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Some Algebraic Inequality Lemmas and Their Application on Problems

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1 Introduction

Lemma 1.1 *$a, b, c, k \in \mathbf{R}^+$ such that $a+b+c=x$ and $k \geq 1$. Then prove that*

$$\frac{a^k+k-1}{b+k-1}+\frac{b^k+k-1}{c+k-1}+\frac{c^k+k-1}{a+k-1} \geq \frac{3xk}{x+3k-3}$$

Proof. By effective AM-GM and Bergstrom Inequality

$$\sum_{cyc}\frac{a^k+k-1}{b+k-1}=\sum_{cyc}\frac{\overbrace{a^k+1+1+\cdots+1}^{k}}{b+k-1} \overbrace{\geq}^{AM-GM} \sum_{cyc}\frac{ka}{b+k-1}$$

$$=k\left(\frac{a}{b+k-1}\right) \overbrace{\geq}^{Bergstorm} \frac{k(a+b+c)^2}{ab+ac+bc+(k-1)(a+b+c)}$$

$$=k\left(\frac{a+b+c}{\frac{a+b+c}{3}+k-1}\right)=\frac{3xk}{x+3k-3}$$

Application on Problem- Macedonia 2010 1.2

$a, b, c \in \mathbf{R}^+$ such that $a+b+c=3$. Then prove that

$$\frac{a^3+2}{b+2}+\frac{b^3+2}{c+2}+\frac{c^3+2}{a+2}\geq 3$$

Proof. By **Lemma 1.1**, placing $x=3$ and $k=3$, we get the following :

$$\frac{a^3+2}{b+2}+\frac{b^3+2}{c+2}+\frac{c^3+2}{a+2}\geq \frac{3xk}{x+3k-3}=3$$

Lemma 2.1.
$x,y,z,a,b,c\in\mathbf{R}^+$ such that $x+y+z=k$. Then prove that

$$\frac{1}{\sqrt{x(ay+bz)}}+\frac{1}{\sqrt{y(az+bx)}}+\frac{1}{\sqrt{z(ax+by)}}\geq\frac{9}{k\sqrt{a+b}}$$

Proof. By Bergstorm Inequality and Jensen Inequality

$$\sum_{cyc}\frac{1}{\sqrt{x(ay+bz)}}\overbrace{\geq}^{Bergstrom}\frac{9}{\sqrt{x(ay+bz)}+\sqrt{y(az+bx)}+\sqrt{z(ax+by)}}$$

$$\overbrace{\geq}^{Jensen}\frac{9}{3\sqrt{\frac{axy+ayz+axz+bxy+byz+bxz}{3}}}=\frac{9}{3\sqrt{\frac{(a+b)(xy+yz+zx)}{3}}}$$

$$\geq\frac{9}{3\sqrt{\frac{(a+b)(\frac{(x+y+z)^2}{3})}{3}}}=3\sqrt{\frac{9}{k^2(a+b)}}=\frac{9}{k\sqrt{a+b}}$$

Lemma 2.2.
$x,y,z\in\mathbf{R}^+$ such that $x+y+z=3$. Then prove that

$$\frac{1}{\sqrt{x(2y+3z)}}+\frac{1}{\sqrt{y(2z+3x)}}+\frac{1}{\sqrt{z(2x+3y)}}\geq\frac{3}{k\sqrt{5}}$$

Proof.

By **Lemma 2.1.**, placing $x+y+z=3$ and $a+b=5$, we get the following:

$$\frac{1}{\sqrt{x(2y+3z}}+\frac{1}{\sqrt{y(2z+3x}}+\frac{1}{\sqrt{z(2x+3y}}\geq\frac{3}{k\sqrt{5}}\geq\frac{9}{k\sqrt{a+b}}=\frac{3}{\sqrt{5}}$$

Lemma 3.1. *$y_1, y_2, ..., y_n, x, p, n \in \mathbf{R}^+$ positive reels and $n > p \geq 2$ integers , $a_1, a_2, ..., a_n \geq 1$ integers, $m \geq 1$. Then prove that*

$$\sum_{y_i,i=1}^{i=p}\sqrt[\binom{n-1}{p-1}]{\frac{xy_1^{\frac{4m(n-1)!}{(n-p)!}}+k^2x}{\prod_{a_1+a_2+...+a_p=n}(a_1y_1+a_2y_2+...+a_py_p)}}\geq\sqrt[\binom{n-1}{p-1}]{2xk}.\binom{n+1}{p-1}\frac{2.\left(\frac{y_1+y_2+...+y_n}{p}\right)^{2m(p-1)!-1}}{n(n+1)}$$

Determine when does the equality occurs ?

Proof By effective AM-GM and Bergstrom
Let us show

$$xy_1^t+k^2x\geq 2xky_1^{\frac{y}{2}}$$

.

$$xy_1^t+k^2x=\overbrace{\frac{y_1^t}{k}+\frac{y_1^t}{k}+\cdots+\frac{y_1^t}{k}}^{xk}+\overbrace{k+k+\cdots+k}^{xk}\geq 2xky^{\frac{txk}{2xk}}=2xky^{\frac{t}{2}}$$

Then for the denominator

$$xy_1^{\frac{4m(n-1)!}{(n-p)!}}+k^2x\geq 2xky_1^{\frac{2m(n-1)!}{(n-p)!}}$$

$$\sum_{y_i,i=1}^{i=p}\binom{n-1}{p-1}\sqrt{\frac{xy_1^{\frac{4m(n-1)!}{(n-p)!}}+k^2x}{\prod_{a_1+a_2+...+a_p=n}(a_1y_1+a_2y_2+...+a_py_p)}}$$

$$\geq\sum_{y_i,i=1}^{i=p}\binom{n-1}{p-1}\sqrt{\frac{2xky_1^{\frac{2m(n-1)!}{(n-p)!}}}{\prod_{a_1+a_2+...+a_p=n}(a_1y_1+a_2y_2+...+a_py_p)}}$$

$$= \frac{\sqrt[\binom{n-1}{p-1}]{2xk}\left(y_1^{2m(p-1)!} + y_2^{2m(p-1)!} + \cdots + y_p^{2m(p-1)!}\right)}{\sqrt[\binom{n-1}{p-1}]{\prod_{a_1+a_2+\cdots+a_p=n}\left(a_1y_1 + a_2y_2 + \cdots + a_py_p\right)}}$$

$$\geq \frac{\sqrt[\binom{n-1}{p-1}]{2xk}.2\left(y_1^{2m(p-1)!} + y_2^{2m(p-1)!} + \cdots + y_p^{2m(p-1)!}\right)}{\left[\frac{(n-p+1)(n-p+2)(y_1+y_2+\cdots+y_p)}{\binom{n-1}{p-1}}\right]}$$

$$\geq \frac{\sqrt[\binom{n-1}{p-1}]{2xk}.2\left(\frac{(y_1+y_2+\cdots+y_p)^{2m(p-1)!}}{p^{2m(p-1)!}.p}\right)}{\left[\frac{(n-p+1)(n-p+1)(y_1+y_2+\cdots+y_p)}{\binom{n-1}{p-1}}\right]}$$

$$= \sqrt[\binom{n-1}{p-1}]{2xk}.2\left[\frac{\binom{n-1}{p-1}}{(n-p+1)(n-p+2)}\frac{(y_1+y_2+\cdots+y_p)^{2m(p-1)!-1}}{p^{2m(p-1)!-1}}\right]$$

$$= \sqrt[\binom{n-1}{p-1}]{2xk}.\frac{(n-1)!}{(n-p)!(n-p+1)(n-p+2)(p-1)!}.2\left(\frac{y_1+y_2+\cdots+y_p}{p}\right)^{2m(p-1)!-1}$$

$$= \sqrt[\binom{n-1}{p-1}]{2xk}.\frac{(n-1)!}{(n-p+2)!(p-1)!}.2\left(\frac{y_1+y_2+\cdots+y_p}{p}\right)^{2m(p-1)!-1}$$

$$= \sqrt[\binom{n-1}{p-1}]{2xk}.\frac{(n+1)!}{(n-p+2)!(p-1)!}.\frac{(n-1)!}{(n+1)!}.2\left(\frac{y_1+y_2+\cdots+y_p}{p}\right)^{2m(p-1)!-1}$$

$$= \binom{n-1}{p-1}\sqrt{2xk}.\binom{n+1}{p-1}.\frac{1}{n(n+1)}.2\left(\frac{y_1+y_2+\cdots+y_p}{p}\right)^{2m(p-1)!-1}$$

$$= \binom{n-1}{p-1}\sqrt{2xk}.\binom{n+1}{p-1}.2\frac{\left(\frac{y_1+y_2+\ldots+y_n}{p}\right)^{2m(p-1)!-1}}{n(n+1)}$$

Important Places We know the π includes how many terms inside of it by binomial coefficients => $\binom{n-1}{p-1}$. In the problem, $a_1, a_2, \cdots, a_p \geq 1$ integers which allows us to use ball-pocket binomial formula.) After using GM-AM on the denominator, $min_{a_1} = 1$, $max_{a_1} = n-p+1$ because of other $p-1$ has at least one.

Equality occurs if and if $y_1^{\frac{4m(n-1)!}{(n-p)!}} = y_2^{\frac{4m(n-1)!}{(n-p)!}} = \cdots = y_p^{\frac{4m(n-1)!}{(n-p)!}}$ and $a_1 = a_2 = \cdots = a_p = y_1 = \cdots = y_p$

Lemma 3.2
$a, b, c, x, n \in \mathbf{R}^+$ positive reels, $i, j, s \geq 1$ integers, $n \geq 4$ and $k \geq 2$. Then prove that

$$\sum_{k=a}^{c}\binom{n-1}{2}\sqrt{\frac{xa^{4m(n-1)!}+4x}{\prod_{i+j+s=n}(ia+jb+sc)}} \geq \binom{n-1}{2}\sqrt{4x}.\binom{n+1}{2}.2\frac{\left(\frac{a+b+c}{3}\right)^{4m-1}}{n(n+1)}$$

Proof.
By **Lemma 3.1.**, placing $k=2$, $p=3$, we get the following:

$$\sum_{k=a}^{c}\binom{n-1}{2}\sqrt{\frac{xa^{4m(n-1)!}+4x}{\prod_{i+j+s=n}(ia+jb+sc)}} \geq \binom{n-1}{p-1}\sqrt{2xk}.\binom{n+1}{p-1}\frac{2.\left(\frac{y_1+y_2+\ldots+y_n}{p}\right)^{2m(p-1)!-1}}{n(n+1)}$$

$$= \binom{n-1}{2}\sqrt{4x}.\binom{n+1}{2}.2\frac{\left(\frac{a+b+c}{3}\right)^{4m-1}}{n(n+1)}$$

Lemma 3.3
$a, b, c, x, n \in \mathbf{R}^+$ such that $n \geq 3$. Then prove that

$$\sum_{q=a}^{c} \sqrt[(n-1)!]{\frac{xa^{8k(n-1)!}+9x}{\prod_{i+j=n}(ia+jb)}} \geq \sqrt[(n-1)!]{6x}.(n-2)!\frac{(\frac{a+b+c}{3})^{4k-1}}{6n}$$

Proof.
By $xa^t + k^2x \geq 2xka^{\frac{t}{2}}$

$$\sum_{cyc} \sqrt[n-1]{\frac{xa^{8m(n-1)}+9x}{\prod_{i+j=n}(ia+jb)}} \overbrace{\geq}^{AM-GM} \sum_{cyc} \sqrt[n-1]{6x.\frac{a^{4m(n-1)}}{\prod_{i+j=n}(ia+jb)}}$$

$$= \sqrt[n-1]{6x}\left(\sum_{cyc} \sqrt[n-1]{\frac{a^{4m(n-1)}}{\prod_{i+j=n}(ia+jb)}}\right)$$

$$= \sqrt[n-1]{6x}\left(\sum_{cyc} \frac{a^{4m}}{\sqrt[n-1]{\prod_{i+j=n}(ia+jb)}}\right) \overbrace{\geq}^{GM-AM} \sqrt[n-1]{6x}\left(\sum_{cyc} \frac{a^{4m}}{\frac{n(n-1)(a+b)}{(n-1)}}\right)$$

$$= \sqrt[n-1]{6x}\left(\sum_{cyc} \frac{a^{4m}}{n(a+b)}\right) \geq \sqrt[n-1]{6x}\left(\frac{(a+b+c)^{4m}}{2n(a+b+c).3^{4m-2}}\right)$$

$$= \sqrt[n-1]{6x}\left(\frac{\left(\frac{a+b+c}{3}\right)^{4m-1}}{6n}\right)$$

Lemma 4.1.
$a, b, c, x, t, y \in \mathbf{R}^+$, $a+b+c = k$ and $x \geq 2$. Then prove that following inequality holds

$$\frac{t-y\sqrt{a}}{\sqrt{c+xa}}+\frac{t-y\sqrt{b}}{\sqrt{a+xb}}+\frac{t-y\sqrt{c}}{\sqrt{b+xc}}\geq\frac{3}{\sqrt{x+1}}(t\sqrt{\frac{k}{3}}-y)$$

$$\sum_{cyc}\frac{t-\sqrt{a}}{\sqrt{c+xa}}=\sum\frac{t}{\sqrt{c+xa}}-\sum_{cyc}\sqrt{\frac{a}{c+xa}}$$

.

$$\sum_{cyc}\frac{t}{\sqrt{c+xa}}\overbrace{\geq}^{Bergstorm}t(\frac{9}{\sqrt{c+xa}+\sqrt{a+xb}+\sqrt{b+xc}})\overbrace{\geq}^{Jensen}t(\frac{9}{3\sqrt{\frac{(x+1)(a+b+c)}{3}}})$$

$$=\frac{3t}{\sqrt{(x+1)(\frac{k}{3})}}$$

$$\sum_{cyc}\frac{y\sqrt{a}}{\sqrt{c+xa}}\overbrace{\leq}^{Jensen}3y\sqrt{\frac{\frac{a}{c+xa}+\frac{b}{a+xb}+\frac{c}{b+xc}}{3}}$$

Let d satysfie $\frac{a}{c+xa}+\frac{b}{a+xb}+\frac{c}{b+xc}\leq\frac{1}{d}$.

$d[a(a+xb)(b+xc)+b(b+xc)(c+xa)+c(c+xa)(a+xb)]\leq(c+xa)(a+xb)(b+xc)$

After expanding :

$d(a(ab+axc+xb^2+x^2bc)+b(bc+xab+xc^2+x^2ac)+c(ac+xcb+xa^2+x^2ab))$

$$=d[(a^2b+b^2c+c^2a)+2x(a^2c+b^2a+c^2b)+3x^2abc]$$

$$= d(a^2b + b^2c + c^2a) + 2dx(a^2c + b^2a + c^2b) + 3dx^2(abc)$$

$$(c + xa)(a + xb)(b + xc) = (c + xa)(ab + xac + xb^2 + x^2bc)$$

$$= abc + xac^2 + xcb^2 + xba^2 + x^2bc^2 + x^2ca^2 + x^2ab^2 + x^3abc$$

$$= x(a^2b + b^2c + c^2a) + x^2(a^2c + b^2a + c^2b) + (x^3 + 1)(abc)$$

That implies

$$d(a^2b+b^2c+c^2a)+2dx(a^2c+b^2a+c^2b)+3dx^2(abc) \leq x(a^2b+b^2c+c^2a)+x^2(a^2c+b^2a+c^2b)+(x^3+1)(abc)$$

$$0 \leq (x-d)(a^2b+b^2c+c^2a)+(x^2-2dx)(a^2c+b^2a+c^2b)+(x^3-3dx^2+1)(abc)$$

$$(x-d)(\sum_{cyc} a^2b) + (x^2 - 2dx)(\sum_{cyc} a^2c) + (x^3 - 3dx^2 + 1)(abc)$$

$$\overbrace{\geq}^{AM-GM} (x-d)(3abc)+(x^2-2xd)(3abc)+(x^3-3x^2d+1)abc = abc(3x-3d+3x^2-6xd+x^3-3x^2d+1)$$

$$= abc(x+1)^2(x+1-3d) \geq 0$$

Because of $a, b, c, x \in \mathbf{R}^+$, $x + 1 - 3d$ should be greater than 0.

$$d \leq \frac{x+1}{3}$$

.

We have found d in a x way. Placing this in inequality holds. If we place this in inequality:

$$3y\sqrt{\frac{\frac{a}{c+xa} + \frac{b}{a+xb} + \frac{c}{b+xc}}{3}} \geq 3y\sqrt{\frac{\frac{1}{d}}{3}} \geq 3\sqrt{\frac{\frac{3}{x+1}}{3}} = y\frac{3}{\sqrt{x+1}}$$

.

Unit both inequalities:

$$\sum_{cyc} \frac{t - y\sqrt{a}}{\sqrt{c+xa}} \geq \frac{3t}{\sqrt{\frac{k(x+1)}{3}}} - y\frac{3}{\sqrt{x+1}} = \frac{3}{\sqrt{x+1}}(t\sqrt{\frac{3}{k}} - y)$$

.

Lemma 4.2.
$a, b, c, x, t \in \mathbf{R}^+$ and $a + b + c = k$. Additionally $x \geq 2$.Then prove that

$$\frac{t - \sqrt{a}}{\sqrt{c+xa}} + \frac{t - \sqrt{b}}{\sqrt{a+xb}} + \frac{t - \sqrt{c}}{\sqrt{b+xc}} \geq \frac{3}{\sqrt{x+1}}(t\sqrt{\frac{3}{k}} - 1)$$

Proof.
If we place $y = 1$ in **Lemma 4.1.**, we get the following:

$$\frac{t - \sqrt{a}}{\sqrt{c+xa}} + \frac{t - \sqrt{b}}{\sqrt{a+xb}} + \frac{t - \sqrt{c}}{\sqrt{b+xc}} \geq \frac{3}{\sqrt{x+1}}(t\sqrt{\frac{3}{k}} - y) = \frac{3}{\sqrt{x+1}}(t\sqrt{\frac{3}{k}} - 1)$$

Lemma 4.3.
$a, b, c \in \mathbf{R}^+$ and $a + b + c = 3$. Then prove that

$$\frac{2-\sqrt{a}}{\sqrt{c+3a}}+\frac{2-\sqrt{b}}{\sqrt{a+3b}}+\frac{2-\sqrt{c}}{\sqrt{b+3c}}\geq\frac{3}{2}$$

Proof.
Placing $y=1$, $t=2$, $k=3$ and $x=3$, we get the following:

$$\frac{2-\sqrt{a}}{\sqrt{c+3a}}+\frac{2-\sqrt{b}}{\sqrt{a+3b}}+\frac{2-\sqrt{c}}{\sqrt{b+3c}}\geq\frac{3}{\sqrt{x+1}}(t\sqrt{\frac{3}{k}}-1)=\frac{3}{2}$$

Lemma 5.1.
$a,b,c,x,p,s,k\in\mathbf{R}^+$, $k\geq\frac{1}{2}$ and $a+b+c=t$. Then prove that

$$\frac{a^{2k-1}+p}{xb+s}+\frac{b^{2k-1}+p}{xc+s}+\frac{c^{2k-1}+p}{xa+s}\geq\frac{9p+\frac{t^{2k-1}}{3^{2k-3}}}{xt+3s}$$

First Proof (Author).

$$\frac{a^{2k-1}+p}{xb+s}+\frac{b^{2k-1}+p}{xc+s}+\frac{c^{2k-1}+p}{xa+s}=\sum_{cyc}\frac{a^{2k-1}}{xb+s}+p\left(\sum_{cyc}\frac{1}{xb+s}\right)$$

$$=\sum_{cyc}\frac{a^{2k}}{xab+as}+p\left(\sum_{cyc}\frac{1}{xb+s}\right)\geq\frac{(a+b+c)^{2k}}{3^{2k-2}.\,(x(ab+bc+ca)+s(a+b+c))}+p\left(\frac{9}{x(a+b+c)+3s}\right)$$

$$\geq\frac{(a+b+c)^{2k}}{3^{2k-2}.\left(x(\frac{(a+b+c)^2}{3})+s(a+b+c)\right)}+p\left(\frac{9}{x(a+b+c)+3s}\right)\geq\frac{9p+\frac{(a+b+c)^{2k-1}}{3^{2k-3}}}{x(a+b+c)+3s}=\frac{9p+\frac{t^{2k-1}}{3^{2k-3}}}{xt+3s}$$

$$\frac{(a+b+c)^{2k}}{3^{2k-2}.\left(x(\frac{(a+b+c)^2}{3})+s(a+b+c)\right)}\geq\frac{\frac{(a+b+c)^{2k-1}}{3^{2k-3}}}{x(a+b+c)+3s}$$

$$\frac{a+b+c}{3(a+b+c)(\frac{x(a+b+c)}{3}+s)} \geq \frac{1}{x(a+b+c)+3s}$$

$$\frac{1}{3(x(\frac{a+b+c}{3})+s)} \geq \frac{1}{x(a+b+c)+3s}$$

$$\frac{1}{x(a+b+c)+3s} \geq \frac{1}{x(a+b+c)+3s}$$

Second Proof (Davros)

$$\sum_{cyc} \frac{p}{xa+s} \geq \frac{9p}{xt+3s}$$

We need to prove:

$$\sum_{cyc} \frac{a^{2k-1}}{xb+s} \geq \frac{t^{2k-1}}{3^{2k-3}(xt+3s)}$$

$$\sum_{cyc} \frac{a^{2k-1}}{xb+s} \overbrace{\geq}^{Rearrangement} \sum_{cyc} \frac{a^{2k-1}}{xa+s}$$

Let $f(a) = \dfrac{a^{2k-1}}{xa+s}$.

$$f''(a) = \frac{2a^{2k-1}(k(2k-1)x^2a^2+(2k-1)(2k+1)sxa+(2k^2+k)s^2)}{(xa+s)^3} \geq 0$$

That means function is strictly convex on $k \in [\frac{1}{2}, \infty)$

$$\sum_{cyc} \frac{a^{2k-1}}{xa+s} \geq \frac{t^{2k-1}}{3^{2k-3}(xt+3s)}$$

Lemma 5.2.
$a, b, c, x, p, s \in \mathbf{R}^+$ and $a+b+c=t$. Then prove that

$$\frac{a^7+2}{xb+s} + \frac{b^7+2}{xc+s} + \frac{c^7+2}{xa+s} \geq \frac{\frac{t^7}{3^5}+18}{xt+3s}$$

Proof.
Placing $k=4$, $p=2$ in **Lemma 5.1.**, we get the following:

$$\frac{a^7+2}{xb+s} + \frac{b^7+2}{xc+s} + \frac{c^7+2}{xa+s} \geq \frac{9p+\frac{t^{2k-1}}{3^{2k-3}}}{xt+3s} = \frac{\frac{t^7}{3^5}+18}{xt+3s}$$

Lemma 5.3.(Marius Stanean)
$a, b, c \in \mathbf{R}^+$ and $a+b+c=3$. Then prove that

$$\frac{a^3+5}{b+2} + \frac{b^3+5}{c+2} + \frac{c^3+5}{a+2} \geq 6$$

Proof.
If we place $k=2$, $x=1$, $t=3$, $s=2$ and $p=5$, we get the following:

$$\frac{a^3+5}{b+2} + \frac{b^3+5}{c+2} + \frac{c^3+5}{a+2} \geq \frac{a^{2k-1}+p}{xb+s} + \frac{b^{2k-1}+p}{xc+s} + \frac{c^{2k-1}+p}{xa+s} \geq \frac{9p+\frac{t^{2k-1}}{3^{2k-3}}}{xt+3s} = 6$$