A dominance intuitionistic fuzzy-rough set approach and its applications

Bing Huang a,b,⇑, Yu-liang Zhuang a,b, Hua-xiong Li c, Da-kuan Wei d

a School of Information Sciences, Nanjing Audit University, Nanjing 211815, PR China
b Information Systems Auditing Experimental Center, Nanjing Audit University, Nanjing 211815, PR China
c School of Engineering and Management, Nanjing University, Nanjing 210093, PR China
d School of Information Engineering, Hunan Science and Technology University, Yongzhou 425100, PR China

Article info

Article history:
Received 15 November 2011
Received in revised form 24 October 2012
Accepted 18 December 2012
Available online 27 December 2012

Keywords:
Intuitionistic fuzzy set
Rough set model
Dominance relation
Reduction
Rule extraction

Abstract

Although the rough set and intuitionistic fuzzy set both capture the same notion, imprecision, studies on the combination of these two theories are rare. Rule extraction is an important task in a type of decision systems where condition attributes are taken as intuitionistic fuzzy values and those of decision attribute are crisp ones. To address this issue, this paper makes a contribution of the following aspects. First, a ranking method is introduced to construct the neighborhood of every object that is determined by intuitionistic fuzzy values of condition attributes. Moreover, an original notion, dominance intuitionistic fuzzy decision tables (DIFDT), is proposed in this paper. Second, a lower/upper approximation set of an object and crisp classes that are confirmed by decision attributes is ascertained by comparing the relation between them. Third, making use of the discernibility matrix and discernibility function, a lower and upper approximation reduction and rule extraction algorithm is devised to acquire knowledge from existing dominance intuitionistic fuzzy decision tables. Finally, the presented model and algorithms are applied to audit risk judgment on information system security auditing risk judgement for CISA, candidate global supplier selection in a manufacturing company, and cars classification.

1. Introduction

It is known that rough set theory is introduced by Pawlak as an extension of the classical set theory [1,2]. The basic tools are relations which are the representatives of information systems or decision tables. In Pawlak’s rough set theory, the relation is an equivalence, whereas the relation is maybe a dominance, covering, similarity, tolerance, or other indiscernibility one within various generalized rough set models [3–7], in which the dominance rough set model and the fuzzy rough set theory are two types of the most important extended rough set models.

To consider the ranking properties of criteria, Greco et al. [8–11] proposed a dominance-based rough set approach (DRSA) based on the substitution of the indiscernibility relation with a dominance relation. In the DRSA, condition attributes are the criteria used and classes are ranked by preference; therefore, the knowledge approximated is a collection of upward and downward unions of classes, and the dominance classes are sets of objects defined in a dominance relation. In recent years, a number of studies on DRSA have been conducted [12–17].
Fuzzy rough set is also a generalization of classic rough sets that are used to manage real-valued decision tables. Dubois and Prade [18] investigated the fuzzification of rough sets. The concepts of rough fuzzy set and fuzzy rough sets were proposed, in which crisp binary relations are replaced with fuzzy relations in the universe of discourse. When these concepts are used as bases, a proposed fuzzy rough set theory compensates for the deficiencies of the traditional rough set theory in several aspects [19–24].

Another important mathematical tool that can address imperfect and/or imprecise information is the intuitionistic fuzzy (IF) set initiated by Atanassov [25,26]. The IF set is naturally considered as an extension of Zadeh’s fuzzy sets, and is defined by a pair of membership functions: a fuzzy set provides only the a degree to which an element belongs to a universe, whereas an IF set yields both a membership degree and a non-membership degree. The membership and non-membership values generate an indeterminacy index that models the hesitancy in deciding the degree to which an object satisfies a particular property. Recently, IF set theory has been successfully applied in decision analysis and pattern recognition [27–36]. Although rough sets and Atanassov’s intuitionistic fuzzy sets both capture particular facets of the same notion-imprecision, studies on the combination of intuitionistic fuzzy set theory and rough set theory are rare. In [37], Coker showed that a fuzzy rough set is in fact an intuitionistic L-fuzzy set, which appears to eliminate the likelihood of a new hybrid theory.

Rough set approximations have recently been introduced into intuitionistic fuzzy sets [38–41,23,33]. On the basis of fuzzy sets, and the fuzzy rough set is in fact an intuitionistic L-fuzzy set, which appears to eliminate the likelihood of a new hybrid theory. Rough set approximations have recently been introduced into intuitionistic fuzzy sets [38–41,23,33]. On the basis of fuzzy sets, and the fuzzy rough set is in fact an intuitionistic L-fuzzy set, which appears to eliminate the likelihood of a new hybrid theory. Rough set approximations have recently been introduced into intuitionistic fuzzy sets [38–41,23,33]. On the basis of fuzzy sets, and the fuzzy rough set is in fact an intuitionistic L-fuzzy set, which appears to eliminate the likelihood of a new hybrid theory. Rough set approximations have recently been introduced into intuitionistic fuzzy sets [38–41,23,33]. On the basis of fuzzy sets, and the fuzzy rough set is in fact an intuitionistic L-fuzzy set, which appears to eliminate the likelihood of a new hybrid theory. Rough set approximations have recently been introduced into intuitionistic fuzzy sets [38–41,23,33]. On the basis of fuzzy sets, and the fuzzy rough set is in fact an intuitionistic L-fuzzy set, which appears to eliminate the likelihood of a new hybrid theory. Rough set approximations have recently been introduced into intuitionistic fuzzy sets [38–41,23,33]. On the basis of fuzzy sets, and the fuzzy rough set is in fact an intuitionistic L-fuzzy set, which appears to eliminate the likelihood of a new hybrid theory.

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2. Preliminaries

In this section, we introduce intuitionistic fuzzy values and basic operations, recall some of their properties, and propose some related notations of the intuitionistic fuzzy-valued decision table.

Definition 2.1. Let \((\mu, \gamma)\) be an order pair, where \(0 \leq \mu, \gamma \leq 1\) and \(0 \leq \mu + \gamma \leq 1\). Then we call \((\mu, \gamma)\) an intuitionistic fuzzy value.

Definition 2.2 ([25,26]). Let \(a_i = (\mu_i, \gamma_i)\) (1 ≤ i ≤ 2) be two intuitionistic fuzzy values, then

1. \(a_1 = a_2 \iff \mu_1 = \mu_2 \land \gamma_1 = \gamma_2\);
2. \(a_1 \sqcap a_2 = < \min\{\mu_1, \mu_2\}, \max\{\gamma_1, \gamma_2\}>\);
3. \(a_1 \sqcup a_2 = < \max\{\mu_1, \mu_2\}, \min\{\gamma_1, \gamma_2\}>\).

If \(a_1\) and \(a_2\) are two intuitionistic fuzzy values, then so are \(a_1 \sqcap a_2\) and \(a_1 \sqcup a_2\). Given intersection and union of two intuitionistic fuzzy values, generalizing the arbitrary finite intuitionistic fuzzy values as follows is easily obtained.
Definition 2.3. Let $\mathbf{x}_i = (\mu_i, \gamma_i)$ ($1 \leq i \leq n$) be $n$ intuitionistic fuzzy values. Their intersection, denoted by $\bigcap_{i \in \mathcal{C}} \mathbf{x}_i$, and their union, denoted by $\bigcup_{i \in \mathcal{C}} \mathbf{x}_i$, are defined as follows:

\[
\bigcap_{i \in \mathcal{C}} \mathbf{x}_i = (\min_{i \in \mathcal{C}} \mu_i, \max_{i \in \mathcal{C}} \gamma_i),
\]

\[
\bigcup_{i \in \mathcal{C}} \mathbf{x}_i = (\max_{i \in \mathcal{C}} \mu_i, \min_{i \in \mathcal{C}} \gamma_i).
\]

Definition 2.4. An intuitionistic fuzzy-valued decision table (IFDT) is a quadruple $S = (U, \mathcal{C} \cup D, V, f)$, where $U$ is a non-empty and finite set of objects called the universe, $\mathcal{C}$ is a non-empty and finite set of conditional attributes, $D = \{d\}$ denotes a singleton of decision attribute $d$, and $\mathcal{C} \cap D = \emptyset$. $V = V_1 \cup V_2$, where $V_1$ and $V_2$ are domains of condition and decision attributes, respectively. Information function $f$ is a map from $U \times \mathcal{A}$ onto $V$, such that $f(x, c) \in V_c$ for all $c \in \mathcal{C}, V_c \subseteq V_1$, and $f(x, d) \in V_2$ for $D = \{d\}$, where $f(x, c)$ is an intuitionistic fuzzy values denoted by $f(x, c) = (\mu_c(x), \gamma_c(x))$, and $f(x, d)$ is a crisp value.

We call $f(x, c)$ the intuitionistic fuzzy value of object $x$, under the condition attribute $c$ and $f(x, d)$ the intuitionistic fuzzy value of $x$ under the decision attribute $d$. In particular, $f(x, c)$ would degenerate into fuzzy value if $\mu_c(x) = 1 - \gamma_c(x)$ for every $x \in U$. Under this consideration, we regard a fuzzy decision table as a special form of intuitionistic fuzzy-valued decision tables. In practical decision-making analysis, we always consider only a binary dominance relation between objects that are possibly dominant in terms of values of an attributes set in an intuitionistic fuzzy-valued decision table. In general, the preference for an increasing or decreasing rank is determined by a decision maker. If the domain of an attribute is arranged according to a decreasing or increasing order, then the attribute is a criterion. Without any loss of generality, we now consider only the increasing rank in dominance-based intuitionistic fuzzy-valued decision tables.

Definition 2.5. We call an intuitionistic fuzzy-valued decision table a DIFDT if all the condition attributes and the decision attribute are criteria.

Example 2.1. A DIFDT is presented in Table 1, where, $U = \{x_1, x_2, \ldots, x_{10}\}$, $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_5\}$.

In a DIFDT, key tasks are to compare and rank objects in terms of a condition attribute subset where every attribute takes intuitionistic fuzzy values. In the following, we introduce a kind of ranking method for two intuitionistic fuzzy values as follows:

Definition 2.6 [34]. Let $\mathbf{x}_1 = (\mu_1, \gamma_1)$ and $\mathbf{x}_2 = (\mu_2, \gamma_2)$ be intuitionistic fuzzy values, $s(\mathbf{x}_1) = \mu_1 - \gamma_1$ and $s(\mathbf{x}_2) = \mu_2 - \gamma_2$ the scores of $\mathbf{x}_1$ and $\mathbf{x}_2$. Let $h(\mathbf{x}_1) = \mu_1 + \gamma_1$ and $h(\mathbf{x}_2) = \mu_2 + \gamma_2$ be the accuracy of $\mathbf{x}_1$ and $\mathbf{x}_2$. We can rank $\mathbf{x}_1$ and $\mathbf{x}_2$ with the following criteria:

1. If $s(\mathbf{x}_1) < s(\mathbf{x}_2)$, then $\mathbf{x}_1 < \mathbf{x}_2$;
2. If $s(\mathbf{x}_1) = s(\mathbf{x}_2)$ and
   a. $h(\mathbf{x}_1) = h(\mathbf{x}_2)$, then $\mathbf{x}_1 = \mathbf{x}_2$;
   b. $h(\mathbf{x}_1) < h(\mathbf{x}_2)$, then $\mathbf{x}_1 < \mathbf{x}_2$;
   c. $h(\mathbf{x}_1) > h(\mathbf{x}_2)$, then $\mathbf{x}_1 > \mathbf{x}_2$.

For two intuitionistic fuzzy values, $\mathbf{x}_1 = (0.3, 0.5)$ and $\mathbf{x}_2 = (0.4, 0.6)$, there are $s(\mathbf{x}_1) = s(\mathbf{x}_2) = -0.2$ and $h(\mathbf{x}_1) = 0.8/h(\mathbf{x}_2) = 1.0$; thus, $\mathbf{x}_1 < \mathbf{x}_2$. On the basis of Definition 2.6, we can design a method for ranking two objects whose attribute characters are described by means of intuitionistic fuzzy values.

### Table 1

A DIFDT.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$c_5$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>(0.4, 0.5)</td>
<td>(0.3, 0.5)</td>
<td>(0.8, 0.2)</td>
<td>(0.4, 0.5)</td>
<td>(0.7, 0.1)</td>
<td>2</td>
</tr>
<tr>
<td>$x_2$</td>
<td>(0.3, 0.5)</td>
<td>(0.4, 0.5)</td>
<td>(0.6, 0.1)</td>
<td>(0.4, 0.5)</td>
<td>(0.7, 0.3)</td>
<td>2</td>
</tr>
<tr>
<td>$x_3$</td>
<td>(0.3, 0.5)</td>
<td>(0.4, 0.5)</td>
<td>(0.8, 0.1)</td>
<td>(0.4, 0.5)</td>
<td>(0.7, 0.3)</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>(0.1, 0.8)</td>
<td>(0.1, 0.8)</td>
<td>(0.4, 0.5)</td>
<td>(0.1, 0.8)</td>
<td>(0.8, 0.2)</td>
<td>1</td>
</tr>
<tr>
<td>$x_5$</td>
<td>(0.7, 0.3)</td>
<td>(0.4, 0.5)</td>
<td>(0.9, 0.1)</td>
<td>(0.5, 0.5)</td>
<td>(0.8, 0.1)</td>
<td>2</td>
</tr>
<tr>
<td>$x_6$</td>
<td>(0.3, 0.6)</td>
<td>(0.4, 0.6)</td>
<td>(0.7, 0.2)</td>
<td>(0.5, 0.5)</td>
<td>(0.8, 0.2)</td>
<td>3</td>
</tr>
<tr>
<td>$x_7$</td>
<td>(0.4, 0.5)</td>
<td>(0.4, 0.5)</td>
<td>(0.8, 0.2)</td>
<td>(0.4, 0.5)</td>
<td>(0.8, 0.2)</td>
<td>4</td>
</tr>
<tr>
<td>$x_8$</td>
<td>(0.4, 0.6)</td>
<td>(0.4, 0.5)</td>
<td>(0.9, 0.1)</td>
<td>(0.7, 0.3)</td>
<td>(0.8, 0.2)</td>
<td>4</td>
</tr>
<tr>
<td>$x_9$</td>
<td>(0.4, 0.6)</td>
<td>(0.7, 0.3)</td>
<td>(0.9, 0.1)</td>
<td>(0.4, 0.5)</td>
<td>(0.9, 0.0)</td>
<td>5</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>(0.7, 0.3)</td>
<td>(0.7, 0.3)</td>
<td>(0.8, 0.2)</td>
<td>(0.9, 0.0)</td>
<td>(0.4, 0.5)</td>
<td>5</td>
</tr>
</tbody>
</table>
Definition 2.7. Let \( D \subseteq (U, C \cup D, V, f) \) and \( B \subseteq C \), for \( x, y \in U \), denoted by
\[
x \preceq_B y \iff f(x, b) \preceq f(y, b) \iff f(x, b) < f(y, b) \vee f(y, b) = f(x, b), \forall b \in B.
\]
Obviously, \( \preceq_B \) is a binary relation in \( U \), that is
\[
\preceq_B = \{(x, y) \in U \times U | f(x, b) \preceq f(y, b), \forall b \in B\}.
\]
The binary relation defined in Definition 2.7 is called a dominance-based relation in \( D \).

Property 2.1. Let \( D \subseteq (U, C \cup D, V, f) \) and \( E \subseteq B \subseteq C \), then

1. \( \preceq_B \) is reflexive, transitive and non-symmetric;
2. \( \preceq_B = \bigcap (\preceq \subseteq) \);
3. \( \preceq_B \subseteq \bowtie \).

In terms of \( B \subseteq C \), the dominance-based class induced by the dominance-based relation, \( \preceq_B \) is the set of objects dominating \( x \); i.e., \( [x]_B^\preceq = \{ y \in U | (x, y) \in \preceq_B \} \), where \( [x]_B^\preceq \) describes the set of objects that may dominates \( x \) in terms of \( B \subseteq C \) in a \( D \). This set is called the \( B \)-dominating set with respect to \( x \in U \). Meanwhile, the \( B \)-dominated set with respect to \( x \in U \) can be denoted by \( [x]_B^\preceq = \{ y \in U | (y, x) \in \preceq_B \} \).

Example 2.2 (Continued from Example 2.1). Compute the dominating and dominated classes induced by \( \preceq_C \) in Table 1. One can get all the dominating and dominated classes in Table 2 as follows.

3. Rough set approach of \( D \)

In this section, we investigate set approximation and attribute reduction with respect to the dominance-based relation in \( D \).

Definition 3.1. Let \( D \subseteq (U, C \cup \{d\}, V, f) \) and \( B \subseteq C \). The universe, \( U \), is partitioned into \( m \) equivalence classes by the decision attribute, \( d \). That is, \( U/\{d\} = \{U_1, U_2, \ldots, U_m\} \), where \( U_1 < U_2 < \ldots < U_m \), and \( U_i < U_j \) denotes that for \( x \in U_i, y \in U_j \) implies \( f(x, d) < f(y, d) \). Denote \( U_k^\preceq = \bigcup_{i, j \leq k} U_i \), and let:

\[
\text{App}_B(U_k^\preceq) = \{ x | [x]_B^\preceq \subseteq U_k^\preceq \} (1 \leq k \leq m),
\]
\[
\text{App}_B(U_k^\preceq) = \{ x | x \subseteq U_k^\preceq \neq \phi \} = \bigcup_{x \in U_k^\preceq} [x]_B^\preceq (1 \leq k \leq m),
\]
\[
\text{BND}_B(U_k^\preceq) = \text{App}_B(U_k^\preceq) - \text{App}_B(U_k^\preceq),
\]
\[
\text{App}_B(U_k^\preceq) = \{ x | [x]_B^\preceq \subseteq U_k^\preceq \} (1 \leq k \leq m),
\]
\[
\text{App}_B(U_k^\preceq) = \{ x | x \subseteq U_k^\preceq \neq \phi \} = \bigcup_{x \in U_k^\preceq} [x]_B^\preceq (1 \leq k \leq m),
\]
\[
\text{BND}_B(U_k^\preceq) = \text{App}_B(U_k^\preceq) - \text{App}_B(U_k^\preceq).
\]

<table>
<thead>
<tr>
<th>( U )</th>
<th>( [x]_B^\preceq )</th>
<th>( [x]_B^\preceq )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>{ }</td>
<td>{x_1, x_3, x_7}</td>
</tr>
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<td>{x_2}</td>
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<td>{x_4, x_6}</td>
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<td>( x_{10} )</td>
<td>{x_{10}}</td>
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</tbody>
</table>
$$\text{App}_B^g(x) = \bigcap_{x \in U_i} U_i$$,

$$\overline{\text{App}}_B^g(x) = \bigcap_{x \in U_i} U_i$$,

$$BND_B^g(x) = \overline{\text{App}}_B^g(x) - \text{App}_B^g(x)$$,

$$\text{App}_B^g(x) = \bigcap_{x \in U_i} U_i$$,

$$\overline{\text{App}}_B^g(x) = \bigcap_{x \in U_i} U_i$$,

$$BND_B^g(x) = \overline{\text{App}}_B^g(x) - \text{App}_B^g(x)$$.

Then, $\text{App}_B(U_i), \overline{\text{App}}_B(U_i)$ and $BND_B(U_i)$ are the lower approximation, upper approximation and boundary of the dominated class, $U_i$ with respect to condition attribute subset $B$; $\overline{\text{App}}_B(U_i), \overline{\text{App}}_B(U_i)$ and $BND_B(U_i)$ are the lower approximation, upper approximation and boundary of the dominating class, $U_i$ with respect to $B$; In terms of $B$, $\text{App}_B^g(x), \overline{\text{App}}_B^g(x)$ and $BND_B^g(x)$ are the $\geq$ lower approximation, upper approximation and boundary of $x$. $\text{App}_B^g(x), \overline{\text{App}}_B^g(x)$ and $BND_B^g(x)$ are the $\leq$ lower approximation, upper approximation and boundary of $x$. We call Definition 3.1 a dominance-based intuitionistic fuzzy-rough set model.

**Property 3.1.** Let $DIFDT= (U, C \cup \{d\}, V, f), B_1 \subseteq B_2 \subseteq C$, and $x \in U$; thus,

1. $\text{App}_B(U_i) \subseteq \text{App}_B(U_i), \text{App}_B(U_i) \subseteq \text{App}_B(U_i), \text{App}_B(U_i) \subseteq \text{App}_B(U_i)$,
2. $\text{App}_B^g(x) \subseteq \text{App}_B^g(x), \text{App}_B^g(x) \subseteq \text{App}_B^g(x), \text{App}_B^g(x) \subseteq \text{App}_B^g(x)$,
3. $\text{App}_B(U_i) \subseteq \text{App}_B(U_i), \text{App}_B(U_i) \subseteq \text{App}_B(U_i)$.

**Property 3.2.** Let $DIFDT= (U, C \cup \{d\}, V, f)$ and $B_1 \subseteq B_2 \subseteq C$; then,

1. $\text{App}_B(U_i) = \{x \in U|\text{App}_B^g(x) = U_i, t > k\}$,
2. $\text{App}_B(U_i) = \{x \in U|\text{App}_B^g(x) = U_i, t < k\}$,
3. $\text{App}_B(U_i) = \{x \in U|\overline{\text{App}}_B^g(x) = U_i, t \leq k\}$.

**Proof.**

1. $x \in \text{App}_B(U_i) \iff |x|_B \subseteq U_i \iff |x|_B \cap U_i = \phi, t_1 < k \iff \text{App}_B(x) = \bigcap_{|x|_B \subseteq U_i} U_i = U_i, t \geq k$;
2. $x \in \overline{\text{App}}_B(U_i) \iff |x|_B \subseteq U_i \iff |x|_B \cap U_i = \phi, t_2 > k \iff \text{App}_B(x) = \bigcap_{|x|_B \subseteq U_i} U_i = U_i, t \leq k$;
3. $x \in \overline{\text{App}}_B(U_i) \iff |x|_B \cap U_i = \phi \iff \text{App}_B(x) = \bigcap_{|x|_B \subseteq U_i} U_i = U_i, t \leq k$.

Consider an object $y \in |x|_B^g$; then $\forall b \in B, f(y, b) \leq f(x, b)$ and $|y|_B \subseteq |x|_B^g$. Because we can denote the lower approximation of $x$ as $\text{App}_B(x) = \bigcap_{|y|_B \subseteq U_i} U_i = U_i$, then the upper bound of the decision value of $y$ should be $f(z_1, d), z_1 \in U_{k_1}$, i.e., $f(y, d) \leq f(z_1, d), z_1 \in U_{k_1}$; We can also denote the upper approximation of $x$ as $\overline{\text{App}}_B(x) = \bigcap_{|y|_B \subseteq U_i} U_i = U_i$, then the lower bound of the decision value of $y$ should be $f(z_2, d), z_2 \in U_{k_2}$, i.e., $f(y, d) \geq f(z_2, d), z_2 \in U_{k_2}$.

A special subset of the condition attribute set in DIFDT preserves the lower or upper bound rules of all the objects in the universe. To derive this subset, we provide the necessary definitions of attribute reduction in DIFDT as follows.
Definition 3.2. Let DIFDT = (U, C ∪ {d}, V, f) and B ⊆ C.

1) B is considered a ≤ (≥) consistent lower approximation set of DIFDT if ∀x ∈ U, \( \text{App}_{\leq}(x) = \text{App}_{\leq}(x) \). \( \text{App}_{\geq}(x) = \text{App}_{\geq}(x) \).

If B is a ≤ (≥) consistent lower approximation set of DIFDT and no proper subset of B is a ≤ (≥) consistent lower approximation set of DIFDT, then B is considered a ≤ (≥) lower approximation reduce of DIFDT.

2) B is considered a ≤ (≥) consistent upper approximation set of DIFDT if ∀x ∈ U, \( \text{App}_{\leq}(x) = \text{App}_{\leq}(x) \). \( \text{App}_{\geq}(x) = \text{App}_{\geq}(x) \).

If B is a ≤ (≥) consistent upper approximation set of DIFDT and no proper subset of B is a ≤ (≥) consistent upper approximation set of DIFDT, then B is considered a ≤ (≥) upper approximation reduce of DIFDT.

Consider a ≤ consistent lower approximation set B. If \( y \in \{x\}_{B} \), i.e., \( \forall b \in B, f(y, b) \leq f(x, b) \), then \( f(y, d) \leq f(z_1, d), z_1 \in U_k, \text{App}_{\leq}(x) = \text{App}_{\leq}(x) = U_k \). In other words, B is a ≤ consistent lower approximation set, then

\[ \forall b \in B, f(y, b) \leq f(x, b) \Rightarrow f(y, d) \leq f(z_1, d), z_1 \in U_k, \text{App}_{\leq}(x) = U_k \].

Similarly, B is a ≥ consistent upper approximation set, then

\[ \forall b \in B, f(y, b) \geq f(x, b) \Rightarrow f(y, d) \geq f(z_2, d), z_2 \in U_k, \text{App}_{\leq}(x) = U_k \].

Hence, we can conclude that a ≤ consistent lower (upper) approximation set is a subset of condition attribute set that preserves the ≤ lower (upper) approximations of all objects. The ≤ lower (upper) bounds of all decision rules derived from the ≤ consistent lower (upper) approximation set are completely consistent with the ones derived from C; i.e., if two decision rules derived respectively from the reduced and the original systems are supported by the same object, their ≤ lower (upper) bounds must be identical. Analogously, a ≥ lower (upper) consistent set is a subset of condition attribute set that preserves all ≥ upper (lower) bounds of all decision rules.

If B is a ≤ consistent lower approximation set, then

\[ \forall b \in B, f(y, b) \leq f(x, b) \Rightarrow f(y, d) \leq f(z, d), z \in U_{k_i}, \text{App}_{\leq}(x) = U_{k_i} \].

If B is a ≥ upper approximation consistent set, then

\[ \forall b \in B, f(y, b) \geq f(x, b) \Rightarrow f(y, d) \geq f(z, d), z \in U_{k_i}, \text{App}_{\geq}(x) = U_{k_i} \].

For B ⊆ C, if we simplify \( f(z, d), z \in U_{k_i}, \text{App}_{\leq}(x) = U_{k_i} \) as \( \text{App}_{\leq}(x) = f(z, d) \) (referred as to ≤ lower approximation value of x in terms of B), \( f(z, d), z \in U_{k_i}, \text{App}_{\leq}(x) = U_{k_i} \) as \( \text{App}_{\leq}(x) = f(z, d) \) (referred as to ≤ upper approximation value of x in terms of B) and \( f(y, d) \leq f(z, d), z \in U_{k_i}, \text{App}_{\leq}(x) = U_{k_i} \) as \( \text{App}_{\leq}(x) = f(z, d) \) (referred as to ≤ lower approximation value of x in terms of B), respectively, then in a straightforward method, the ≤ (≥) consistent lower (upper) approximation set, B, preserves the following ≤ (≥) upper and lower bounds of all decision rules supported by x ∈ U, respectively:

\[ \forall y \in U, \forall b \in B, f(y, b) \leq f(x, b) \Rightarrow f(y, d) \leq \text{App}_{\leq}(x) \],
\[ \forall y \in U, \forall b \in B, f(y, b) \geq f(x, b) \Rightarrow \text{App}_{\leq}(x) \leq f(y, d) \],
\[ \forall y \in U, \forall b \in B, f(y, b) \leq f(x, b) \Rightarrow \text{App}_{\leq}(x) \geq f(y, d) \],
\[ \forall y \in U, \forall b \in B, f(y, b) \geq f(x, b) \Rightarrow \text{App}_{\leq}(x) \geq f(y, d) \].

Example 3.1 (Continued from Example 2.2). Let U/{d} = \{U_1, U_2, U_3, U_4, U_5\}, where \( U_1 = \{x_3, x_4\}, U_2 = \{x_1, x_2, x_5\}, U_3 =\{x_6\}, U_4 =\{x_7, x_8\}, U_5 =\{x_9, x_{10}\} \). Then, \( U_1^1 = U_1, U_2^1 = U_1 \cup U_3, U_3^1 = U_1 \cup U_2 \cup U_3, U_1^1 = U_1 \cup U_2 \cup U_3 \cup U_4, U_3^1 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \). One can get all the ≤ and ≥ low/upper approximations with respect to C in Table 3 as follows.

<table>
<thead>
<tr>
<th>( \text{App}_{\leq}(x) )</th>
<th>( \text{App}_{\geq}(x) )</th>
<th>( \text{App}_{\leq}(x) )</th>
<th>( \text{App}_{\geq}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( U_1^1 )</td>
<td>( U_2^1 )</td>
<td>( U_3^1 )</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( U_1^1 )</td>
<td>( U_2^1 )</td>
<td>( U_3^1 )</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>( U_2^1 )</td>
<td>( U_3^1 )</td>
<td>( U_4^1 )</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>( U_3^1 )</td>
<td>( U_4^1 )</td>
<td>( U_5^1 )</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>( U_3^1 )</td>
<td>( U_4^1 )</td>
<td>( U_5^1 )</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>( U_3^1 )</td>
<td>( U_4^1 )</td>
<td>( U_5^1 )</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>( U_3^1 )</td>
<td>( U_4^1 )</td>
<td>( U_5^1 )</td>
</tr>
<tr>
<td>( x_8 )</td>
<td>( U_3^1 )</td>
<td>( U_4^1 )</td>
<td>( U_5^1 )</td>
</tr>
<tr>
<td>( x_9 )</td>
<td>( U_3^1 )</td>
<td>( U_4^1 )</td>
<td>( U_5^1 )</td>
</tr>
<tr>
<td>( x_{10} )</td>
<td>( U_3^1 )</td>
<td>( U_4^1 )</td>
<td>( U_5^1 )</td>
</tr>
</tbody>
</table>
4. Approximation reduction approach of DIFDT

Susmaga et al. [48] introduced a discernibility matrix to dominance-based decision tables and addressed the computation of dominance-based reducts using the dominance information table in DRSA. In this section, for simplicity, we only investigate an attribute reduction approach for the ≤ and ≥ lower approximations in DIFDT. This approach can be easily generalized to the ≤ and ≥ upper approximations in DIFDT.

**Theorem 4.1.** Let DIFDT = (U, C ∪ {d}, V, f) and B ⊆ C. Then,

1. B is a ≤ consistent lower approximation set if and only if f(y, d) > [App≤(x)] implies y ∈ [x] for x, y ∈ U.
2. B is a ≥ consistent upper approximation set if and only if f(y, d) < [App≥(x)] implies y ∈ [x] for x, y ∈ U.

**Proof.**

(1) ⇒ If there are x, y ∈ U, such that f(y, d) > [App≤(x)] and y ∈ [x], we obtain [App≤(x)] ≥ f(y, d) > [App≤(x)]; i.e., App≤(x) ≠ App≤(x). Therefore, B is not a ≤ consistent lower approximation set, which contradicts the assumption.

(2) This proof is similar to that in (1). □

**Theorem 4.1** provides an approach to verifying whether a subset of condition attributes is consistent. Unfortunately, we cannot directly use this theorem to determine all the ≤ or ≥ lower approximation reducts. The discernibility matrix is key to various reduction algorithms in rough set theory. Thus, we define additional practical approaches to determining all the ≤ and ≥ lower by constructing discernibility matrices. We provide the definition of discernibility matrices as follows.

**Definition 4.1.** Let DIFDT = (U, C ∪ {d}, V, f) and U = {x1, x2, ..., xn}. We define the ≤ and ≥ lower approximation discernibility matrices of DIFDT as M≤ = (d) and M≥ = (d), where

\[
    d = \begin{cases} 
    \{c ∈ C | f(x, c) > f(x, c)\} & \text{if } f(x, d) > [App≤(x)], \\
    \{c ∈ C | f(x, c) > f(x, c)\} & \text{otherwise,}
    \end{cases}
\]

and

\[
    d = \begin{cases} 
    \{c ∈ C | f(x, c) < f(x, c)\} & \text{if } f(x, d) < [App≤(x)], \\
    \{c ∈ C | f(x, c) < f(x, c)\} & \text{otherwise,}
    \end{cases}
\]

respectively.

**Theorem 4.2.** Let DIFDT = (U, C ∪ {d}, V, f), U = {x1, x2, ..., xn} and B ⊆ C. Then,

1. B is a ≤ consistent lower approximation set if and only if B ∩ d = ∅ for all d (1 ≤ i, j ≤ n).
2. B is a ≥ consistent lower approximation set if and only if B ∩ d = ∅ for all d (1 ≤ i, j ≤ n).

**Proof.** According to **Definition 3.2** and **Theorem 4.1**, these two conclusions are straightforward. □

Now, we use discernibility matrices to acquire all the ≤ and ≥ lower approximation reducts of DIFDT.

**Theorem 4.3.** Let DIFDT = (U, C ∪ {d}, V, f). Transform the minimal disjunction form of the ≤ and ≥ lower approximation discernibility formulas, D≤ = ∨ (d) and D≥ = ∨ (d), to disjunction forms D≤ = ∨(c) and D≥ = ∨(c), respectively. We set B≤ = {c : v = 1, ..., q} and B≥ = {c : l = 1, ..., t}. Then, B≤(r = 1, ..., s) and B≥(r = 1, ..., t) are the ≤ and ≥ lower approximation reducts, respectively.

**Proof.** This is a direct result of **Theorem 4.2** and the definition of the minimal disjunction form. □

Decision rules commonly act as knowledge that aid the decision making. Therefore, rule acquisition is more important than reducts under certain circumstances. We present the methods of deriving ≤ upper and ≥ lower bound rules based on discernibility matrices.
Definition 4.2. Let $\text{DIFDT} = (U, C \cup \{d\}, V, f)$ and $U = \{x_1, x_2, \ldots, x_n\}$. The $\preceq$ lower and upper approximation discernibility matrices of $x_i$ are $M^\preceq = (d\overset{\preceq}{\downarrow}_{ij})_{n \times n}$ and $M^\succeq = (d\overset{\succeq}{\uparrow}_{ij})_{n \times n}$, respectively. We define the $\preceq$ and $\succeq$ lower discernibility functions of $x_i$ as $f^\preceq(x_i)$ and $f^\succeq(x_i)$, where $f^\preceq(x_i) = \bigvee\{d\overset{\preceq}{\downarrow}_{ij}\}$ and $f^\succeq(x_i) = \bigvee\{d\overset{\succeq}{\uparrow}_{ij}\}$.

Theorem 4.4. Let $\text{DIFDT} = (U, C \cup \{d\}, V, f)$. We transform the minimal disjunction form of the $\preceq$ and $\succeq$ lower approximation discernibility formulas of $x_i$, $f^\preceq(x_i) = \bigvee\{d\overset{\preceq}{\downarrow}_{ij}\}$ and $f^\succeq(x_i) = \bigvee\{d\overset{\succeq}{\uparrow}_{ij}\}$, into disjunction forms $f^\preceq(x_i) = \bigvee_{i=1}^{n-1} \{C_{ij} : r = 1, \ldots, u\}$ and $f^\succeq(x_i) = \bigvee_{j=1}^{n-1} \{C_{ij} : \ell = 1, \ldots, w\}$, respectively. Let $B^\preceq(x_i) = \{C_{ij} : l = 1, \ldots, p_j\}$ ($r = 1, \ldots, u$) and $B^\succeq(x_i) = \{C_{ij} : q = 1, \ldots, q_j\}$ ($\ell = 1, \ldots, w$). Then, $B^\preceq(x_i)$ ($r = 1, \ldots, u$) and $B^\succeq(x_i)$ ($\ell = 1, \ldots, w$) are the $\preceq$ and $\succeq$ lower discernibility functions.

5. Comparison of the dominance-based intuitionistic fuzzy-rough set model in DIFDT with the other rough set models

A consistent or inconsistent comparison with related work would be important to emphasize the rationality of the presented rough set model. In this section, we will establish the relationships between the dominance-based intuitionistic fuzzy-rough set model in DIFDT with the other rough set models.

1. Comparison between the presented rough set model in DIFDT and Pawlak’s rough set model [3].

2. Comparison between the presented dominance-based intuitionistic fuzzy-rough set model in DIFDT and DRSA [8–10].

3. Comparison between DIFRSA and S-DRSA [14].

4. Comparison between DIFRSA and fuzzy rough set model in interval-valued fuzzy information systems [48].

5. Comparison between DIFRSA and dominance-based rough set model in intuitionistic fuzzy information systems [46].

6. Application examples

As a useful tool that manages imperfect data and information, as well as imprecise knowledge, the intuitionistic fuzzy approach has been successfully applied to perform multi-criteria decision-making, group decision, and grey relational analysis [59,39,36]. In a similar manner, the proposed DIFRSA can also be used in these domains. To demonstrate its potential, we present three applications: (i) information systems security audit risk judgement for certified information systems auditors.
(CISA), (ii) candidate global supplier selection in a manufacturing company, and (iii) cars classification. In these applications, we use our approach to extract the simplified lower and upper bound intuitionistic fuzzy rules.

Example 6.1. Information systems security audit risk judgement for CISA.

The result of audit designation is significantly influenced by the audit evidence collected when planning the information systems security audit and the amount of audit evidence that depends on the degree of audit risk judgement. Therefore, the more objective and accurate the audit risk judgement rules, the lower the audit costs and risk of the information systems security audit and the amount of audit evidence that depends on the degree of audit risk judgement. Therefore, the information systems security audit risk assessment rules can be drawn using the proposed approach on the basis of the theory of DIFDT proposed in Sections 3 and 4. The information systems security audit risk judgement data we use here comes from the practical research of the first author’s colleagues, as shown in Table 4.

Table 4 shows a information systems security audit risk judgement decision table, which is a DIFDT. In this table, object set $\mathcal{U} = \{x_1, x_2, \ldots, x_{10}\}$ includes 10 audited objects. The condition attribute set $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_5\}$ has five condition attributes, where $c_1$ = “Better Systems Total Security,” $c_2$ = “Better Systems Operation Security,” $c_3$ = “Safer Data Center,” $c_4$ = “Credible Hardware Device,” and $c_5$ = “Credible Network Security”. Every value which condition attribute is taken on has special actual meaning. For example, $f(x_1, c_1) = (0.2, 0.4)$ means that the membership degree of systems total security is 0.2, and non-membership degree of systems total security is 0.4. The decision attribute set, $\mathcal{D}$ = “Risk Judgement Order of Information Systems Security Audit”. The domain of $\mathcal{D}$ is $\{1, 2, 3\}$, where 1 means “Complete Examination,” 2 means “Major Examination,” and 3 means “No Examination”. By using the definitions in the Sections 3 and 4, we can obtain all the lower approximations presented in Table 5.

The $\leq$ and $\geq$ lower approximation discernibility matrix are

$$M^\leq = \begin{pmatrix}
C & c_4 & c_5 & C & C & c_2 & C & C & c_1 & c_2 & c_3 & c_5 & C & C \\
C & c_1 & c_2 & c_3 & C & c_1 & c_2 & c_3 & C & c_1 & c_2 & c_3 & C \\
C & c_2 & c_1 & c_5 & C & c_2 & c_1 & c_5 & C & c_2 & c_1 & c_5 & C \\
C & c_1 & c_2 & c_3 & C & c_1 & c_2 & c_3 & C & c_1 & c_2 & c_3 & C \\
C & c_1 & c_2 & c_3 & c_5 & C & c_1 & c_2 & c_3 & c_5 & C & c_1 & c_2 & c_3 & C \\
\end{pmatrix}$$

and

$$M^\geq = \begin{pmatrix}
C & c_1 & c_2 & C & c_1 & c_2 & c_1 & C & c_1 & C & c_1 & C & C & C \\
C & c_1 & c_2 & c_3 & c_5 & C & c_1 & c_2 & c_3 & c_5 & c_1 & c_2 & c_3 & c_5 & C \\
C & c_1 & c_2 & c_3 & c_5 & C & c_1 & c_2 & c_3 & c_5 & C & c_1 & c_2 & c_3 & C \\
\end{pmatrix}$$

respectively.

Table 4
An information systems security audit risk judgement decision table.

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$c_5$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>(0.2, 0.4)</td>
<td>(0.1, 0.7)</td>
<td>(0.2, 0.6)</td>
<td>(0.6, 0.4)</td>
<td>(0.2, 0.8)</td>
<td>1</td>
</tr>
<tr>
<td>$x_2$</td>
<td>(0.1, 0.7)</td>
<td>(0.1, 0.8)</td>
<td>(0.3, 0.6)</td>
<td>(0.5, 0.2)</td>
<td>(0.2, 0.7)</td>
<td>2</td>
</tr>
<tr>
<td>$x_3$</td>
<td>(0.1, 0.8)</td>
<td>(0.1, 0.8)</td>
<td>(0.2, 0.8)</td>
<td>(0.5, 0.4)</td>
<td>(0.6, 0.4)</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>(0.1, 0.9)</td>
<td>(0.6, 0.3)</td>
<td>(0.2, 0.7)</td>
<td>(0.2, 0.8)</td>
<td>(0.6, 0.4)</td>
<td>1</td>
</tr>
<tr>
<td>$x_5$</td>
<td>(0.4, 0.6)</td>
<td>(0.2, 0.6)</td>
<td>(0.2, 0.8)</td>
<td>(0.2, 0.8)</td>
<td>(0.2, 0.8)</td>
<td>2</td>
</tr>
<tr>
<td>$x_6$</td>
<td>(0.1, 0.6)</td>
<td>(0.2, 0.6)</td>
<td>(0.2, 0.8)</td>
<td>(0.2, 0.4)</td>
<td>(0.2, 0.8)</td>
<td>1</td>
</tr>
<tr>
<td>$x_7$</td>
<td>(0.6, 0.4)</td>
<td>(0.6, 0.4)</td>
<td>(0.6, 0.4)</td>
<td>(0.7, 0.3)</td>
<td>(0.4, 0.6)</td>
<td>2</td>
</tr>
<tr>
<td>$x_8$</td>
<td>(0.6, 0.2)</td>
<td>(0.6, 0.2)</td>
<td>(0.8, 0.2)</td>
<td>(0.4, 0.6)</td>
<td>(0.4, 0.5)</td>
<td>2</td>
</tr>
<tr>
<td>$x_9$</td>
<td>(0.6, 0.2)</td>
<td>(0.6, 0.2)</td>
<td>(0.8, 0.2)</td>
<td>(0.1, 0.6)</td>
<td>(0.8, 0.2)</td>
<td>3</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>(0.6, 0.4)</td>
<td>(0.6, 0.4)</td>
<td>(0.8, 0.2)</td>
<td>(0.8, 0.2)</td>
<td>(0.6, 0.4)</td>
<td>3</td>
</tr>
</tbody>
</table>
According to Theorem 4.3, only a pair of ≤ and ≥ lower approximation reducts exists; i.e., \( (c_1, c_5) \). On the basis of the obtained ≤ and ≥ lower discernibility matrix, we can derive the ≤ lower discernibility functions of some objects using Definition 4.2 and Theorem 4.4. For example, the ≤ lower discernibility functions of \( x_i (1 \leq i \leq 8) \) are

\[
\begin{align*}
\mathcal{L}(x_1) &= (c_4 \lor c_5) \land (c_1 \lor c_2) \land (c_1 \lor c_2 \lor c_3 \lor c_5) = (c_1 \land c_4) \lor (c_1 \land c_5) \lor (c_2 \land c_4) \lor (c_2 \land c_5), \\
\mathcal{L}(x_2) &= c_1 \lor c_2 \lor c_3 \lor c_5, \\
\mathcal{L}(x_3) &= (c_1 \lor c_2) \land (c_1 \lor c_2 \lor c_4) = c_1 \lor (c_2 \land c_3) \lor (c_2 \land c_4), \\
\mathcal{L}(x_4) &= c_1, \\
\mathcal{L}(x_5) &= c_1 \lor c_2 \lor c_3 \lor c_4 \lor c_5, \\
\mathcal{L}(x_6) &= (c_1 \land c_4) \lor (c_1 \land c_5) \lor (c_1 \land c_2) \lor (c_1 \land c_3), \\
\mathcal{L}(x_7) &= c_3 \lor c_5 \lor (c_1 \land c_4), \\
\mathcal{L}(x_8) &= c_5.
\end{align*}
\]

The simplified ≤ upper bound rules for information systems security audit risk judgement are drawn by the ≤ lower discernibility functions of objects as follows:

\[
\begin{align*}
f(x, c_1) &\leq 0.2 \land f(x, c_4) \leq 0.6 \land 0.4 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_1), \\
f(x, c_1) &\leq 0.2 \land f(x, c_5) \leq 0.2 \lor 0.8 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_1), \\
f(x, c_2) &\leq 0.1 \land f(x, c_4) \leq 0.6 \lor 0.4 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_1), \\
f(x, c_3) &\leq 0.1 \land f(x, c_4) \leq 0.2 \lor 0.8 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_1), \\
f(x, c_4) &\leq 0.1 \land f(x, c_5) \leq 0.1 \lor 0.8 \lor 0.6 \Rightarrow f(x, c_2) \leq 0.3 \lor 0.6 \lor 0.8, \\
f(x, c_5) &\leq 0.2 \land 0.7 \Rightarrow f(x, d) \leq 2 \text{(supported by } x_2), \\
f(x, c_1) &\leq 0.1 \lor 0.8 \lor f(x, c_2) \leq 0.1 \lor 0.8 \land f(x, c_3) \leq 0.2 \lor 0.8 \lor 0.4, \\
f(x, c_2) &\leq 0.1 \lor 0.8 \land f(x, c_4) \leq 0.5 \lor 0.4 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_3), \\
f(x, c_1) &\leq 0.1 \lor 0.9 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_4), \\
f(x, c_1) &\leq 0.4 \lor 0.6 \lor f(x, c_2) \leq 0.2 \lor 0.6 \lor f(x, c_3) \leq 0.2 \lor 0.8 \lor 0.4, \\
f(x, c_4) &\leq 0.2 \lor 0.8 \lor f(x, c_5) \leq 0.2 \lor 0.8 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_5), \\
f(x, c_1) &\leq 0.1 \lor 0.6 \land f(x, c_3) \leq 0.2 \lor 0.8 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_6), \\
f(x, c_1) &\leq 0.1 \lor 0.6 \land f(x, c_4) \leq 0.2 \lor 0.8 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_6), \\
f(x, c_1) &\leq 0.1 \lor 0.6 \land f(x, c_5) \leq 0.2 \lor 0.8 \Rightarrow f(x, d) \leq 1 \text{(supported by } x_6),
\end{align*}
\]
\[ f(x, c_3) \leq (0.6, 0.4) \lor f(x, c_3) \leq (0.4, 0.6) \Rightarrow f(x, d) \leq 2 \text{(supported by } x_7), \]
\[ f(x, c_1) \leq (0.6, 0.4) \land f(x, c_4) \leq (0.7, 0.3) \Rightarrow f(x, d) \leq 2 \text{(supported by } x_7), \]
\[ f(x, c_3) \leq (0.4, 0.5) \Rightarrow f(x, d) \leq 2 \text{(supported by } x_8). \]

The first rule can be interpreted as follows:

If Better Systems Total Security is less than (0.2, 0.4) and Credible Hardware Device is less than (0.6, 0.4), then Risk Judgement on Information Systems Security Audit is Complete Examination.

Similarly, the \( \geq \) lower approximation discernibility functions of \( x_2, x_5, x_7, x_8, x_9, x_{10} \) and their corresponding simplified lower bound rules for information systems security audit risk judgement are obtained by using the \( \geq \) lower approximation discernibility matrix as follows:

\[ l^-(x_2) = c_4 \lor (c_1 \land c_5) \lor (c_2 \land c_6), \]
\[ l^-(x_5) = c_1, \]
\[ l^-(x_7) = c_1 \lor c_3 \lor c_4, \]
\[ l^-(x_8) = c_1 \lor c_2 \lor c_3, \]
\[ l^-(x_9) = c_5, \]
\[ l^-(x_{10}) = c_4 \lor (c_1 \land c_5) \lor (c_3 \land c_5), \]
\[ f(x, c_4) \geq (0.5, 0.2) \lor (f(x, c_1) \geq (0.1, 0.7) \land f(x, c_5) \geq (0.2, 0.7)) \lor, \]
\[ (f(x, c_1) \geq (0.3, 0.6) \land f(x, c_5) \geq (0.2, 0.7)) \Rightarrow f(x, d) \geq 2 \text{(supported by } x_2), \]
\[ f(x, c_1) \geq (0.4, 0.6) \Rightarrow f(x, d) \geq 2 \text{(supported by } x_3), \]
\[ f(x, c_1) \geq (0.6, 0.4) \lor f(x, c_3) \geq (0.6, 0.4) \lor f(x, c_4) \geq (0.7, 0.3), \]
\[ \Rightarrow f(x, d) \geq 2 \text{(supported by } x_7), \]
\[ f(x, c_1) \geq (0.6, 0.4) \lor f(x, c_2) \geq (0.6, 0.2) \lor f(x, c_3) \geq (0.8, 0.2), \]
\[ \Rightarrow f(x, d) \geq 2 \text{(supported by } x_8), \]
\[ f(x, c_3) \geq (0.8, 0.2) \Rightarrow f(x, d) \geq 3 \text{(supported by } x_9), \]
\[ f(x, c_4) \geq (0.8, 0.2) \lor (f(x, c_1) \geq (0.6, 0.4) \land f(x, c_5) \geq (0.6, 0.4)) \lor, \]
\[ (f(x, c_1) \geq (0.6, 0.4) \land f(x, c_5) \geq (0.6, 0.4)) \Rightarrow f(x, d) \geq 3 \text{(supported by } x_{10}), \]

This rule, \( f(x, c_3) \geq (0.8, 0.2) \Rightarrow f(x, d) \geq 3 \) can be explicated as:

If Credible Network Security is greater than (0.8, 0.2), then Risk Judgement on Information Systems Security Audit is No Examination.

**Example 6.2.** Candidate global supplier selection in a manufacturing company

A multi-criteria decision making problem is concerned with a manufacturing company which wants to select the best global supplier according to the core competencies of suppliers. Four types of suppliers are found in a manufacturing company. These are denoted by \( U = \{x_1, x_2, x_3, x_4\} \). The company can evaluate their core competencies from the following aspects (attributes): “Level of Technology Innovation” (\( c_1 \)), “Control Ability of Flow” (\( c_2 \)), “Ability of Management” (\( c_3 \)), and “Level of Service” (\( c_4 \)).

We assume that the manufacturing company is amenable only to providing its attribute values on each attribute as an intuitionistic value and those of the selection results (\( d \)) on the selectors as crisp ones. Here, the candidate global selection rules can generally be extracted using the proposed rule extraction approach in this paper. Inspired by Tan and Chen [61], we build a similar dataset (Table 6); partly taken from [61]. The domain of \( d \) is \( \{1, 2, 3\} \), where 1 means “Rejection,” 2 means “Considerable Selection,” and 3 means “Priority Selection.”
Using the same procedure in Example 5.1, we derive the $\preceq$ and $\succeq$ lower approximation discernibility matrices

$$M^{=} = \begin{pmatrix}
C & c_3c_4 & C & C \\
C & C & C & C \\
c_3 & c_3 & C & c_1c_2c_4 \\
C & c_2c_3 & C & C \\
\end{pmatrix}$$

and

$$M^{-} = \begin{pmatrix}
C & C & c_3 & C \\
c_3c_4 & C & c_3 & c_2c_3 \\
C & C & C & C \\
C & C & c_1c_2c_4 & C \\
\end{pmatrix},$$

respectively.

There is only $\preceq$ and $\succeq$ lower approximation reduct $\{c_3\}$. All the simplified $\preceq$ upper and $\succeq$ lower bound rules for the selection of candidate global supplier are as follows:

$$f(x, c_3) \preceq \langle 0.7, 0.2 \rangle \land f(x, c_4) \preceq \langle 0.3, 0.1 \rangle \Rightarrow f(x, d) \preceq 2 (\text{supported by } x_1),$$

$$f(x, c_3) \preceq \langle 0.5, 0.4 \rangle \Rightarrow f(x, d) \leq 1 (\text{supported by } x_3),$$

$$f(x, c_2) \preceq \langle 0.2, 0.5 \rangle \land f(x, c_3) \preceq \langle 0.4, 0.2 \rangle \Rightarrow f(x, d) \leq 2 (\text{supported by } x_4),$$

$$f(x, c_3) \preceq \langle 0.7, 0.2 \rangle \Rightarrow f(x, d) \geq 2 (\text{supported by } x_1),$$

$$f(x, c_3) \preceq \langle 0.9, 0.1 \rangle \Rightarrow f(x, d) \geq 3 (\text{supported by } x_2),$$

$$f(x, c_1) \preceq \langle 0.6, 0.2 \rangle \land f(x, c_3) \preceq \langle 0.4, 0.2 \rangle \land f(x, c_4) \preceq \langle 0.7, 0.1 \rangle,$$

$$\Rightarrow f(x, d) \geq 2 (\text{supported by } x_4).$$

This rule, $f(x, c_3) \preceq \langle 0.7, 0.2 \rangle \Rightarrow f(x, d) \geq 2$ can be interpreted as:

If Ability of Management is greater than $\langle 0.7, 0.2 \rangle$, then Candidate Global Supplier Selection is at least Considerable Selection.

**Example 6.3. Cars classification**

The car dataset contains the information of ten new cars to be classified in the Guangzhou car market in Guangdong, China. Let $U = \{x_i | 1 \leq i \leq 10\}$ be the cars, each of which is described by six attributes: fuel economy ($c_1$), aerodynamics degree ($c_2$), price ($c_3$), comfort ($c_4$); design ($c_5$), and safety ($c_6$). The characteristics of the ten new cars under the six attributes are represented (Table 7; partly taken from [35]).

Similar to Examples 6.1 and 6.2, some simplified $\preceq$ upper and $\succeq$ lower bound rules can be drawn as follows (Owing to space limitations, we only list partial rules):

$$f(x, c_2) \preceq \langle 0.2, 0.7 \rangle \land f(x, c_3) \preceq \langle 0.4, 0.5 \rangle \Rightarrow f(x, d) \leq 1 (\text{supported by } x_1),$$

$$f(x, c_2) \preceq \langle 0.3, 0.5 \rangle \land f(x, c_3) \preceq \langle 0.2, 0.6 \rangle \Rightarrow f(x, d) \leq 1 (\text{supported by } x_6),$$

... $f(x, c_1) \preceq \langle 0.8, 0.1 \rangle \land f(x, c_6) \preceq \langle 0.8, 0.2 \rangle \land f(x, c_2) \preceq \langle 0.7, 0.2 \rangle \land f(x, c_3) \preceq \langle 0.2, 0.6 \rangle,$

$$\Rightarrow f(x, d) \geq 4 (\text{supported by } x_5).$$

**Table 6**

<table>
<thead>
<tr>
<th>U</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>(0.5, 0.3)</td>
<td>(0.5, 0.4)</td>
<td>(0.7, 0.2)</td>
<td>(0.3, 0.1)</td>
<td>2</td>
</tr>
<tr>
<td>$x_2$</td>
<td>(0.4, 0.3)</td>
<td>(0.3, 0.4)</td>
<td>(0.9, 0.1)</td>
<td>(0.5, 0.2)</td>
<td>3</td>
</tr>
<tr>
<td>$x_3$</td>
<td>(0.4, 0.1)</td>
<td>(0.5, 0.3)</td>
<td>(0.5, 0.4)</td>
<td>(0.6, 0.2)</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>(0.6, 0.2)</td>
<td>(0.2, 0.5)</td>
<td>(0.4, 0.2)</td>
<td>(0.7, 0.1)</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 7
Cars classification.

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>(0.3, 0.4)</td>
<td>(0.2, 0.7)</td>
<td>(0.4, 0.5)</td>
<td>(0.8, 0.1)</td>
<td>(0.4, 0.5)</td>
<td>(0.2, 0.7)</td>
</tr>
<tr>
<td>$x_2$</td>
<td>(0.4, 0.3)</td>
<td>(0.5, 0.1)</td>
<td>(0.6, 0.2)</td>
<td>(0.2, 0.7)</td>
<td>(0.3, 0.6)</td>
<td>(0.7, 0.2)</td>
</tr>
<tr>
<td>$x_3$</td>
<td>(0.4, 0.2)</td>
<td>(0.6, 0.1)</td>
<td>(0.8, 0.1)</td>
<td>(0.2, 0.6)</td>
<td>(0.3, 0.7)</td>
<td>(0.5, 0.2)</td>
</tr>
<tr>
<td>$x_4$</td>
<td>(0.3, 0.4)</td>
<td>(0.9, 0.0)</td>
<td>(0.8, 0.1)</td>
<td>(0.7, 0.1)</td>
<td>(0.1, 0.8)</td>
<td>(0.2, 0.8)</td>
</tr>
<tr>
<td>$x_5$</td>
<td>(0.8, 0.1)</td>
<td>(0.7, 0.2)</td>
<td>(0.7, 0.0)</td>
<td>(0.4, 0.1)</td>
<td>(0.8, 0.2)</td>
<td>(0.2, 0.6)</td>
</tr>
<tr>
<td>$x_6$</td>
<td>(0.4, 0.3)</td>
<td>(0.3, 0.5)</td>
<td>(0.2, 0.6)</td>
<td>(0.7, 0.1)</td>
<td>(0.5, 0.4)</td>
<td>(0.3, 0.6)</td>
</tr>
<tr>
<td>$x_7$</td>
<td>(0.6, 0.4)</td>
<td>(0.4, 0.2)</td>
<td>(0.7, 0.2)</td>
<td>(0.3, 0.6)</td>
<td>(0.3, 0.7)</td>
<td>(0.6, 0.1)</td>
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<tr>
<td>$x_8$</td>
<td>(0.9, 0.1)</td>
<td>(0.7, 0.2)</td>
<td>(0.7, 0.1)</td>
<td>(0.4, 0.5)</td>
<td>(0.4, 0.5)</td>
<td>(0.8, 0.0)</td>
</tr>
<tr>
<td>$x_9$</td>
<td>(0.4, 0.4)</td>
<td>(1.0, 0.0)</td>
<td>(0.9, 0.1)</td>
<td>(0.6, 0.2)</td>
<td>(0.2, 0.7)</td>
<td>(0.1, 0.8)</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>(0.9, 0.1)</td>
<td>(0.8, 0.0)</td>
<td>(0.6, 0.3)</td>
<td>(0.5, 0.2)</td>
<td>(0.8, 0.1)</td>
<td>(0.6, 0.4)</td>
</tr>
</tbody>
</table>

\[
f(x, c_1) \geq (0.9, 0.1) \lor f(x, c_6) \geq (0.8, 0.0) \lor (f(x_2) \geq (0.7, 0.2) \land f(x, c_3) \geq ,
\]
\[
(0.7, 0.1) \land f(x, c_5) \geq (0.4, 0.5) \Rightarrow f(x, d) \geq 4 \text{(supported by } x_3) ,
\]
\[
f(x, c_1) \geq (0.9, 0.1) \lor f(x, c_3) \geq (0.8, 0.1) \lor (f(x_2) \geq (0.8, 0.0) \land f(x, c_6) \geq ;
\]
\[
(0.6, 0.4) \lor (f(x, c_1) \geq (0.6, 0.3) \land f(x, c_4) \geq (0.5, 0.2) \land f(x, c_6) \geq (0.6, 0.4) ,
\]
\[
\Rightarrow f(x, d) \geq 4 \text{(supported by } x_{10}) \]

7. Conclusion

Rough set theory is a useful mathematical tool for dealing with uncertain information. However, when we consider ranking fuzzy-valued objects rather than classifying them, conventional rough set theory is unable to solve these problems. One of the extensions of the classic rough set approach is the dominance fuzzy-valued rough set approach. The intuitionistic fuzzy decision table is an important type of data one, and is a generalized form of fuzzy-valued information systems [62,63]. This paper focuses on the construction of a fuzzy-rough set model and rule extraction in DIFDT, which aids decision making under dominance-based intuitionistic fuzzy settings. The main contribution of this paper is the introduction of the ranking method for comparing two intuitionistic fuzzy values in DIFDT. First, we defined the notion of DIFDT and introduced a ranking method for all objects with the dominance-based relation between objects. Second, on the basis of the dominance-based relation, we established a fuzzy-rough set approach in DIFDT, which are grounded primarily based on the substitution of the indiscernibility relation with the dominance-based relation. Third, to extract the simplest dominance intuitionistic fuzzy lower and upper bound rules, we used the discernibility matrices to propose two attribute reduction approaches for eliminating redundant information. Finally, we applied these approaches to information systems security audit risk judgement for CISA, candidate global supplier selection in a manufacturing company, and cars classification. The application examples yielded valuable rules. The approaches simplified a DIFDT and enabled the direct identification of considerably simpler intuitionistic fuzzy lower and upper bound rules. Moreover, the derived rules can facilitate knowledge acquisition from DIFDT.

Acknowledgements

The authors thank the anonymous referees for their constructive suggestions. This paper was supported by the Natural Science Foundation of China (Grant Nos. 61170105 and 71201076), Qing Lan Project of Jiangsu Province, and the Priority Academic Program Development of Jiangsu Higher Education Institutions (Audition Science and Technology).

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