solutions to this problem, but there is only one “good” solution which is a fair curve devoid of undesirable loops. To select a “good” solution among those four solutions, several criteria have been devised including the absolute rotation number, elastic bending energy, or absence of anti-parallel tangents (Farouki, 1996; Farouki and Neff, 1995; Moon et al., 2001), etc. Besides these, the authors together with Farouki and Moon proposed a topological quantity, called the hodograph winding number, and suggested that it is better to select as “good” solution the one with zero hodograph winding number (Choi et al., 2007). This solution is seen to be the one that is topologically closest to the unique cubic Bézier curve \( r(t) \) to the same first-order Hermite data in the sense that thus chosen PH quintic curve does not have unnecessary loops. Fig. 1 illustrates this point. In Fig. 1, four possible solutions to the PH quintic Hermite interpolation problem are depicted against the (same) corresponding cubic curve. It is readily seen that the PH quintic curves in the first, third and fourth cases have unnecessary loops and only that in the second case has no such loop. The method presented in Choi et al. (2007) picks out the likes of the ones shown in the second case.

Since the hodograph winding number, to be more accurate, the absolute hodograph winding number, is the basic concept we employ in this paper, it is better for us to review it here.

**Definition 2.1.** Let \( X(t) \) be a complex-valued continuous curve for \( t \in [0, 1] \). If \( X(t) \neq 0 \) for \( t \in [0, 1] \), there is a continuous real-valued function \( \theta(t) \) such that
\[
X(t) = |X(t)| \exp \sqrt{i \theta(t)}.
\]
For such an \( X(t) \), the angle variation \( \Delta \theta_X \) of \( X \) is defined by
\[
\Delta \theta_X = \theta(1) - \theta(0),
\]
and the winding number \( \text{wind}(X) \) is defined by
\[
\text{wind}(X) = \frac{1}{2\pi} \Delta \theta_X
\]
when \( X \) is a closed curve, i.e., \( X(0) = X(1) \).
Definition 2.2. Let \( r(t) \) be a complex plane to clarify which point of view is taken. We call the complex plane in which \( \alpha, \beta, z_i \) for some \( c(t) \) Fig. 2. Reversely-concatenated hodographs right, we have wind\((r' \oplus \gamma')(t)\) = \(-2\), wind\((r' \oplus \gamma')(t)\) = \(0\), wind\((r' \oplus \gamma')(t)\) = \(-1\), wind\((r' \oplus \gamma')(t)\) = \(1\).

We then need the following concept:

Definition 2.3 (Hodograph winding number). Let \( \gamma(t) \) be a PH quintic curve interpolating a first-order Hermite data as given in Eq. (1) and let \( r(t) \) be the unique cubic Bézier curve interpolating the same first-order Hermite data. Since \( r'(0) = \gamma'(0) \) and \( r'(1) = \gamma'(1) \), \( r'(t) \) and \( \gamma'(t) \) can be combined to form the reversely concatenated curve \( c(t) = (r' \oplus \gamma')(t) \). We call the winding number of \( c = r' \oplus \gamma' \) with respect to the origin the hodograph winding number and denote it by wind\((r' \oplus \gamma')\).

Since the same complex plane is often treated in a different point of view, we give three different names for the complex plane in which their square-root curve \( s(t) \) and \( \chi(t) \) reside the double covering space.

For the examples in Fig. 1, the reversely-concatenated hodographs \( c(t) = (r' \oplus \gamma')(t) \) are shown in Fig. 2, with their hodograph winding numbers.

In Choi et al. (2007), it is proved that there always exists a solution with zero hodograph winding number, i.e., \( \gamma \) such that wind\((r' \oplus \gamma')\) = \(0\). It is possible that two different such solutions exist. In that case, we also gave a simple formula to select a better solution, which is, by visual inspection, seen to be closer to the cubic Bézier \( r \).

So far we have touched upon the problem of finding a single piece of the PH quintic curve. Now we turn our attention to the \( C^2 \) PH quintic spline interpolation problem. From the condition that a spline \( \gamma \) is \( C^2 \) continuous, we can easily verify the following lemma.

Lemma 2.4. Let \( \gamma_1(t), \ldots, \gamma_N(t) \) be planar PH quintic curves, which constitute a \( C^2 \) planar quintic PH spline \( \gamma \) in this order. Then, \( \gamma_i'(t) \) can be expressed as

\[
\begin{align*}
\gamma_1'(t) & = \alpha(1-t)^2 + z_12(1-t)t + \lambda_1 t^2, \\
\gamma_i'(t) & = [\lambda_{i-1}(1-t)^2 + z_i 2(1-t)t + \lambda_i t^2] (i = 2, \ldots, N - 1), \\
\gamma_N'(t) & = [\lambda_{N-1}(1-t)^2 + z_N2(1-t)t + \beta t^2] \\
\end{align*}
\]

for some \( \alpha, \beta, z_i \in \mathbb{C} \), where \( \lambda_i \) is 0 or \( \lambda_i = (z_i + z_{i+1})/2 \) (\( i = 1, \ldots, N \)).
Theorem. Since $y_1(t)$ a PH quintic curve, it is expressed as
\[ y_1(t) = \left[ a(1-t)^2 + z_1(1-t)t + \lambda_1t^2 \right]^2. \]
for some $a, z_1, \lambda_1 \in \mathbb{C}$, and $y_2(t)$ is expressed as
\[ y_2(t) = \left[ a_2(1-t)^2 + z_2(1-t)t + b_2t^2 \right]^2. \]
for some $a_2, z_2, b_2 \in \mathbb{C}$. From the $C^2$ continuity, $y_1'(1) = y_2'(0)$, i.e., $a_2 = \pm \lambda_1$. Therefore, $y_2'(t)$ can be expressed as
\[ y_2'(t) = \left[ \lambda_1(1-t)^2 + z_2(1-t)t + \lambda_2t^2 \right]^2. \]
for some $z_2, \lambda_2 \in \mathbb{C}$. (In the case that $a_2 = -\lambda_1$,
\[ y_2'(t) = \left[ -\lambda_1(1-t)^2 + z_2(1-t)t + \lambda_2t^2 \right]^2. \]
Then, we rewrite $-z_2, -\lambda_2$ by $z_2, \lambda_2$, respectively.) From the fact that $y_1''(1) = y_2''(0)$, we have
\[ \lambda_1(\lambda_1 - 1) = \lambda_1(z_2 - \lambda_1). \]
Thus, $\lambda_1 = 0$ or $\lambda_1 = (z_1 + z_2)/2$. In this way, we can show the result.

To avoid complication, we now deal only with the case when $\lambda_i = (z_i + z_{i+1})/2$ ($i = 1, \ldots, N$), which we call the generic PH quintic spline.

**Definition 2.5 (Generic planar $C^2$ PH quintic spline).** We call a planar $C^2$ PH quintic spline $\gamma(t)$ whose segment $\gamma_i(t)$ is given by $\gamma_i'(t) = \chi_i^2$, where $\chi_i$ is of the form
\[
\chi_1(t) = a(1-t)^2 + z_1(1-t)t + \frac{z_1 + z_2}{2}t^2, \\
\chi_i(t) = \frac{z_{i-1} + z_i}{2}(1-t)^2 + z_i(1-t)t + \frac{z_i + z_{i+1}}{2}t^2 \quad (i = 2, \ldots, N-1), \\
\chi_N(t) = \frac{z_{N-1} + z_N}{2}(1-t)^2 + z_N(1-t)t + \frac{z_N + z_1}{2}t^2
\]
for some $a, \beta, z_i \in \mathbb{C}$ a generic planar $C^2$ PH quintic spline.

For the problem to find generic planar $C^2$ PH quintic spline curves that interpolates a sequence of points $p_0, \ldots, p_N$, Albrecht and Farouki suggested three different types of end conditions to be satisfied and showed that there exist $2^{N+k}$ solutions ($k \in \{-1, 0, +1\}$) depending on the end conditions (Albrecht and Farouki, 1996). Since it is essential in the subsequent sections, we present relevant results about the solutions of the interpolation problem for three different end conditions from Albrecht and Farouki (1996).

If two end-derivatives $a, b$ at the points $p_0, p_N$ are given (this end condition is called the prescribed end condition), the problem is corresponding to finding $a, \beta$, and $z_i$’s in equations of Definition 2.5 such that
\[ a^2 = a, \quad \beta^2 = b, \quad \int_0^1 \chi_i(t)^2 \, dt = \Delta p_i = p_i - p_{i-1} \quad (i = 1, \ldots, N). \]
(4)
For each fixed pair $(a, \beta)$ of square-roots, this problem is essentially solving a system of $N$ quadratic equations with unknowns $z_1, \ldots, z_N$,
\[
f_1(z_1, \ldots, z_N) = 17z_1^2 + 12z_1z_2 + 3z_2^2 + 14\alpha z_1 + 2\alpha z_2 + 12\alpha^2 - 60\Delta p_1 = 0, \\
f_i(z_1, \ldots, z_N) = 3z_{i-1}^2 + 27z_{i-1}^2 + 3z_{i+1}^2 + 13(z_{i-1} + z_{i+1})z_i + z_{i-1}z_{i+1} - 60\Delta p_i = 0 \quad (i = 2, \ldots, N-1), \\
f_N(z_1, \ldots, z_N) = 17z_N^2 + 12z_Nz_{N-1} + 3z_{N-1}^2 + 14\beta z_N + 2\beta z_{N-1} + 12\beta^2 - 60\Delta p_N = 0, \]
(5)
and it has $2^N$ distinct solutions of $\{z_1, \ldots, z_N\}$. Since there are 4 different pairs of $(\alpha, \beta)$, there are $2^{N+2}$ sets of $\{\alpha, \beta, z_1, \ldots, z_N\}$ satisfying Eq. (4). But, since $\{-\alpha, -\beta, -z_1, \ldots, -z_N\}$ is a solution of Eq. (4) if $\{\alpha, \beta, z_1, \ldots, z_N\}$ is a solution of Eq. (4) and $\pm \{\alpha, \beta, z_1, \ldots, z_N\}$ give the same PH quintic spline, there are $2^{N+1}$ distinct generic $C^2$ PH quintic splines for a given Hermite data.

For the cubic end spans, which enforces $\gamma_1$ and $\gamma_N$ to be PH cubic curves, $\alpha$ and $\beta$ are given by

$$\alpha = \frac{1}{2}(3z_1 - z_2), \quad \beta = \frac{1}{2}(3z_N - z_{N-1}).$$

(6)

With this replacement, the system of $N$ quadratic equations

$$\int_0^1 \chi_i(t)^2 \, dt = \Delta p_i \quad (i = 1, \ldots, N)$$

(7)

has $2^N$ different solutions, but there are $2^{N-1}$ distinct $C^2$ PH quintic splines satisfying the cubic end spans by the same reason as in the case of prescribed end conditions.

Finally, for the case of periodic end conditions, which enforces $p_0 = p_N$ and $C^2$ continuity at $p_0 = p_N$, $\alpha$ and $\beta$ are given by

$$\alpha = \frac{1}{2}(z_1 + z_N), \quad \beta = \frac{1}{2}(z_N + z_1)$$

or

$$\alpha = \frac{1}{2}(z_1 - z_N), \quad \beta = \frac{1}{2}(z_N - z_1).$$

(8)

(9)

For each case of $(\alpha, \beta)$, Eq. (7) has $2^N$ different solutions, and thus there are $2^{N-1}$ different $C^2$ PH quintic splines for a given data $p_0, \ldots, p_N$. Therefore, there are totally $2^N$ distinct $C^2$ PH quintic splines for the periodic end conditions.

Albrecht and Farouki (1996) developed a very nice algorithm, which is based on a homotopy method, to compute all the solutions to a given data for three different end conditions. While one can find all solution splines by using the algorithm, just like the problem of finding a PH quintic curve for a first-order Hermite data, there are usually only one “good” PH spline that is free of undesirable loops. To select the “good” PH spline, the same criteria such as the absolute rotation number or elastic bending energy have been adopted.

3. Absolute hodograph winding number

A first but necessary step toward constructing a good or best solution to the PH quintic spline problem is to develop a conceptual device that identifies good solutions. Our strategy in this paper is of course to pursue the topological approach that was so successfully employed in Choi et al. (2007). We thus choose to look at the cubic spline interpolating the same Hermite data and satisfying the given end conditions as a reference and try to identify the solution to the PH quintic spline problem with the same end conditions, which is closest to the reference cubic spline. We will denote this corresponding reference cubic spline by $r(t)$ and the $i$th segment curve by $r_i(t)$ hereafter. This “closeness” should then be taken in the topological sense alluded above. So a naive nonetheless natural idea is to try to compare the PH quintic spline with the cubic spline using our old device, namely, the hodograph winding number, and try to choose the PH quintic spline that has the zero hodograph winding number.

But this naive approach is mired with problems. First of all, except for the prescribed end conditions, the hodograph curves of the PH quintic spline and the cubic spline do not form a closed curve, therefore it is impossible to apply the concept of the hodograph winding number. Even in the prescribed end conditions, many complications arise.

Fig. 3 shows all 16 generic PH quintic splines which interpolate 4 points, $p_0, p_1, p_2, p_3$ with prescribed end-tangents $a, b$. Each PH quintic spline is compared with the unique cubic Bézier spline $r$ which interpolates the same data. The first thing to notice is that there are six cases in which the hodograph winding number is zero, and then it is easy to see that only the PH quintic spline in Fig. 3(i), among all 16 possible candidates, has no loop and it is topologically closest to $r$. One reason why this hodograph winding number does not capture the topological closeness between $y$ and $r$ is that in many cases the loops with opposite orientation cancel each other in the hodograph winding.
number computation, which means that the “zeroness” of the hodograph winding number is no guarantee of the absence of the unnecessary loops.

It is therefore necessary to develop a concept that captures the topological nature of the winding number while remedying the problems explained above. We first need the following:

**Definition 3.1.** Let $X_1(t)$ and $X_2(t)$ be continuous complex-valued curves for $t \in [0, 1]$ and suppose that the line segments $L_1(t) = X_1(1)(1 - t) + X_2(1)t$ and $L_0(t) = X_2(0)(1 - t) + X_1(0)t$ do not pass through the origin for $t \in [0, 1]$. The extended reversely concatenated curve $X_1 \ominus^* X_2$ is defined by

$$(X_1 \ominus^* X_2)(t) = \begin{cases} 
X_1(4t), & t \in [0, 1/4], \\
L_1(4t - 1), & t \in [1/4, 1/2], \\
X_2(3 - 4t), & t \in [1/2, 3/4], \\
L_0(4t - 3), & t \in [3/4, 1].
\end{cases}$$

See Fig. 4 for an illustration of the extended reversely concatenated curve.
Table 1
The absolute hodograph winding number \( \text{wind}_A(r' \oplus^* \gamma') \) for the cubic spline \( r \) and PH quintic splines \( \gamma \) in Fig. 3

<table>
<thead>
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<th>(a)–(h)</th>
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<td>1</td>
<td>0</td>
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<td>5</td>
<td>1</td>
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Note that the angle variation of \( X_1 \oplus^* X_2 \) is the same as that of \( X_1 \ominus X_2 \) when \( X_1(0) = X_2(0) \) and \( X_1(1) = X_2(1) \). Thus, in this case, \( \text{wind}(X_1 \ominus X_2) = \text{wind}(X_1 \oplus^* X_2) \).

With this definition, we can now consider the winding number of the extended reversely concatenated curve of each segments, \( \text{wind}(r_i' \ominus^* \gamma_i') \) as well as that of the whole splines, \( \text{wind}(r' \ominus^* \gamma') \) whether the end points of \( r' \) and \( \gamma' \) coincide or not. But, still, this is not enough. We need an additional measure of topological closeness between two splines, \( r(t) \) and \( \gamma(t) \), which we call the semi-topological quantity, absolute hodograph winding number.

**Definition 3.2 (Absolute hodograph winding number).** For a cubic Bézier spline \( r \) and a PH quintic spline \( \gamma \), the absolute hodograph winding number \( \text{wind}_A(r' \ominus^* \gamma') \) of \( r \) and \( \gamma \) is defined by:

\[
\text{wind}_A(r' \ominus^* \gamma') = \sum_{i=1}^{N} |\text{wind}(r_i' \ominus^* \gamma_i')|.
\]

We simply write \( \text{wind}_A \) to denote this number when \( r' \) and \( \gamma' \) are clear from the context.

Fig. 5 illustrates the absolute hodograph winding number of \( r' \) and \( \gamma' \) together with the hodograph winding number of them. In Fig. 5, each of the two spline \( \gamma' \) (solid curve) and \( r' \) (dotted curve) is made up of two pieces: \( \gamma_1' \) and \( \gamma_2' \) for \( \gamma' \); \( r_1' \) and \( r_2' \) for \( r' \), respectively. The dotted line segment is the one that connects \( \gamma_1'(1) \) and \( r_1'(1) \). It is then easy to observe that \( \text{wind}(r_1' \ominus^* \gamma_1') = 1 \) and \( \text{wind}(r_2' \ominus^* \gamma_2') = -1 \). Thus, we have \( \text{wind}(r' \ominus \gamma') = \text{wind}(r' \ominus^* \gamma') = 0 \), but \( \text{wind}_A(r' \ominus^* \gamma') = 2 \). This example shows that \( r'(t) \) and \( \gamma'(t) \) may behave quite differently even though the hodograph winding number of \( r \) and \( \gamma \) is zero. As a result, we can conclude that the hodograph winding number does not correctly capture the topological difference between \( r' \) and \( \gamma' \). On the other hand, the absolute hodograph winding number is not zero and it seems to correctly reveal the topological difference between \( r' \) and \( \gamma' \).

Table 1 shows the absolute hodograph winding numbers for PH quintic splines in Fig. 3. The first row is the absolute hodograph winding numbers for splines in Fig. 3(a) through Fig. 3(h), and the second row for splines in Fig. 3(i) through Fig. 3(p). From Fig. 3 and this table, one can easily find that the absolute hodograph winding number is the same as the number of loops in this example. Actually, this is not a special phenomenon to this example. We
have found by extensive testing that the absolute hodograph winding number roughly captures the number of loops a PH quintic spline has in it.

We have performed an extensive testing. As reported in Table 2, which is to be expounded below, there about 10 to 20% of the solutions of the PH quintic spline problem have zero hodograph winding number, but the number of zero absolute hodograph winding number solutions is relatively small—it is reported there are at most 4 zero absolute hodograph winding number solutions for all tested data.

All these observations lead us to believe that the correct topological criterion is the absolute hodograph winding number rather than the hodograph winding number of the corresponding cubic spline \( r(t) \) and a PH quintic spline \( \gamma(t) \) when one tries to identify the good PH quintic spline. With this idea, we will call a PH quintic spline with zero absolute hodograph winding number a zero absolute hodograph winding number spline and we will take a zero absolute hodograph winding number spline as the good PH quintic spline among all solution splines.

The first attempt to identify a “fairer” PH quintic spline among multiple solution splines was pioneered by Farouki, who suggested a nice and intuitive way using the elastic bending energy (Farouki, 1996). In most cases, it determines a nice and “looking-fairer” solution and gives the same solution spline as the absolute hodograph winding number criterion does. However, in some cases, the absolute hodograph winding number criterion behaves somewhat differently from the least bending energy criterion. As an example, Fig. 6 shows this difference. The selected PH quintic splines are displayed together with the corresponding cubic spline for Hermite data 1 and its a bit perturbed data, Hermite data 2: Hermite data 1 is used for splines in Fig. 6(a) and Hermite data 2 for splines in Fig. 6(b). As in Fig. 6(a), for Hermite data 1, two criteria pick up one PH quintic spline as the good spline: the least bending energy solution and the zero absolute hodograph winding number solution coincide. However, as one can see in Fig. 6(b), two criteria select different splines as the good spline for Hermite data 2. By the nature of the absolute hodograph winding number, we can observe that the selected spline by the absolute hodograph winding number criterion stays consistently when the Hermite data is perturbed as in this example. Also, it is observed that the spline determined by the semi-topological criterion stays closer to the corresponding cubic spline regardless of the Hermite data.

This absolute hodograph winding number of \( r \) and \( \gamma \) relates with the winding number of the extended reversely concatenated curve \( r' \ominus \gamma' \). From its definition, we can easily prove the following lemma.

**Lemma 3.3.**

(i) If \( \text{wind}_A(r' \ominus^* \gamma') \) is zero, then

\[
\text{wind}(r' \ominus^* \gamma') = \sum_{i=1}^{N} \text{wind}(r'_i \ominus^* \gamma'_i) = 0.
\]

(ii) \( \text{wind}(r' \ominus^* \gamma') \) is even (resp., odd) if and only if \( \text{wind}_A(r' \ominus^* \gamma') \) is even (resp., odd).

Now with the above lemma, we investigate more about the possibility of the absolute hodograph winding number. First, let us consider the problem to find a PH quintic spline interpolating \( p_0, \ldots, p_N \) and two prescribed end-derivatives \( a, b \). Let us fix a root of \( a \), say \( \alpha \). And, let \( s(t) \) be the continuous spline whose \( i \)th segment \( s_i(t) \) satisfies

\[
s_i(t)^2 = r'_i(t) \quad (i = 1, \ldots, N), \quad s_1(0) = \alpha,
\]
where \( r \) is the unique cubic Bézier spline interpolating the same Hermite data. As in the case of finding a PH quintic curve satisfying a first-order Hermite data, the end point of \( s(t) \), i.e., \( s_N(1) \) plays an important role in determining if \( \text{wind}(r' \ominus s) \) is even or odd. Suppose \( \gamma(t) \) is a PH quintic spline interpolating given data and \( \chi(t) \) is a continuous square root curve of \( \gamma'(t) \) such that

\[
\chi_i(t)^2 = \gamma_i'(t), \quad \chi_1(0) = \alpha,
\]

where \( \chi_i \) is the \( i \)th segment of \( \chi \). If \( \chi_N(1) = s_N(1) \), then \( \text{wind}(r' \ominus s') = \text{wind}(r' \ominus \gamma') = 2\text{wind}(s \ominus \chi) \). Thus, \( \text{wind}(r' \ominus s') \) is even. Otherwise, \( \text{wind}(r' \ominus s') = \text{wind}(r' \ominus \gamma') = (\Delta \theta_\chi - \Delta \theta_\gamma)/\pi = 2n + 1 \) for some integer \( n \) since \( s_1(0) = \chi_1(0) \), but \( s_N(1) = -\chi_N(1) \). This observation is summarized in the following lemma.

**Lemma 3.4.** If \( s_N(1) = \chi_N(1) \), then \( \text{wind}_A(r' \ominus s') \) is even. If \( s_N(1) = -\chi_N(1) \), then \( \text{wind}_A(r' \ominus s') \) is odd. There are \( 2^N \) PH quintic splines whose absolute hodograph winding number is even, and \( 2^N \) PH quintic splines whose absolute hodograph winding number is odd.

Fig. 3 and Table 1 illustrate this lemma. Actually, the splines in Fig. 3(a) through 3(h) are those PH quintic splines such that \( s_N(1) = -\chi_N(1) \), and the splines in Fig. 3(i) through 3(p) are those such that \( s_N(1) = \chi_N(1) \). Fig. 7 shows 16 square root splines \( \chi(t) \) corresponding to \( \gamma(t) \) in Fig. 3 in the same order. This figure illustrates the above lemma.
Table 2
Average number of solution splines such that wind\( (r' \odot^* \gamma') = 0 \)

<table>
<thead>
<tr>
<th></th>
<th>( N = 5 )</th>
<th>( N = 7 )</th>
<th>( N = 9 )</th>
<th>( N = 11 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prescribed End Conditions</td>
<td>11.83(18.48%)</td>
<td>41.67(16.28%)</td>
<td>145.13(14.17%)</td>
<td>537.35(13.12%)</td>
</tr>
<tr>
<td>Cubic End Spans</td>
<td>3.52(22.00%)</td>
<td>12.40(19.38%)</td>
<td>41.38(16.16%)</td>
<td>143.00(13.96%)</td>
</tr>
<tr>
<td>Periodic End Conditions</td>
<td>4.98(15.56%)</td>
<td>20.64(16.13%)</td>
<td>76.83(15.01%)</td>
<td>268.12(13.09%)</td>
</tr>
</tbody>
</table>

Prescribed End Conditions 11.83(18.48%) 41.67(16.28%) 145.13(14.17%) 537.35(13.12%)
Cubic End Spans 3.52(22.00%) 12.40(19.38%) 41.38(16.16%) 143.00(13.96%)
Periodic End Conditions 4.98(15.56%) 20.64(16.13%) 76.83(15.01%) 268.12(13.09%)

in more detail. Also, it is very clear to see that the topologically closest solution (Fig. 7(i)) is the closest spline to \( s(t) \) in the sense of Hausdorff distance in this space, which is called the double covering space (Choi et al., 2007).

Now let us examine what kind of value the absolute hodograph winding number can have for the problem with the periodic end conditions. In this case, the solution splines are one of two types: one group of solutions obtained by using Eq. (8) and another group of solutions by Eq. (9). For the splines in the first group, \( \text{wind}(\gamma') = 2\text{wind}(\chi) \), and \( \text{wind}(\gamma') \) is odd for the splines in the second group. Since the reference cubic Bézier spline \( r \) has a closed hodograph for the periodic end conditions, \( \text{wind}(r') \) is even if \( s_1(0) = s_N(1) \), and it is odd if \( s_1(0) = -s_N(1) \). By this observation, we can easily get the following lemma.

**Lemma 3.5.**

(i) If \( s_1(0) = s_N(1) \), then \( \text{wind}_A(r' \odot^* \gamma') \) is even for PH quintic splines obtained by Eq. (8) and \( \text{wind}_A(r' \odot^* \gamma') \) is odd for PH quintic splines obtained by Eq. (9).

(ii) If \( s_1(0) = -s_N(1) \), then \( \text{wind}_A(r' \odot^* \gamma') \) is odd for PH quintic splines obtained by Eq. (8) and \( \text{wind}_A(r' \odot^* \gamma') \) is even for PH quintic splines obtained by Eq. (9).

### 4. Semi-topological selection criterion

In the previous section, we introduced a new semi-topological quantity, the absolute hodograph winding number to identify a good solution. In this section we develop a criterion, and thus a method applicable for small size problem, of selecting the best solution to the PH quintic spline problem.

To put this question in perspective, we first performed an extensive test by examining the zero absolute hodograph winding number solutions for a large number of Hermite data. For this test, we randomly generated 100 Hermite data without any restriction on the generated data for each of three end conditions \( (N = 5, 7, 9, 11) \).

Tables 3, 4 and 5 show the empirical results we obtained. As one can see in Table 3, although it is rare, there exist some configurations for which there is no zero absolute hodograph winding number solution. Also, for some cases of Hermite data, there are multiple zero absolute hodograph winding number solutions even though the zero absolute hodograph winding number solution is uniquely determined for most of cases (over 90%). Thus, the absolute hodograph winding number criterion need to be extended to cover some special cases maintaining the semi-topological

Table 3
The number of Hermite data out of 100 for which there is no zero absolute hodograph winding number solution

<table>
<thead>
<tr>
<th></th>
<th>( N = 5 )</th>
<th>( N = 7 )</th>
<th>( N = 9 )</th>
<th>( N = 11 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prescribed End Conditions</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Cubic End Spans</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Periodic End Conditions</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4
The number of Hermite data out of 100 for which there are multiple zero absolute hodograph winding number solutions. In this case, the number of zero absolute hodograph winding number solutions is 2 or 4 for all cases of tested data

<table>
<thead>
<tr>
<th></th>
<th>( N = 5 )</th>
<th>( N = 7 )</th>
<th>( N = 9 )</th>
<th>( N = 11 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prescribed End Conditions</td>
<td>15</td>
<td>9</td>
<td>17</td>
<td>15</td>
</tr>
<tr>
<td>Cubic End Spans</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Periodic End Conditions</td>
<td>5</td>
<td>5</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>
Table 5
The number of Hermite data out of 100 for which there is only one zero absolute hodograph winding number solution

<table>
<thead>
<tr>
<th></th>
<th>$N = 5$</th>
<th>$N = 7$</th>
<th>$N = 9$</th>
<th>$N = 11$</th>
</tr>
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<td>Prescribed End Conditions</td>
<td>85</td>
<td>91</td>
<td>83</td>
<td>85</td>
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<tr>
<td>Cubic End Spans</td>
<td>98</td>
<td>99</td>
<td>96</td>
<td>95</td>
</tr>
<tr>
<td>Periodic End Conditions</td>
<td>95</td>
<td>95</td>
<td>93</td>
<td>93</td>
</tr>
</tbody>
</table>

property. Since the basic idea in selecting the good PH quintic spline is comparison with the corresponding cubic spline, we first define a simplified pseudo-Hausdorff distance in order to quantify closeness between a cubic spline and a PH quintic spline.

**Definition 4.1.** For a $C^2$ cubic spline $r(t)$ and a $C^2$ PH quintic spline $\gamma(t)$ consisting of $N$ segments, the simplified pseudo-Hausdorff distance $\|r - \gamma\|^*$ is defined by

$$\|r - \gamma\|^* = \min \left\{ \sum_{i=1}^{N} |s_i(1/2) - \chi_i(1/2)|, \sum_{i=1}^{N} |s_i(1/2) + \chi_i(1/2)| \right\}$$

where $s_i(t)$ is the $i$th segment of a continuous square-root spline $s(t)$ of $r'(t)$, and $\chi_i(t)$ is the $i$th segment of a continuous square-root spline $\chi(t)$ of $\gamma'(t)$.

With this definition, we devise a semi-topological selection criterion extended from the absolute hodograph winding number criterion.

**Semi-topological selection criterion**

For a given Hermite data, we select the PH quintic spline $\gamma$ given below as the good one. If there exists at least one zero absolute hodograph winding number solution, $\gamma$ is selected to be the one of least simplified pseudo-Hausdorff distance among zero absolute hodograph winding number solutions. Otherwise, $\gamma$ is selected to be the one of least simplified pseudo-Hausdorff distance among all solutions.

By this criterion, the selected good PH quintic spline is always zero absolute hodograph winding number solution if there exists any such a solution. When there is unique such a solution, this selection criterion is just the same as the absolute hodograph winding number criterion. When there are multiple such solutions, this criterion seems to pick up the closest PH quintic spline to the corresponding cubic spline $r$ in terms of Hausdorff distance in the double covering space maintaining topological closeness. This criterion also identifies the closest solution to $r$ even when there is no zero absolute hodograph winding number solution.

To verify this selection criterion, we performed extensive numerical experiments for the set of Hermite data used in the test for Tables 3, 4 and 5. From the experiment results, it turns out that this selection criterion works very well for any data set: it always picks up the closest PH quintic spline to $r$ in terms of shape similarity.

5. Modified Albrecht–Farouki–Kuspa–Manni–Sestini method

Though the semi-topological selection criterion introduced in the previous section works quite well, it becomes unworkable when the number of points to be interpolated gets big if one needs to enumerate all solution splines to apply the selection criterion: the number of solutions increases exponentially. Thus, it is desirable to have an algorithm to get the selected spline by the criterion directly without solving all the other solutions. It turns out that the novel methods developed by Albrecht, Farouki, Kuspa, Manni, and Sestini (Albrecht and Farouki, 1996; Farouki et al., 2001) can be successfully modified for our purpose. The method they have devised is basically a Newton–Raphson method, but with very good initial point selection strategy to apply Newton–Raphson iteration. We will adopt the same method, but devise a new initial point selection strategy for our purpose.

In Farouki et al. (2001), they use Newton–Raphson method with an initial starting points $z_0^1, \ldots, z_0^N$ to get one good solution of Eq. (4). Since the final solution that Newton–Raphson method gives after all the iterations is completely
determined by the initial starting point, it is very critical to choose the very right initial approximation to get the “good” final solution in this method: the initial spline determined by \( z_1^0, \ldots, z_N^0 \) should be close enough to the desired spline. So, in this paper, we give another method to get a starting approximation which is closest to a zero absolute hodograph winding number solution.

The basic idea behind the concept of the absolute hodograph winding number is to compare PH quintic splines with the reference cubic Bézier spline in terms of semi-topological quantity in order to find the closest PH quintic spline to the cubic Bézier spline. Thus, it is very natural to take a spline which is close to the reference curve as a starting approximation.

But, we need to clarify the space under which PH quintic splines and the reference cubic spline are compared with each other. Since we are interested in the hodograph winding number, clearly the curve space is not an appropriate space under which we compare reference splines and PH quintic splines. Also, since \( C^2 \) PH quintic spline is determined by \( \alpha, \beta \) in solving Eq. (10). For the case of periodic end conditions, we use Eq. (8) if \( s_N \) is replaced by linear combinations of \( \chi_1, \ldots, \chi_N \) in the double covering space, we first need to get a continuous square-root spline \( s(t) \) of \( r'(t) \). By using this square-root spline, we get an initial starting \( C^1 \) quadratic spline \( \chi(t) \) equating

\[
\chi_i(1/2) = s_i(1/2) \quad (i = 1, \ldots, N),
\]

where \( \chi_i \) and \( s_i \) are \( i \)th segment curve of \( \chi \) and \( s \). Since the above equations are reduced into

\[
2\alpha + 5z_1 + z_2 = 8s_1(1/2),
\]

\[
z_{i-1} + 6z_i + z_{i+1} = 8s_i(1/2) \quad (i = 2, \ldots, N-1),
\]

\[
z_{N-1} + 5z_N + 2\beta = 8s_N(1/2),
\]

we need only solve a system of linear equations to get the initial starting points \( z_1^0, \ldots, z_N^0 \) with \( \alpha, \beta \) replaced by some constants and \( s_i \) with \( \chi_i \) in Eq. (11) are replaced by linear combinations of \( z_1, \ldots, z_N \) depending on the types of end conditions.

Since we want a zero-absolute hodograph winding number solution, we set \( \alpha = s_1(0) \) and \( \beta = s_N(1) \) in Eq. (11) when we are given two-end derivatives to solve. Lemma 3.4 justifies this choice of \( \alpha, \beta \) for the case of prescribed end conditions. For the case of cubic end spans, \( \alpha, \beta \) in Eq. (11) are replaced by linear combinations of \( z_1, \ldots, z_N \) as in Eq. (6). For the case of periodic end conditions, we use Eq. (8) if \( s_1(0) = s_N(1) \), and Eq. (9) if \( s_1(0) = -s_N(1) \) to replace \( \alpha, \beta \) in Eq. (11). This choice of \( \alpha, \beta \) is justified by Lemma 3.5. With this choice of \( \alpha, \beta \), there is no ambiguity in solving Eq. (10).

The initial approximation scheme in this paper is superficially very similar to that in Farouki et al. (2001), which is given by

\[
\gamma_i'(1/2) = r_i'(1/2) \quad (i = 1, \ldots, N).
\]

But, it deals with a different space, the hodograph space, under which it tries to approximate the reference curve. Eq. (10) implies Eq. (12), but the converse is not true. Also, in the scheme in Farouki et al. (2001), there are ambiguities in determining square-roots of quantities involved in Eq. (12). Since Eq. (12) yields the system of linear equations

\[
z_{i-1} + 6z_i + z_{i+1} = 8\sqrt{r_i'(1/2)} \quad (i = 2, \ldots, N-1),
\]

one needs to determine the square root of \( r_i'(1/2) \). In Farouki et al. (2001), it is treated by taking either square root of \( r_i'(1/2) \) and choosing the subsequent square root so as to minimize \( |\sqrt{r_i'(1/2)} - \sqrt{r_{i-1}'(1/2)}| \). And, this ambiguity results in producing different solutions depending on the choice of square-root \( \sqrt{r_i'(1/2)} \).

Fig. 8 shows two different PH quintic splines with the cubic Bézier spline for the same Hermite data. Newton–Raphson method with the starting spline given by Eq. (10) gives \( \chi_a \), but it gives \( \chi_a \) or \( \chi_b \) depending on the choice of
Fig. 8. Two different PH quintic splines with the unique cubic spline. We have wind\(_A(r'(\odot^*\gamma'_a)) = 0\) and wind\(_A(r'(\odot^*\gamma'_b)) = 2\). Left: Double covering space. Right: Curve space.

Fig. 9. Three different PH quintic splines with the unique cubic spline. \(s(t)\) and \(r(t)\) are displayed by using dotted curves. Left: \(\chi_a(t)\) and \(\gamma_a(t)\). Center: \(\chi_b(t)\) and \(\gamma_b(t)\). Right: \(\chi_c(t)\) and \(\gamma_c(t)\). We have wind\((r'(\odot^*\gamma'_a)) = 0\), but wind\((r'(\odot^*\gamma'_b)) = \) wind\((r'(\odot^*\gamma'_c)) = 2\). Top: Double covering space. Bottom: Curve space.

square-root \(\sqrt{r'_1(1/2)}\) when Eq. (12) is used. For some cases of Hermite data, the PH quintic spline obtained from the initial starting spline given by Eq. (10) cannot be obtained by changing the square-root \(\sqrt{r'_1(1/2)}\). Fig. 9 illustrates this case. The initial spline given by Eq. (10) gives \(\chi_a(t)\) through the Newton–Raphson method while two initial splines given by Eq. (12) depending on the choice of \(\sqrt{r'_1(1/2)}\) give \(\chi_b(t)\) and \(\chi_c(t)\).

We summarize the modified Albrecht–Farouki–Kuspa–Manni–Sestini algorithm as follows.

**Modified Albrecht–Farouki–Kuspa–Manni–Sestini algorithm**

We suppose we are given \(N + 1\) points \(p_0, p_1, \ldots, p_N\) to be interpolated with one of three types of end conditions: prescribed end conditions, cubic end spans, or periodic end conditions. When we are to find the PH quintic spline \(\gamma(t)\) selected by the semi-topological selection criterion, we follow the algorithm stated below.

Step 1. Find the unique cubic spline \(r(t)\) interpolating the given Hermite data in accordance with the given end conditions: when prescribed end conditions or periodic end conditions are given for PH quintic spline, the same
end conditions are also enforced for \( r(t) \), and quadratic end spans is required for \( r(t) \) when cubic end spans is given for PH quintic spline. Denote the  \( i \)th segment of \( r(t) \) by \( r_i(t) \).

**Step 2.** Fix a square-root \( \alpha_0 \) of \( r_i'(0) \) and let \( s(t) \) be the continuous, square-root spline of \( r'(t) \) such that

\[
s_1(0) = \alpha_0,
\]

where \( s_1(t) \) is the first piece of \( s(t) \).

**Step 3.** Find \( z^{(0)} = (z_1^{(0)}, \ldots, z_N^{(0)})^T \) satisfying Eq. (11) with appropriately replaced \( \alpha, \beta \):

(a) Prescribed End Conditions: \( \alpha \) is replaced by \( s_1(0) \), and \( \beta \) by \( s_N(1) \).

(b) Cubic End Spans: \( \alpha \) is replaced by \( \frac{1}{2}(3z_1 - z_2) \), and \( \beta \) by \( \frac{1}{2}(3z_N - z_{N-1}) \).

(c) Periodic End Conditions: When \( s_1(0) = s_N(1) \), \( \alpha \) is replaced by \( \frac{1}{2}z_1 + z_N \), and \( \beta \) by \( \frac{1}{2}z_N + z_1 \). When \( s_1(0) = -s_N(1) \), \( \alpha \) is replaced by \( \frac{1}{2}(z_1 - z_N) \), and \( \beta \) by \( \frac{1}{2}(z_N - z_1) \).

Set \( n = 0 \).

**Step 4.** If \( \text{err}(z^{(n)}) = \sum_{k=1}^{N} f_k(z^{(n)})^2 \geq \text{ERR}_\text{BOUND} \) and \( n \leq \text{ITER}_\text{LIMIT} \), do Step 5. Otherwise, go to Step 6.

**Step 5.** Find \( z^{(n+1)} \) and \( h_n \) given by

\[
\begin{align*}
    z^{(n+1)} &= z^{(n)} - M(z^{(n)})^{-1}f(z^{(n)}), \\
    h_n &= 120 \| M(z^{(n)})^{-1} \|_\infty \| M(z^{(n)})^{-1}f(z^{(n)}) \|_\infty,
\end{align*}
\]

where \( f = (f_1, \ldots, f_N)^T \) and \( M(z^{(n)}) \) is the Jacobian matrix of \( f \) at \( z^{(n)} \). Increase \( n \) by 1 and go to Step 4.

**Step 6.** If \( \text{err}(z^{(n)}) < \text{ERR}_\text{BOUND} \) and \( \min h_n \leq 1/2 \), then we construct the PH quintic spline \( \gamma(t) \) given by Eq. (4) with \( z^{(n)} \) substituted for \((z_1, \ldots, z_N)\). Otherwise, do appropriate actions according to the exception handling policy.

We have run extensive tests of the above algorithm with \( \text{ERR}_\text{BOUND} = 10^{-10} \) for the set of randomly generated Hermite data used in Section 4. For every case, the convergence from the initial spline is observed in 3–7 iterations. Tables 6, 7 and 8 show the test results about the convergence of the algorithm. The convergence is verified by Kantorovich convergence test introduced in Farouki et al. (2001): since the modified algorithm is the same iterative method as the efficient algorithm in Farouki et al. (2001), the same convergence test can be applied for this modified algorithm also. Kantorovich convergence test tells that the sequence starting from an initial point finally converges to a solution if the Kantorovich parameter \( h_0 \) at the initial point is less than 0.5. Even though \( h_0 \) is larger than 0.5 for some cases, \( h_i \), the Kantorovich parameter after \( i \)th iteration, finally becomes less than 0.5. (For the case of periodic end conditions, the maximum value of \( h_4 \) is 0.16 for \( N = 11 \).)

As well as the modified algorithm always converges, it successfully finds the same PH quintic spline selected by the semi-topological selection criterion for all tested data: the modified algorithm gives the unique zero absolute hodograph winding number solution when there is only one such solution, the one of least simplified pseudo-Hausdorff

### Table 6

<table>
<thead>
<tr>
<th>( N )</th>
<th>( h_0 )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
<th># of iterations</th>
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<td>0.00 ~ 0.04</td>
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<td>0.00 ~ 0.12</td>
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<td>0.00 ~ 0.02</td>
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<td>3 ~ 5</td>
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<tr>
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<td>0.01 ~ 1.30</td>
<td>0.00 ~ 0.19</td>
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### Table 7

<table>
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<tr>
<th>( N )</th>
<th>( h_0 )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
<th># of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.20 ~ 17.32</td>
<td>0.00 ~ 1.53</td>
<td>0.00 ~ 0.50</td>
<td>0.00 ~ 0.03</td>
<td>3 ~ 6</td>
</tr>
<tr>
<td>7</td>
<td>0.37 ~ 20.79</td>
<td>0.01 ~ 2.11</td>
<td>0.00 ~ 1.27</td>
<td>0.00 ~ 0.17</td>
<td>3 ~ 6</td>
</tr>
<tr>
<td>9</td>
<td>0.39 ~ 4.59</td>
<td>0.01 ~ 0.85</td>
<td>0.00 ~ 0.07</td>
<td>0.00 ~ 0.00</td>
<td>3 ~ 5</td>
</tr>
<tr>
<td>11</td>
<td>0.53 ~ 8.63</td>
<td>0.01 ~ 1.56</td>
<td>0.00 ~ 0.51</td>
<td>0.00 ~ 0.03</td>
<td>3 ~ 5</td>
</tr>
</tbody>
</table>
Table 8
Periodic End Conditions

<table>
<thead>
<tr>
<th>( N )</th>
<th>( h_0 )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
<th># of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.16 ∼ 1.85</td>
<td>0.00 ∼ 0.49</td>
<td>0.00 ∼ 0.04</td>
<td>0.00 ∼ 0.00</td>
<td>3 ∼ 5</td>
</tr>
<tr>
<td>7</td>
<td>0.22 ∼ 6.95</td>
<td>0.00 ∼ 1.30</td>
<td>0.00 ∼ 0.2</td>
<td>0.00 ∼ 0.00</td>
<td>3 ∼ 5</td>
</tr>
<tr>
<td>9</td>
<td>0.27 ∼ 2.37</td>
<td>0.01 ∼ 0.40</td>
<td>0.00 ∼ 0.02</td>
<td>0.00 ∼ 0.00</td>
<td>3 ∼ 5</td>
</tr>
<tr>
<td>11</td>
<td>0.34 ∼ 5.96</td>
<td>0.01 ∼ 16.45</td>
<td>0.00 ∼ 1.77</td>
<td>0.00 ∼ 1.16</td>
<td>3 ∼ 7</td>
</tr>
</tbody>
</table>

Fig. 10. Two PH quintic splines \( \gamma \) and \( \tilde{\gamma} \) for the same Hermite data such that wind \( A = 0 \). \( r(t) \) is the unique cubic spline to the same Hermite data, and \( s(t) \) is a square-root spline of \( r'(t) \). \( \chi \) is a square-root spline of \( \gamma'(t) \) and \( \tilde{\chi} \) is a square-root spline of \( \tilde{\gamma}'(t) \). \( s(t) \) and \( r(t) \) are displayed by dotted curves, and \( \chi(t) \), \( \gamma(t) \), \( \tilde{\chi}(t) \), and \( \tilde{\gamma}(t) \) are by solid curves. We have \( \|r - \gamma\|^* = 0.28 \) and \( \|r - \tilde{\gamma}\|^* = 3.60 \).

distance among zero absolute hodograph winding number solutions when there are multiple such solutions, and the one of least simplified pseudo-Hausdorff distance among all solutions when there is no zero absolute hodograph winding number solution.

For example, Fig. 10 shows two different PH quintic splines satisfying the same Hermite data. The absolute hodograph winding numbers are zeros for both of splines, but the simplified pseudo-Hausdorff distance of the PH quintic spline in Fig. 10(a) is less than that of the PH quintic spline in Fig. 10(b). In fact, the semi-topological selection criterion picks up the PH quintic spline in Fig. 10(a) as the good one since there are only two zero absolute hodograph winding number solutions for this Hermite data. Clearly, the PH quintic spline in Fig. 10(a) is closer to \( r(t) \) and looks better. For this example, this modified numerical algorithm also gives the PH quintic spline in Fig. 10(a).

Fig. 11 shows some examples of tests with prescribed end-derivatives: it shows the unique cubic spline \( r(t) \) and the PH quintic spline \( \gamma(t) \) found by the modified algorithm in this section, displayed in both of the Double covering space (top) and the Curve space (bottom). As in the extensive tests above, all the solution splines obtained by the modified algorithm are of zero absolute hodograph winding number solution and the same splines selected by the semi-topological selection criterion in Section 4.

6. Conclusion

We have developed a new semi-topological quantity, called the absolute hodograph winding number, which is used to devise a criterion to select the best solution the PH quintic spline problems. This criterion can be used as an algorithm in case the number of points to be interpolated is small. We then modified the numerical method of Albrecht, Farouki, Kuspa, Manni, and Sestini (Albrecht and Farouki, 1996; Farouki et al., 2001) to devise a numerical method of finding the best solution in terms of the semi-topological selection criterion. The main difference in our case is that the numerical computation is done in the double covering space in our case while the method of Albrecht, Farouki,
Kuspa, Manni, and Sestini does it in the hodograph space. This seemingly minor difference seems to produce better result.

We also did extensive numerical test on our method and found our results quite satisfactory. We believe that our method may be profitably employed in practice.

References


