Maximum Likelihood Direction of Arrival Estimation Using Spherical Harmonics

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Problem overview

- COMINT/ELINT ESM application.
- Passive array consisting of multiple omni-directional sensing elements (antennas) arranged in an arbitrary geometric configuration.
- Intercepting CW signals of short wavelength (relative to the array dimensions).
- Want to estimate the Direction of Arrival (DOA) of a single radiation source.
- Typically solved using Phase Interferometry (PI) [1].
Mathematical definition of Spherical Harmonics (SH) [2]

\[ Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l + 1)(l - m)!}{4\pi(l + m)!}} e^{im\phi} P^m_l(\cos \theta) \]

with \( l \geq 0 \) and \(-l \leq m \leq +l\)

where

Associated Legendre function, of the first kind.
Visual representation of spherical harmonics

$l = 0, m = 0$
Visual representation of spherical harmonics (cont.)

\[ l = 1, m = -1 \]

\[ l = 1, m = 0 \]

\[ l = 1, m = +1 \]
Visual representation of spherical harmonics (cont.)

- $l = 2$, $m = -2$
- $l = 2$, $m = -1$
- $l = 2$, $m = 0$
- $l = 2$, $m = +1$
- $l = 2$, $m = +2$
Visual representation of spherical harmonics (etc …)

\[ l = 3, m = -3 \]
\[ l = 3, m = -2 \]
\[ l = 3, m = -1 \]
\[ l = 3, m = 0 \]
\[ l = 3, m = +1 \]
\[ l = 3, m = +2 \]
\[ l = 3, m = +3 \]
Properties of spherical harmonics [2]

- Form an **Orthonormal** set.
- The **Product** of two spherical harmonics (with $l_1$ and $l_2$) yields a finite expansion of spherical harmonics (with a maximum $l$ parameter of $l_1+l_2$). Use the Wigner $3jm$ coefficients.
- **Rotation** of a spherical harmonic yields a finite sum of spherical harmonics (with the maximum $l$ parameter unchanged). Use the Wigner $D$-function.
- **Convolution** of a linear combination of spherical harmonics with another, yields a linear combination of spherical harmonics.
Example applications

– Physics and chemistry [2&3].
– Computer graphics [4].
– Acoustics [5].
– Antenna modelling [6&7].
– Beamforming [8-10].
– Tracking …
Bayesian estimation

\[ p(x|z) = \frac{p(z|x)p(x)}{\int p(z|x)p(x)dx} \]

\[ \hat{x}_{MAP} = \arg \max_x p(z|x)p(x) \]
Gaussians in Cartesian Bayesian estimation (the Kalman filter)

Use of Gaussians greatly simplifies the Cartesian problem.

Similar benefits in the polar problem when SH are used ...
Spherical harmonics in polar Bayesian estimation

- Consider a simple hypothetical scenario ...
- Single stationary emitter in the far field.
- Unobstructed stationary array in free space.
- 1 central sensor, 4 peripheral sensors, 4 baselines of equal length.
- 2-channel fast A/D converter.
- 5 digitized samples are collected in the time it takes for light to travel along the axis of a baseline.
- The measured group delay error is assumed to be Gaussian distributed with a mean of zero and a standard deviation of half the sample period.
- Accuracy depends on the signal wavelength.
  - E.g. 1: Wavelength of the intercepted signal is greater than twice the baseline length ($\lambda = 2.5D$). Measurements are conically ambiguous.
  - E.g. 2: Wavelength of the intercepted signal is less than twice the baseline length ($\lambda = 1.5D$). Measurements may have multiple conical ambiguities.
Spherical harmonics in polar Bayesian estimation e.g. 1

Baseline axis (dashed green). Sensor location (green asterisk).

True DOA (dashed magenta).
Spherical harmonics in polar Bayesian estimation e.g. 1

Measurement on 1st baseline (on +ve x axis).

Likelihood function. Gaussian in lag space. Conical in polar coordinates. LSQ fit SH, with \( m=0 \) (real) and up to \( l=10 \), to 22 specified “control” points (blue dots). Multiply by complex conjugate to ensure +ve everywhere.

\( \theta' \) is angle relative to baseline axis.

True \( \theta' \) (red circle). Meas. \( \theta' \) (green circle).
Spherical harmonics in polar Bayesian estimation e.g. 1

Prior pdf (red).
Likelihood fcn (green).
Posterior pdf (blue).
All +ve and real.

Spherically ambiguous (uniform) prior.
Posterior pdf equals likelihood function.

Rotate likelihood function from baseline-centric coordinate system into array-centric coordinate system.

The DOA estimate is the maximum of the posterior pdf.
Spherical harmonics in polar Bayesian estimation e.g. 1

Measurement on 2nd baseline (on +ve x axis).
Spherical harmonics in polar Bayesian estimation e.g. 1

- Likelihood function is a “fuzzy” cone
- Posterior pdf (the product of the red and green fuzzy cones) has two maxima – above and below the x-y plane.
Spherical harmonics in polar Bayesian estimation e.g. 1

Measurement on 3rd baseline (on +ve z axis).
Incorporation of this measurement resolves the up/down ambiguity.
Spherical harmonics in polar Bayesian estimation e.g. 1

Measurement on 4th baseline (midway between +ve x, y and z axes).
Spherical harmonics in polar Bayesian estimation e.g. 1

Incorporation of this redundant measurement “sharpens” and shifts the “beam” slightly.

Prune SH in all functions, with coefficients less than 0.001. 340 non-negligible SH basis functions, with a max \( l \) of 20, are required to represent this solution.
Spherical harmonics in polar Bayesian estimation e.g. 2
Spherical harmonics in polar Bayesian estimation e.g. 2

The likelihood function is composed of two fuzzy cones.
Spherical harmonics in polar Bayesian estimation e.g. 2
Spherical harmonics in polar Bayesian estimation e.g. 2

At this frequency, not all measurements are ambiguous.
Spherical harmonics in polar Bayesian estimation e.g. 2

Fat cone (green), intersects with concave dumbbells (red) to form butterfly wings (blue).
Spherical harmonics in polar Bayesian estimation e.g. 2
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Four feasible DOAs in the prior pdf (red), reduced to two in the posterior pdf (blue).
Spherical harmonics in polar Bayesian estimation e.g. 2
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Final disk-like measurement resolves the ambiguity.

Still have an ambiguous lobe of lesser probability density.
Spherical harmonics in polar Bayesian estimation

- Monte Carlo simulation results, RMSE (°).
- SH compared with PI (in parentheses).

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<tbody>
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<tr>
<td><strong>2.5D</strong></td>
<td>4.22 (5.02)</td>
<td>6.54 (5.88)</td>
<td>6.36 (5.32)</td>
<td>1.62 (1.21)</td>
<td>8.54 (7.87)</td>
<td>4.79 (6.11)</td>
<td>7.06 (7.21)</td>
<td>4.73 (3.32)</td>
<td>4.04&lt;sup&gt;a&lt;/sup&gt; (4.01)</td>
</tr>
<tr>
<td><strong>1.5D</strong></td>
<td>4.13 (5.02)</td>
<td>6.58 (5.88)</td>
<td>6.24 (5.32)</td>
<td>153 (153)</td>
<td>8.5 (7.87)</td>
<td>4.82 (6.11)</td>
<td>7.07 (7.21)</td>
<td>4.67&lt;sup&gt;b&lt;/sup&gt; (3.32)</td>
<td>4.36 (4.01)</td>
</tr>
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<sup>a</sup> e.g. 1
<sup>b</sup> e.g. 2

Neither method resolves the ambiguity in this case. Both may have been lucky in the other 1.5D cases (50/50 chance of choosing the correct maxima when two are present).
Conclusions

– Use of spherical harmonics allows the direction-of-arrival problem to be solved using Bayesian methods.
– This provides a framework for …
  – Refining the estimate over time.
  – Visualising uncertainty and ambiguity.
– Accuracy is similar to phase interferometry.
– Greater complexity and execution time.
References