A Necessary and Sufficient Condition for the Stabilization of Decentralized Time-Delay Systems by Time-Delay Controllers

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Abstract: It has recently been shown that a necessary and sufficient condition for the \( \mu \)-stabilization of a linear time-invariant (LTI) retarded time-delay system by LTI decentralized finite-dimensional controllers is that the system should not have any \( \mu \)-decentralized fixed modes with respect to decentralized static feedback. In the present work, we consider the stabilization of such systems by decentralized time-delay controllers and show that the same condition is also necessary and sufficient in this case too. This extends a previously known fact for centralized control to decentralized control. Namely, a LTI retarded time-delay system can be stabilized by decentralized LTI time-delay controllers if and only if it can be stabilized by decentralized LTI finite-dimensional controllers. We then consider the extension of these results to LTI neutral time-delay systems. We show that the necessity continues to hold for this case as well. Furthermore, sufficiency also continues to hold provided that the system has finitely many modes to the right of a vertical line passing through \( \mu - \epsilon \) for some positive \( \epsilon \). A number of examples are also presented to demonstrate the applications of the results obtained.

Keywords: Decentralized control; time-delay systems; time-delay controllers; decentralized fixed modes.

1. INTRODUCTION

For many large-scale systems, it may be very costly, if not impossible, to collect all the information in a centralized place, process it there, and dispatch the control commands from there. Decentralized control is either preferable or necessary for such systems (Siljak (1978, 1991); Jamshidi (1983, 1997); Lunze (1992)). In the stabilization and mode placement of decentralized control systems the notion of decentralized fixed modes, which was first introduced by Wang and Davison (1973), plays a central role. A decentralized fixed mode (DFM) is a mode of a linear time-invariant (LTI) dynamic system which can not be moved by decentralized static output feedback. Furthermore, a complex number is said to be a \( \mu \)-DFM if it is a DFM with real part greater than or equal to \( \mu \) (Momeni and Aghdam (2008a)). It was established by Wang and Davison (1973) that a necessary and sufficient condition for the \( \mu \)-stabilization of a LTI decentralized finite-dimensional dynamic system by LTI decentralized finite-dimensional dynamic controllers is that it should not have any \( \mu \)-DFMs, where \( \mu \) defines the border of the stability region on the complex plane (a LTI dynamic system is said to be stable or \( \mu \)-stable if all of its modes have real parts less than \( \mu \)).

Many dynamic systems may involve time-delays either inherently or due to delays in communication channels, etc. (Niculescu (2001)). The controller design for such systems, which are called time-delay systems, is more challenging because they are infinite-dimensional (Gu et al. (2003); Zhong (2006); Michiels and Niculescu (2007)). Although the subject of decentralized control of finite-dimensional systems has found place in the literature for the past four decades, the consideration of the same problem for time-delay systems has been relatively new (Xu and Lam (1999); Bakule (2005, 2008); Bakule et al. (2005a,b); Mahmoud and Almutairi (2009)). It was established by Momeni and Aghdam (2008a) that a LTI decentralized retarded time-delay system with commensurate time-delays can be \( \mu \)-stabilized by LTI decentralized finite-dimensional dynamic controllers if and only if it does not have any \( \mu \)-DFMs. The same result was generalized to systems with incommensurate time-delays by Momeni et al. (2010).

It has been shown that a LTI centralized retarded time-delay system can be stabilized by a LTI time-delay controller if and only if it can be stabilized by a LTI finite-dimensional controller (Kamen et al. (1985)). To the best knowledge of the authors, the decentralized counterpart of this important result, however, has not been proved in the literature. Therefore, in the present work, we consider \( \mu \)-stabilization of decentralized LTI time-delay systems by time-delay controllers. We first restrict ourselves to the retarded case, where the derivative of the state vector is not subject to time-delays. We consider the cases of commensurate and incommensurate time-delays together. We prove that a LTI decentralized retarded time-delay system can be \( \mu \)-stabilized by LTI decentralized time-delay controllers if and only if it does not have any \( \mu \)-DFMs.

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Combining this result with the result of Momeni et al. (2010), we also conclude that a LTI decentralized retarded time-delay system can be μ-stabilized by LTI decentralized time-delay controllers if and only if it can be μ-stabilized by LTI decentralized finite-dimensional controllers, which extends the result of Kamen et al. (1985) to the decentralized case.

After establishing the above mentioned results for retarded time-delay systems, we consider their extension to neutral time-delay systems, where the derivative of the state vector may be subject to time-delays. We show that the necessity continues to hold for this case as well. Furthermore, sufficiency also continues to hold if the system has finitely many modes with real parts greater than or equal to μ − ε, for some ε > 0.

Throughout the paper, C, R, Z, and N denote the sets of, respectively, complex numbers, real numbers, integers, and non-negative integers. For s ∈ C, Re(s) and Im(s) denote, respectively, the real and the imaginary parts of s. i denotes the imaginary unit. For k ∈ N, R^k denotes the space of k-dimensional real vector functions of a real variable. For x ∈ R^k, ̇x denotes the derivative of x. For k, l ∈ N, I_k and 0_{k×l} respectively denote the k × k-dimensional identity and the k × l-dimensional zero matrices. When the dimensions are apparent, we use I and 0 to denote respectively the identity and the zero matrices. Finally, det(·) and rank(·) respectively denote the determinant and the rank of (·) and bdiag(· · · ) denotes a block diagonal matrix with (· · · ) on the main diagonal.

2. PROBLEM STATEMENT

Consider a decentralized LTI retarded time-delay system Σ, with ν control agents, described as

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=0}^{\infty} \left( A_i x(t-h_i) + \sum_{j=1}^{\nu} B_{j,i} u_j(t-h_i) \right), \\
y_j(t) &= \sum_{i=0}^{\infty} C_{j,i} x(t-h_i), \quad j = 1, \ldots, \nu
\end{align*}
\]

(1)

where t is the time variable, x ∈ R^n is the state, and u_j ∈ R^{P_j} and y_j ∈ R^{Q_j} are, respectively, the input and the output accessible by the jth control agent (j = 1, . . . , ν). The matrices A_i, B_{j,i}, and C_{j,i} (i = 0, . . . , σ, j = 1, . . . , ν) are constant real matrices. 0 = h_0 < h_1 < . . . < h_σ are the time-delays (for notational convenience, we use h_0 = 0), where σ is the number of distinct time-delays involved. Note that we do not make any distinction between the commensurate and incommensurate time-delays; i.e., some of the time-delays h_1, . . . , h_σ may be commensurate, while others are incommensurate.

Relating to Σ, given by (1), let us first present the following definitions.

Definition 1: For any given μ ∈ R, the set of μ-modes of the system Σ is defined as

\[\Omega_\mu(Σ) := \{s ∈ C \mid \text{Re}(s) ≥ \mu \text{ and } φ_Σ(s) = 0\}\]

(2)

where \(φ_Σ(s) := \text{det}(sI − ̄A(s))\) is the characteristic function of the system Σ, where

\[̄A(s) := \sum_{i=0}^{\infty} A_i e^{-sh_i}.
\]

(3)

Definition 2: For any given μ ∈ R, the system Σ is said to be μ-stable if Ω_μ(Σ) = ∅. Furthermore, a controller K is said to μ-stabilize the system Σ, if the closed-loop system obtained by applying the controller K to system Σ is μ-stable.

The objective is to design decentralized controllers (where only feedback from y_j to u_j is allowed for j = 1, . . . , ν) so that the closed-loop system is μ-stable, for some given real μ (normally μ < 0). For this purpose, we define three different classes of controllers:

(1) K_s: the class of decentralized static LTI controllers is all the controllers of the form:

\[u_j(t) = K_j y_j(t), \quad j = 1, \ldots, \nu,
\]

(4)

where K_j are real constant matrices.

(2) K_F: the class of decentralized finite-dimensional dynamic LTI controllers is all the controllers of the form:

\[\dot{z}_j(t) = F_j z_j(t) + G_j y_j(t) \quad \text{and} \quad u_j(t) = H_j z_j(t) + K_j y_j(t), \quad j = 1, \ldots, \nu,
\]

(5)

where z_j ∈ R^{m_j} is the state of the jth controller and F_j, G_j, H_j, and K_j are real constant matrices. Here, the dimension of the controller, m_j ∈ N, is arbitrary. Note that when m_j = 0, for all j = 1, . . . , ν, such a controller reduces to a decentralized static LTI controller; thus, K_s ⊂ K_F.

(3) K_d: the class of decentralized LTI time-delay controllers is all the controllers of the form:

\[\dot{z}_j(t) = \sum_{i=0}^{\rho_j} (F_{j,i} z_j(t-h_{j,i}) + G_{j,i} y_j(t-h_{j,i})) \quad \text{and} \quad u_j(t) = \sum_{i=0}^{\rho_j} (H_{j,i} z_j(t-h_{j,i}) + K_{j,i} y_j(t-h_{j,i})),
\]

(6)

where j = 1, . . . , ν, where z_j ∈ R^{m_j} is the state and 0 = h_{j,0} < h_{j,1} < . . . < h_{j,\rho_j} are the time-delays of the jth controller. Here, both the dimension of the controller, m_j ∈ N, and the number of distinct time-delays involved, \(\rho_j \in N\), as well as the time-delays, h_{j,1}, . . . , h_{j,\rho_j}, are arbitrary (note that the time-delays of the controller need not be the same as those of the system; however, some or all of them may be chosen so). Furthermore, F_{j,i}, G_{j,i}, H_{j,i}, and K_{j,i} are real constant matrices. Note that when \(\rho_j = 0\), for all j = 1, . . . , ν, such a controller reduces to a decentralized finite-dimensional dynamic LTI controller; thus, K_F ⊂ K_d.

As in Wang and Davison (1973) and Momeni et al. (2010), the set of μ-DFMs of the system Σ with respect to a given class of controllers is defined as follows.

Definition 3: For any given μ ∈ R, the set of μ-DFMs of the system Σ with respect to the class of controllers K (where K may be K_s, K_F, or K_d) is defined as

\[\Lambda_\mu(Σ, K) := \{s ∈ C \mid \text{Re}(s) ≥ \mu \text{ and } φ_{Σ,K}(s) = 0, \forall K \in K\}
\]

(7)
where \( \phi_{\Sigma,K}(\cdot) \) is the characteristic function of the closed-loop system obtained by applying controller \( K \) to system \( \Sigma \).

It is clear that \( \Lambda_\mu(\Sigma, K) \subset \Omega_\mu(\Sigma) \) for any class of controllers \( K \) which includes the zero controller (i.e., the controller which applies \( u_i(t) = 0 \), \( \forall t, j = 1, \ldots, \nu \), independent of \( y_i(t), i = 1, \ldots, \nu \); note that each of \( K_s, K_f, \) and \( K_d \) include the zero controller). To determine \( \Lambda_\mu(\Sigma, K_d) \), a numerical procedure, which was originally proposed by Wang and Davison (1973) for finite-dimensional systems and presented for time-delay systems by Momeni et al. (2010), may be used. Alternatively, the algebraic test given by the following lemma may also be used. The advantage of this test over the numerical procedure is that, it gives the desired set with certainty, while the numerical procedure gives it only with probability 1.

**Lemma 1:** Let \( \text{Re}(s_0) \geq \mu \). \( s_0 \in \Lambda_\mu(\Sigma, K_d) \) if and only if there exists \( k \in \{0, \ldots, \nu \} \) and \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, \nu \} \), where \( i_1, \ldots, i_k \) are distinct, such that

\[
\begin{align*}
\begin{bmatrix}
s_0 I - \bar{A}(s_0) B_i(s_0) & \cdots & B_k(s_0) \\
C_{i,k+1}(s_0) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
C_{i,\nu}(s_0) & \cdots & 0
\end{bmatrix}
\end{align*}
\]

\[
\text{rank} \begin{bmatrix}
s_0 I - \bar{A}(s_0) B_i(s_0) & \cdots & B_k(s_0) \\
C_{i,k+1}(s_0) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
C_{i,\nu}(s_0) & \cdots & 0
\end{bmatrix} < n \quad (8)
\]

where \( \{i_k,\ldots,i_1\} := \{1, \ldots, \nu \} \setminus \{i_1, \ldots, i_k\} \),

\[
B_j(s) := \sum_{i=0}^\nu B_{j,i} e^{-s_i h_i}, \quad j = 1, \ldots, \nu, \quad (9)
\]

and

\[
C_j(s) := \sum_{i=0}^\nu C_{j,i} e^{-s_i h_i}, \quad j = 1, \ldots, \nu. \quad (10)
\]

**Proof:** This lemma was presented without proof in Momeni et al. (2010). As indicated there, it can be proved along the lines of the proof of the original result, which was presented for finite-dimensional systems by Anderson and Clements (1981). □

It should be noted that, since \( \Lambda_\mu(\Sigma, K_d) \subset \Omega_\mu(\Sigma) \), the test in the above lemma need to be applied only for \( s_0 \in \Omega_\mu(\Sigma) \), which is a finite set for any finite real \( \mu \) (Nicolescu (2001)). It is also worth to note that when \( k = \nu \) the test becomes a test for spectrally controllability and when \( k = 0 \), it becomes a test for spectral observability (see Richard (2003) for the definitions of spectrality controllability and observability).

As long as the direct computation of \( \Lambda_\mu(\Sigma, K_f) \) or of \( \Lambda_\mu(\Sigma, K_d) \) is concerned, both the applicability of the numerical procedure and of the test in Lemma 1 requires a more complicated proof. Instead, in the next section we will show that \( \Lambda_\mu(\Sigma, K_f) = \Lambda_\mu(\Sigma, K_d) = \Lambda_\mu(\Sigma, K_s) \). Thus, once \( \Lambda_\mu(\Sigma, K_s) \) is found either by the numerical procedure or the test in Lemma 1, this also gives \( \Lambda_\mu(\Sigma, K_f) \) and \( \Lambda_\mu(\Sigma, K_d) \).

3. MAIN RESULTS

We first present the following results.

**Lemma 2:** For any given \( \mu \in \mathbb{R} \),

\[
\Lambda_\mu(\Sigma, K_s) = \Lambda_\mu(\Sigma, K_f) = \Lambda_\mu(\Sigma, K_d). \quad (11)
\]

**Proof:** \( \Lambda_\mu(\Sigma, K_s) \subset \Lambda_\mu(\Sigma, K_f) \) was proved by Momeni et al. (2010). \( \Lambda_\mu(\Sigma, K_f) \subset \Lambda_\mu(\Sigma, K_d) \) follows from \( K_e \subset K_d \) (since any decentralized static LTI control is a decentralized finite-dimensional LTI controller, a mode which can not be moved by any decentralized finite-dimensional LTI controller, can not be moved by a decentralized static LTI controller). □

**Lemma 3:** For any given \( \mu \in \mathbb{R} \),

\[
\Lambda_\mu(\Sigma, K_f) = \Lambda_\mu(\Sigma, K_d). \quad (12)
\]

**Proof:** Consider a controller in the class \( K_d \), described as in (6). Let \( m := \sum_{j=1}^\nu m_j \) and define

\[
\{h_0, \ldots, h_\rho\} := \cup_{j=1}^\nu \{h_{j,0}, \ldots, h_{j,\rho}\},
\]

where \( h_0 = 0 \) and \( h_1, \ldots, h_\rho \) are all nonzero and distinct. Next define

\[
\hat{F}_{j,i} := \begin{cases} F_{j,i} & \text{if } h_i = h_{j,k} \\ 0_{m_j \times m_k} & \text{if } h_i \notin \{h_{j,0}, \ldots, h_{j,\rho}\} \end{cases}
\]

and

\[
\hat{F}_i = \text{bdiag} \{\hat{F}_{1,i}, \ldots, \hat{F}_{\nu,i}\}, \quad i = 0, \ldots, \rho.
\]

Define \( \tilde{G}_i, \tilde{H}_i, \) and \( \tilde{K}_i \) similarly. Then let

\[
\tilde{K}_i := \begin{bmatrix} K_i & H_i \end{bmatrix},
\]

and

\[
\hat{K}(s) := \sum_{i=0}^\nu e^{-s_i h_i} \tilde{K}_i. \quad (\text{Also define})
\]

\[
\check{A}(s) := \begin{bmatrix} \hat{A}(s) & 0 \\ 0 & 0_{m \times m} \end{bmatrix}, \quad \check{B}(s) := \begin{bmatrix} \hat{B}(s) & 0 \\ 0 & I_m \end{bmatrix},
\]

and

\[
\check{C}(s) := \begin{bmatrix} \check{C}(s) & 0 \\ 0 & I_m \end{bmatrix},
\]

where \( \check{A}(s) \) is defined in (3),

\[
\check{B}(s) := \begin{bmatrix} B_{1}(s) & \cdots & B_{\nu}(s) \end{bmatrix},
\]

where \( \check{B}_j(s), j = 1, \ldots, \nu, \) are given by (9), and

\[
\check{C}(s) := \begin{bmatrix} \check{C}_1(s) \\ \vdots \\ \check{C}_\nu(s) \end{bmatrix},
\]

where \( \check{C}_j(s), j = 1, \ldots, \nu, \) are given by (10). Then, the characteristic function of the closed-loop system is given by

\[
\phi_{\Sigma,K}(s) = \det \begin{bmatrix} sI - \hat{A}(s) - \hat{B}(s)\hat{K}(s)\check{C}(s) \end{bmatrix}.
\]

Now, consider \( s_0 \in \Lambda_\mu(\Sigma, K_f) \). Then

\[
\det \begin{bmatrix} s_0 I - \hat{A}(s_0) - \hat{B}(s_0)\hat{K}_0\check{C}(s_0) \end{bmatrix} = 0 \quad (13)
\]

for any \( \hat{K}_0 \) with the above defined structure. However, since \( s_0 \) is fixed, \( \hat{K}(s_0) \) is also a fixed matrix which has the same structure as \( \hat{K}_0 \), except that \( \hat{K}_0 \) is assumed to be real, whereas \( \hat{K}(s_0) \) may be non-real for a non-real \( s_0 \). However, if (13) holds for all real \( \hat{K}_0 \) with a given structure, then it should also hold for all complex \( \hat{K}_0 \) with the same structure. This implies that \( \phi_{\Sigma,K}(s_0) = 0 \). Thus \( \Lambda_\mu(\Sigma, K_f) \subset \Lambda_\mu(\Sigma, K_d) \). On the other hand, since
Now we can prove our main result.

**Proof:** Follows from Lemmas 2 and 3.

Next we prove our main result.

**Theorem 1:** There exists a controller $K \in K_d$ which $\mu$-stabilizes the system $\Sigma$ if and only if $\Lambda_\mu(\Sigma, K_d) = \emptyset$.

**Proof:** To prove the only if part, assume that $\Lambda_\mu(\Sigma, K_d) \neq \emptyset$. Then there exists $s_0 \in \Lambda_\mu(\Sigma, K_d)$. By Lemma 4, then, $s_0 \in \Lambda_\mu(\Sigma, K_d)$. This implies that $\phi_{\Sigma, K}(s_0) = 0$ for all $K \in K_d$. Consequently, there can exist no controller $K \in K_d$ which $\mu$-stabilizes $\Sigma$.

To prove the if part, we note that it was proved by Momeni et al. (2010) that there exists a controller $K \in K_f$ which $\mu$-stabilizes the system $\Sigma$ if and only if $\Lambda_\mu(\Sigma, K_f) = \emptyset$ (actually, only the case $\mu = 0$ was considered by Momeni et al. (2010); however, all the arguments are valid for any finite real $\mu$). If the part of the present theorem then follows from this result, since $K_f \subset K_d$.

By combining the above result with the main result of Momeni et al. (2010) (mentioned in the second paragraph of the above proof), we also obtain the following result, which is the decentralized counterpart of the main result of Kamen et al. (1985).

**Corollary 1:** There exists a controller $K \in K_d$ which $\mu$-stabilizes the system $\Sigma$ if and only if there exists a controller $K \in K_f$ which $\mu$-stabilizes it.

## 4. NEUTRAL SYSTEMS

Consider the system described by (1). Suppose that the term $\sum_{i=1}^\sigma E_i \hat{z}(i - h_i)$ is added to the left-hand side of the first equation, where $E_i$, $i = 1, \ldots, \sigma$, are constant real matrices. Such a system is called a neutral time-delay system. Let us denote this system by $\Sigma_n$. Note that the characteristic function of $\Sigma_n$ is given by $\phi_{\Sigma_n}(s) = \det (sE(s) - A(s))$, where $E(s) := I + \sum_{i=1}^{\sigma} E_i e^{-sh_i}$.

An important difference between $\Sigma_n$ and $\Sigma$ is that, for any given finite real $\mu$, $\Omega_n(\Sigma)$ is a finite set, whereas $\Omega_\mu(\Sigma_n)$ may have infinitely many elements (Niculescu (2001)). Despite this difference, however, the proofs of all the lemmas given above remain valid. In particular, we can establish $\Lambda_\mu(\Sigma_n, K_d) = \Lambda_\mu(\Sigma_n, K_f) = \Lambda_\mu(\Sigma_n, K_s)$.

Furthermore, either the numerical procedure mentioned in Section 2 or the test in Lemma 1 (the term $s_0I - A(s_0)$ in (8) must now be replaced by $s_0E(s_0) - A(s_0)$) can be used to determine $\Lambda_\mu(\Sigma_n, K_d)$ at least in the case when $\Omega_\mu(\Sigma_n)$ is a finite set (when $\Omega_\mu(\Sigma_n)$ has infinitely many elements, the test in Lemma 1 need to be applied for infinitely many $s_0$).

Since, we can establish $\Lambda_\mu(\Sigma_n, K_d) = \Lambda_\mu(\Sigma_n, K_f) = \Lambda_\mu(\Sigma_n, K_s)$, we can easily show that (by using the same argument as in the first paragraph of the proof of Theorem 1) a necessary condition for the existence of a controller $K \in K_d$ (or in $K_f$ or in $K_s$) which $\mu$-stabilizes the system $\Sigma_n$ is that $\Lambda_\mu(\Sigma_n, K_d) = \emptyset$. Furthermore, we can also apply the proof of the if part of Theorem 1 as long as $\Omega_n(\Sigma_n)$ is a finite set for some $\epsilon > 0$ (see Momeni and Aghdam (2008b) for the proof in the case of retarded time-delay systems with commensurate time-delays and $\mu = 0$; the relevant part of the proof there equally applies in the present case). Therefore, Theorem 1 and Corollary 1 hold, not only for $\Sigma$, but also for $\Sigma_n$ with the property that $\Omega_{\mu-\epsilon}(\Sigma_n)$ is a finite set for some $\epsilon > 0$. For the more general case (i.e., when $\Omega_\mu(\Sigma_n)$ is not a finite set or when there exists no $\epsilon > 0$ such that $\Omega_{\mu-\epsilon}(\Sigma_n)$ is a finite set), however, to determine whether or not the condition in Theorem 1 is sufficient for the existence of any type of decentralized controllers which $\mu$-stabilizes the system $\Sigma_n$ or whether there exists any other meaningful sufficient conditions, remains to be an open problem.

### 5. EXAMPLES

**Example 1:** Consider a LTI retarded time-delay system described as in (1) with $\nu = 2$, $\sigma = 4$, $h_1 = 0.5$, $h_2 = 0.75$, $h_3 = 1$, $h_4 = 1.2$.

$$A_0 = \begin{bmatrix} 9.06 & 0 & 0 \\ -3 & 6.81 & 2.76 \\ -1.36 & 1.02 & 3 \end{bmatrix}, \quad B_{1,0} = \begin{bmatrix} 0 \\ 0 \\ -1.38 \end{bmatrix}$$

$$B_{2,0} = \begin{bmatrix} -7.43 \\ 0 \\ 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 4.96 & -3.72 & 1.74 \\ -1.84 & 1.38 & 0 \end{bmatrix},$$

$$B_{1,3} = \begin{bmatrix} 0 \\ 0 \\ -0.87 \end{bmatrix}, \quad B_{2,4} = \begin{bmatrix} 2.57 \\ 0 \\ 0.296 \end{bmatrix}$$

$$C_{1,0} = [0 \ 1.32 \ -10.54], \quad C_{2,0} = [2.32 \ -1.74 \ 0]$$

and all other matrices being zero. Let us denote this system as $\Sigma_1$. By using the Matlab toolbox DDE-BIFTOOL (Engelborghs et al. (2002)), we obtain $\Omega_0(\Sigma_1) = \{9.06, 7.4423, 2.1737, 0.1111\}$, which indicates that the system is not stable (from now on we consider 0-stability). It can, however, be verified that

$$\text{rank} \begin{bmatrix} s_0I - A(s_0) & B_1(s_0) \\ B_2(s_0) \end{bmatrix} = 3$$

and

$$\text{rank} \begin{bmatrix} s_0I - A(s_0) \\ C_1(s_0) \\ C_2(s_0) \end{bmatrix} = 3$$

for all $s_0 \in \Omega_0(\Sigma_1)$. Thus, the system is spectrally stabilizable and detectable and there exists a centralized controller which stabilizes it (Richard (2003)). However, for $s_0 = 9.06$, we note that

$$\text{rank} \begin{bmatrix} s_0I - A(s_0) & B_1(s_0) \\ C_2(s_0) \end{bmatrix} = 2 < 3.$$ Therefore, by Lemma 1, $s_0 \in \Omega_0(\Sigma_1, K_d)$, and hence, $\alpha(\Sigma_1, K_d) \neq \emptyset$. By Theorem 1, this implies that there exists no controller in the class $K_d$ (or in $K_f$ or in $K_s$) which stabilizes this system.

**Example 2:** Consider another LTI retarded time-delay system described as in (1) with $\nu = 2$, $\sigma = 5$, $h_1 = 0.7$, $h_2 = 1.2$, $h_3 = 1.8$, $h_4 = 2.6$, $h_5 = 2.8$,

$$A_0 = \begin{bmatrix} -5 & 4.8 & -9.5 \\ 1.914 & -8.95 & 0.1 \\ 8.65 & 0.127 & -0.5 \end{bmatrix}$$

$$B_{1,4} = \begin{bmatrix} 0.43 \\ -6 \\ -8.22 \end{bmatrix}$$


Example 3: Now, consider a LTI neutral time-delay system described as

\[ B_{2,2} = \begin{bmatrix} 3.38 \\ 9.45 \\ 5.47 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 4 & -1 & 9.7 \\ -0.5 & 1.31 & 0 \\ 0 & -10.8 & 0 \end{bmatrix} \]

\[ C_{1,3} = \begin{bmatrix} 6.86 & -1 & 0.51 \end{bmatrix}, \quad C_{2,1} = \begin{bmatrix} 0.29 & 0.38 & -4 \end{bmatrix} \]

and all other matrices being zero. Let us denote this system as \( \Sigma_2 \). We obtain \( \Omega_0(\Sigma_2) \) for \( \mu = -0.65 \) using the Matlab toolbox DDE-BIFTOOL (Engelborghs et al. (2002)) and plot it in Fig. 1 using ‘*’ (in red). In particular we identify \( \Omega_0(\Sigma_2) = \{0.007316, 0.031 \pm 2.2213i\} \), which indicates that the open-loop system is unstable. In order to find \( \Lambda_0(\Sigma_1, K_d) \), let us apply

\[ u_1(t) = \frac{1}{4} y_1(t) \]

and

\[ u_2(t) = \frac{1}{4} y_2(t) \]

Let us denote the closed-loop system with this feedback as \( \Sigma^0_2 \). We find that

\[ \Omega_0(\Sigma^0_2) = \{0.1314 \pm 1.9278i, 0.0801 \pm 1.034i\} \]

Comparing this with \( \Omega_0(\Sigma_2) \), we conclude that \( \Lambda_0(\Sigma_2, K_a) = \emptyset \). Thus, by Theorem 1, there exists a controller in the class \( K_d \) (and, by Corollary 1, also in the class \( K_f \)) which stabilizes this system. In fact, if we apply the following controller:

\[
\begin{align*}
\dot{z}_1(t) &= -8z_1(t) - 0.2y_1(t - 2.8) \\
u_1(t) &= 0.1z_1(t - 3.4) + 0.03y_1(t - 1.4)
\end{align*}
\]

and

\[
\begin{align*}
\dot{z}_2(t) &= -0.5z_2(t) - 0.03y_2(t - 1.6) \\
u_2(t) &= 0.5z_2(t - 1.8) + 0.2y_2(t - 3.3)
\end{align*}
\]

and calculate \( \Omega_0(\Sigma^1_2) \), where \( \mu = -0.65 \) and \( \Sigma^1_2 \) denotes the closed-loop system obtained by applying the controller (15)–(16) to system \( \Sigma_2 \), using DDE-BIFTOOL, we obtain the closed-loop modes shown in Fig. 1 with ‘+’ (in red). From this figure, we observe that \( \Omega_0(\Sigma^1_2) = \emptyset \), which means that the controller (15)–(16) stabilizes the system \( \Sigma_2 \).

Example 2. We find that

\[
\begin{align*}
\dot{x}(t) + \begin{bmatrix} 3 & 6 \\ -1.5 & -3 \end{bmatrix} \dot{x}(t - 1) &= \begin{bmatrix} 8 & 0 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 3 & 6 \\ -2 & -4 \end{bmatrix} x(t - 1) \\
&+ \begin{bmatrix} -8 \\ 4.25 \end{bmatrix} u_1(t) + \begin{bmatrix} -7.5 \\ 2 \end{bmatrix} u_1(t - 1) \\
&+ \begin{bmatrix} 2.4 \\ -1 \end{bmatrix} u_2(t) + \begin{bmatrix} 1.9 \\ -0.9 \end{bmatrix} u_2(t - 1)
\end{align*}
\]

\[
y_1(t) = [1 1] x(t) \quad y_2(t) = [0 2] x(t)
\]

Let us denote this system by \( \Sigma_3 \). The characteristic function of this system can be found as

\[ \phi_{\Sigma_3}(s) = s^2 + 7se^{-s} - 10s - 14e^{-s} + 16. \]

The roots of \( \phi_{\Sigma_3}(s) = 0 \), i.e., the open-loop modes of \( \Sigma_3 \), are found as

\[
\left\{ \begin{array}{l}
2, \\
W_k \left( -\frac{7}{e^s} \right) + 8, k \in \mathbb{Z}
\end{array} \right.
\]

where \( W_k \) represents the \( k \)th branch of the Lambert W-function (Corless et al. (1996)). We plot these values in Fig. 2 using ‘*’ (in blue), where we calculate the Lambert W-function up to \( \pm 100 \) branches (roots corresponding to higher branches stay in the open left-half complex plane and diverge to infinity along vertical axes). We find that

\[ \Omega_0(\Sigma_3) = \{2, 7.9977646\} \]

Furthermore, we calculate the right-most root in the left-half complex plane as \(-0.152403\). Thus, we can find an \( \epsilon > 0 \) (e.g., \( \epsilon = 0.1 \)) such that \( \Omega_{-\epsilon}(\Sigma_3) \) has finitely many (two) elements. Therefore, by the arguments presented in Section 4, there exists a controller in the class \( K_d \) (and in \( K_f \)) which stabilizes this system if and only if \( \Lambda_0(\Sigma_3, K_a) = \emptyset \). To check whether this condition is satisfied or not, we can apply the test in Lemma 1 (where the term \( s_0 I - A(s_0) \) in (8) is replaced by \( s_0 E(s_0) - A(s_0) \)) and show that neither \( s_1 = 2 \) nor \( s_2 = 7.9977646 \) is a DFM. Thus, there indeed exists a controller in the class \( K_d \) (and in \( K_f \)) which stabilizes this system. In fact, if we let

\[ u_1(t) = 2y_1(t) \]

and
the closed-loop characteristic function can be found as

\[ \phi_{\Sigma_3,K}(s) = s^2 + 7se^{-s} + 11s + 7e^{-s} + 10 \]

whose roots (i.e., the closed-loop modes) can be calculated as

\[ \{-1, W_k(-7e^{10}) - 10, k \in \mathbb{Z}\}. \]

These roots are shown in Fig. 2 with ‘+’ (in red), where we again calculate the Lambert W-function up to \( \pm 100 \)th branches (roots corresponding to higher branches again stay in the open left-half complex plane and diverge to infinity along vertical axes). We observe that there are no closed-loop modes with non-negative real parts. Thus the controller (17)–(18) stabilizes the system \( \Sigma_3 \).

6. CONCLUSIONS

Stabilization of LTI time-delay systems by decentralized LTI time-delay controllers has been considered. It has first been shown that the sets of decentralized fixed modes of a LTI time-delay system under LTI decentralized static, dynamic, or time-delay controllers are equivalent. Then it is proved that, if a LTI retarded time-delay system can be \( \mu \)-stabilized by LTI decentralized time-delay controllers if and only if it does not have any \( \mu \)-DFMs with respect to LTI decentralized static controllers. By combining this result with an earlier result, we also obtain the decentralized version of the result by Kamen et al. (1985): a LTI retarded time-delay system can be \( \mu \)-stabilized by LTI decentralized time-delay controllers if and only if it can be stabilized by LTI dynamic finite-dimensional controllers. We note, however, that stabilization of a time-delay system by a finite-dimensional controller may sometimes require a very high dimensional controller. In such a case, it may be easier to design and implement a time-delay controller (although such a controller in theory will have infinite dimension, it may be easily realized by using delay elements and finite-impulse response filters).

The arguments presented in Section 4 shows that the necessary and sufficient conditions for the \( \mu \)-stabilization of LTI neutral time-delay systems are essentially same as those for LTI retarded time-delay systems as long as the system has finitely many modes with real parts greater than or equal to \( \mu - \epsilon \), for some \( \epsilon > 0 \). When this is not satisfied, the conditions presented in this work continues to be necessary. However, further work is required to determine meaningful sufficient conditions in this case.

REFERENCES


