Global dynamics of equilibrium point for delayed competitive neural networks with different time scales and discontinuous activations

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1. Introduction

In recent years, various neural networks such as Hopfield, cellular, Cohen–Grossberg, and bidirectional associative memory neural networks have been extensively studied, and also have been successfully applied to many fields such as image processing, parallel computing, optimization, associative memory, see [1–8] and the references therein. In these applications, the properties of stability and convergence of the equilibrium point are important in design and application of these neural networks and many researchers have done extensive works on this subject in the literature (see [9–11] and the references therein). Moreover, due to the finite switching speed of the neuron amplifiers and the finite speed of signal propagation, delays are actually unavoidable in the electronic implementation. It is well known, in both biological and artificial neural networks, that the delay is a potential cause of the loss of stability, since it may originate the onset of nonvanishing oscillations [12–15]. Therefore, the study of neural dynamics with consideration of the delayed problem becomes extremely important to manufacture high-quality neural networks.

Over the past decades, the neural network models considered in the dynamics analysis in most papers are the ones with single time scale, which means that in these models only the neuron activity is taken into consideration, there exists only one type of variables, that is, the state variables of the neural neurons. However, in a dynamical neural network, the synaptic weights also vary with respect to time due to the learning process, the variation of connection weights may have influence on the dynamics of neural networks, and so competitive neural networks with different scales introduced by Meyer-Bäse et al. in [16–18], in which the dynamics of neuron states are governed by a set of differential equations as in the usual neural networks. In addition to dynamics of neuron states, connection weights also vary with time under the Hebbian learning law, whose dynamics are described by another set of differential equations. In this model, there are two types of state variables that of the short-term memory (STM) variables describing the fast neural activity and that of the long-term memory (LTM) variables describing the slow unsupervised synaptic modifications. Thus, there are two time scales in these neural networks, in which one corresponds to the fast changes of the neural network states and another corresponds to the slow changes of the synapses by external stimuli. The general neural network equations describing the temporal evolution of the STM and LTM states for the ith neuron of a n-neuron network are

\[
\begin{align*}
\text{STM:} & \quad \frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} D_{ij} f_j(x_j(t)) + \sum_{k=1}^{P} B_{ik} y_k(t) + \sum_{k=1}^{P} \sum_{\tau=1}^{\tau_{\max}} m_{ik}(t) y_k(t-	au), \\
\text{LTM:} & \quad \frac{dm_{ik}(t)}{dt} = -m_{ik}(t) + y_k f_j(x_j(t)), \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, P.
\end{align*}
\]
where $x_i > 0$ represents the neuron current activity level, $a_i > 0$ is a positive function representing the time constant of the neuron, $f_j(x_i(t))$ is the output of the neuron, $m_u(t)$ is the synaptic efficiency, $y_k$ is the constant external stimulus, $D_0$ represents the connection weight between the $i$th neuron and the $j$th neuron, and $D_j(t)$ denotes the synaptic weight of delayed feedback and $B_i$ is the strength of the external stimulus, $\varepsilon > 0$ is the fast time scale of STM state. Recently, considerable effort has been devoted to study the dynamical behaviors of competitive neural networks with different scales, one may refer to [11,19–25] and the references therein.

However, all of the above works were based on the assumption that the activation functions are continuous, even globally Lipschitz. As pointed out by Forti and Nistri in [26,27], a brief review of some common neural network models reveals that neural networks with discontinuous activations are important and frequently arise in practice. Furthermore, the analysis of the ideal discontinuous case is able to reveal crucial features of the dynamics, such as the presence of sliding modes along discontinuity surfaces, the phenomenon of convergence in finite time toward the equilibrium point and the ability to compute the exact global minimum of the underlying energy function, which make these networks especially attractive for the solution of global optimization problems in real world. Thus, considerable attention has been paid on the study of discontinuous neural networks theory and a large body of work has been reported in the literature. In [28], for example, a class of nonsmooth gradient-like systems was considered, by virtue of Lyapunov function and topological degree theory, they investigated the dynamical behaviors of this system. In [29], the authors use the concept of the Filippov solution to study the almost periodic dynamics of a class of delayed dynamical systems with discontinuous right-hand side, we refer to [30–40] and the references therein for other interesting works on neural networks with discontinuous activations.

In addition, according to Nie and Cao [41], when activation functions do not satisfy continuity, we do not know whether the global solution and an equilibrium point of neural networks are existent. Therefore, it is more difficult to study the dynamics of the system (1.1) with discontinuous right-hand side. On the other hand, to the best of our knowledge, a fewer results have been obtained on the global dynamics of equilibrium point for delayed competitive neural networks with different time scales and discontinuous activations.

Motivated by the above discussion, the main contribution of this paper is to investigate the existence, global asymptotic stability and convergence in finite time, global exponential stability of delayed competitive neural networks with different time scales and discontinuous activations. Our analysis is based on the fixed point theorem of differential inclusion theory, linear matrix inequality technique and Lyapunov functional method. The results obtained in the present paper improve and extend previous works in the literature to some extent.

The structure of this paper is outlined as follows. **Section 2** discusses the neural network model studied in this paper and presents some preliminaries related to our main results. **Section 3** presents the main results on the dynamical behavior for system (2.2). **Section 4** gives two examples to demonstrate the effectiveness of the main results. Finally, the study in this paper is concluded in **Section 5**.

**Notations:** Given the column vector $x = (x_1, x_2, \ldots, x_n)^T$, in which the superscript $T$ denotes the transpose of vector, $\|x\| = \sqrt{x^T x}$ is the Euclidean vector norm, i.e., $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. By $x \geq 0$ (respectively, $x > 0$) we mean $x_i \geq 0$ (respectively, $x_i > 0$) for all $i = 1, 2, \ldots, n$.

For any matrix $A, A^T$ and $A^{-1}$ denote the transpose and the inverse of $A$, respectively. If $A$ is a symmetric matrix, $A > 0$ ($A \geq 0$) means that $A$ is positive definite (nonnegative definite). Similarly, $A < 0$ ($A \leq 0$) means that $A$ is negative definite (negative semidefinite). $\lambda_{\text{min}}(A)$ and $\lambda_{\text{max}}(A)$ represent the minimum and maximum eigenvalues of matrix $A$, respectively. We also write $\|A\|_2$ to denote the 2-norm of $A$, i.e., $\|A\|_2 = \sqrt{\lambda(A^T A)}$, where $\lambda(A^T A)$ denotes the spectral radius of $A^T A$. Finally, $E$ denotes the identity matrix.

Given a set $\Omega \subset \mathbb{R}^m$, $K(\Omega)$ denotes the closure of the convex hull of $\Omega$, $\partial_k(\Omega)$ denotes the collection of all nonempty, compact and convex subset of $\Omega$.

### 2. Preliminaries

In this section, we present some definitions and lemmas which will be used throughout the paper.

Firstly, we shall simplify system (1.1) as follows: Setting

$$S_i(t) = \sum_{k=1}^{p} m_k(t)y_k = y_i^T m_i(t),$$

where

$$y = (y_1, \ldots, y_p)^T,
\quad m_i(t) = (m_{i1}(t), m_{i2}(t), \ldots, m_{ip}(t))^T,$$

and summing up the LTM over $k$, the neural networks (1.1) can be rewritten as the state-space form

$$\begin{align*}
\text{STM} : \frac{dx(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^{n} D_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} D_{ij} f_j(x_j(t-\tau(t))) + B_i S_i(t), \\
\text{LTM} : \frac{dS_i(t)}{dt} &= -S_i(t) + \|y\|^2 f_i(x_i(t)), \quad i = 1, 2, \ldots, n,
\end{align*}$$

(2.1)

where $\|y\|^2 = y_1^2 + y_2^2 + \cdots + y_p^2$ is the constant. Without loss of generality, the input stimulus $y$ is assumed to be normalized with unit magnitude $\|y\|^2 = 1$, then the above neural networks are simplified as

$$\begin{align*}
\text{STM} : \frac{dx(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^{n} D_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} D_{ij} f_j(x_j(t-\tau(t))) + B_i S_i(t), \\
\text{LTM} : \frac{dS_i(t)}{dt} &= -S_i(t) + f_i(x_i(t)), \quad i = 1, 2, \ldots, n,
\end{align*}$$

(2.2)

or equivalently the vector form

$$\begin{align*}
\text{STM} : \frac{dx(t)}{dt} &= -Ax(t) + Df(x(t)) + Df(x(t-\tau(t))) + B_0 S(t), \\
\text{LTM} : \frac{dS(t)}{dt} &= -S(t) + f(x(t)),
\end{align*}$$

(2.2)

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $S(t) = (S_1(t), S_2(t), \ldots, S_n(t))^T$, $A = \text{diag}(a_1, a_2, \ldots, a_n)$, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T$, $f(x(t-\tau(t))) = (f_1(x_1(t-\tau(t))), f_2(x_2(t-\tau(t))), \ldots, f_n(x_n(t-\tau(t))))^T$, $D = (D_{ij})_{n \times n}$, $B_0 = \text{diag}(B_1, B_2, \ldots, B_n)$ and the time-varying delay $\tau(t)$ is differentiable function satisfying $0 \leq \tau(t) \leq \tau, \tau(t) \leq \tau^* < 1$,

(2.3)

where $\tau$ and $\tau^*$ are positive constants.

**Definition 2.1 (Class $\mathcal{F}$ of functions).** We call $f \in \mathcal{F}$, if for all $i = 1, 2, \ldots, n$, $f_i(\cdot)$ satisfies: $f_i$ is continuous except a countable set of points $\rho_k$, where the right and left limits $f_i(\rho_k^+)$ and $f_i(\rho_k^-)$ satisfy $f_i(\rho_k^+) > f_i(\rho_k^-), k = 1, 2, \ldots$. Moreover, $f_i$ has only finite discontinuous points on every compact set of it.

**Remark 2.1.** If $f \in \mathcal{F}$, then we have

$$K(f(x)) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^T.$$
constants $a$ and $b$ such that
\[
\|K[f(x)]\| = \sup_{y \in K[f(x)]} \|y\| \leq a\|x\| + b, \tag{2.4}
\]
where
\[
K[f(x)] = (K[f_1(x_1)], K[f_2(x_2)], ..., K[f_n(x_n)])^T,
\]
and
\[
K[f_i(x_i)] = \left[ f_i(x_i^+), f_i(x_i^-) \right] \quad \text{for } i = 1, 2, ..., n.
\]

**Remark 2.2.** If $f(x)$ satisfies the growth condition (g.c.), the discontinuous function $f(x)$ abandons the restriction of boundedness and monotonicity. Therefore, the growth condition (g.c.) is more general and more practical.

**Definition 2.3** (See Filippov [42]): Let $X$ and $Y$ be topological Hausdorff spaces and $P(Y)$ be all nonempty subsets of $Y$. We say that $F: X \rightarrow P(Y)$ is upper semicontinuous (U.S.C.) at $x \in X$ if for every neighborhood $U$ of $x$, there exists a neighborhood $V$ of $x$ such that $F(U) \subset U$ for every $x \in V$. A set-valued function $F: X \rightarrow P(Y)$ is called U.S.C. on $X$ if it is U.S.C. at every $x \in X$.

Now we introduce the concept of Filippov solution.

**Definition 2.4** (See Filippov [42]): Given a set-valued function $F: X \rightarrow P(Y)$, a function $f: X \rightarrow Y$ is said to be a selector for $F$ if $f(x) \in F(x)$ for all $x \in X$.

**Definition 2.5** (Filippov solutions). Function $(x^T, S^T)^T$ is said to be a solution of (2.2) on $[-\tau, T]$ ($T \in (0, +\infty)$) if

1. $(x(t), S(t))$ are continuous on $[-\tau, T)$ and absolutely continuous on any compact interval of $[0, T]$;
2. there exists a measurable function $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)^T: [-\tau, T) \rightarrow \mathbb{R}^n$ such that $\gamma(t) \in K[f(x(t))]$ for a.a. $t \in [-\tau, T)$ and
\[
\begin{align*}
\frac{dx(t)}{dt} &= -Ax(t) + D\gamma(t) + D^T\gamma(t - \tau(t)) + BS(t), \\
\frac{dS(t)}{dt} &= -S(t) + \gamma(t).
\end{align*}
\tag{2.5}
\]
Any function $\gamma(t)$ satisfying (2.5) is called an output solution associated with the state $x(t)$. With this definition it turns out that $x(t), S(t)$ are solutions of (2.2) in the sense of Filippov since it satisfies
\[
\begin{align*}
\frac{dx(t)}{dt} &= -Ax(t) + DK(x(t) + D^T\gamma(t)) + BS(t), \\
\frac{dS(t)}{dt} &= -S(t) + K[\gamma(x(t))].
\end{align*}
\]

**Definition 2.6.** An equilibrium point of (2.2) is a vector $(x^*, S^*)^T \in \mathbb{R}^{2n}$ that satisfies
\[
\begin{align*}
0 &= -Ax^* + (D + D^T)K(x^*) + BS^*, \\
0 &= -S^* + K[\gamma(x^*)].
\end{align*}
\]
Equivalently, $(x^T, S^T)^T \in \mathbb{R}^{2n}$ is an equilibrium point of (2.2) if there exists $x^* \in K[\gamma(x^*)]$ such that
\[
\begin{align*}
-Ax^* + (D + D^T)\gamma^* + BS^* &= 0, \\
-S^* + \gamma^* &= 0. \tag{2.6}
\end{align*}
\]

Since we are interested in studying the time-domain behavior of both states $x, S$ and the output $y$, it is convenient to give the next definition of an initial value problem (IVP) associated to (2.2).

**Definition 2.7** (IVP). For any continuous function $\phi, \psi: [-\tau, 0] \rightarrow \mathbb{R}^n$ and any measurable selection $\Phi: [-\tau, 0] \rightarrow \mathbb{R}^n$, such that $\Phi(s) \in K[\phi(s)]$ for a.a. $s \in [-\tau, 0]$ by an initial value problem associated to (2.2) with initial condition $(\phi(s), \psi)$, we mean the following problem: find a couple of functions $x(t), S(t), \gamma(t)$, such that $(x^T(t), S^T(t))^T$ is a solution of (2.2) on $[-\tau, T)$ for some $T > 0$, $\gamma(t)$ is an output solution associated to $x(t)$, and
\[
\begin{align*}
\frac{dx(t)}{dt} &= -Ax(t) + D\gamma(t) + D^T\gamma(t - \tau(t)) + BS(t), \\
\frac{dS(t)}{dt} &= -S(t) + \gamma(t)
\end{align*}
\]
for a.a. $t \in [0, T)$, $x(s) = \phi(s), \quad \forall s \in [-\tau, 0)$, $S(s) = \psi(s), \quad \forall s \in [-\tau, 0)$, $\gamma(S) = \Phi(s)$ for a.a. $s \in [-\tau, 0)$.

To obtain our main results, we should introduce the following necessary lemmas.

**Lemma 2.1** (Multi-valued version of the Leray–Schauder alternative theorem, see Dugundji and Granas [42]). If $X$ is a Banach space, $C \subset X$ is a nonempty convex set with $0 \in C$ and $G: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a U.S.C. multifunction which maps bounded sets into relatively compact sets, then one of the following statements is true:

1. the set $J = \{x \in C: x \in AG(x), A \in (0, 1)\}$ is unbounded;
2. the $G(.)$ has a fixed point in $C$, i.e., there exists $x \in C$ such that $x \in G(x)$.

**Lemma 2.2** (See Nie and Cao [41]). For any vectors $x, y \in \mathbb{R}^n$ and positive definite matrix $M \in \mathbb{B}_{n \times n}$, the following matrix inequality holds:
\[
2x^T M x + y^T M^{-1} y.
\]

The next technical result is crucial for computing the time derivative along solutions of system (2.2) of the Lyapunov function introduced in the next section.

**Lemma 2.3** (See Qin and Xue [31]).
\[
\frac{d}{dt} \int_0^{u(t)} f_i(s) \, ds = \xi_i u_i(t) \quad \text{for all } \xi_i \in K[f_i(u_i(t))].
\]

3. Main results

In this section, the main result of this paper is stated as follows. For convenience, we split this part into three subsections.

3.1. Existence of the equilibrium point

**Lemma 3.1** (Existence of local solution). If $f \in F$, then any IVP associated with (2.2) has at least one local solution $(x^T(t), S^T(t), \gamma^T(t))^T$ defined on a maximal interval $[0, T)$ for some $T \in (0, +\infty)$.

**Proof.** Similar to Lemma 1 in [27], the proof of this lemma is easy to obtain by the method of step-by-step construction, so we omit it.

Prior to giving our main results, we present additional assumptions and lemmas which are useful for the subsequent proof.

(Ai) For each $i = 1, 2, ..., n$, there exists a constant $L_i$ such that for any two different numbers $u, v \in \mathbb{R}$, $\forall \xi_i \in K[f_i(u)], \xi_i \in K[f_i(v)]$,
\[
\frac{\xi_i - \xi_i}{v - u} \geq -L_i. \tag{3.1}
\]

**Remark 3.1.** $L_i \leq 0$ implies that $f_i(.)$ is a monotonic nondecreasing function.
where \( \mathcal{C}_0 \) is a U.S.C. multifunction which maps bounded sets into relatively compact sets under the growth condition (g.c.).

Suppose that \( f \in \mathcal{G} \) and the assumptions (A1) and (A2) are satisfied, then system (2.2) has at least one equilibrium point.

**Proof.** Let \( F(x) = -(-(1-\lambda)x + \lambda(-Ax + (D + D^T + B)K[f(x)]) \)

Then \( x^* \) is an equilibrium point of system (2.2) which is equivalent to saying that \( x^* \) is a fixed point of \( F(x) \), i.e., \( x^* \in F(x^*) \). It is obvious to see that \( F : \mathbb{R}^n \to \mathbb{P}_n(\mathbb{R}^n) \) is a U.S.C. multifunction which maps bounded sets into relatively compact sets under the growth condition (g.c.).

In order to solve the fixed point problem \( x^* \in F(x^*) \), in fact, due to Lemma 2.1 it is sufficient to show that the set \( \Gamma = \{ x \in \mathbb{R}^n : x \in \bar{J}(\mathcal{F}(x)) \} \)

is a U.S.C. multifunction which maps bounded sets into relatively compact sets under the growth condition (g.c.).

Our first main result can be stated as follows.

**Theorem 3.1.** Suppose that \( f \in \mathcal{G} \) and the assumptions (A1) and (A2) are satisfied, then system (2.2) has at least one equilibrium point.

**Proof.** Let \( F(x) = -(-(1-\lambda)x + \lambda(-Ax + (D + D^T + B)K[f(x)]) \)

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\[ x^T B y \leq \frac{1}{2} C_{\text{max}}(A_1)x^T B B^T x + \frac{1}{2} \lambda_{\text{min}}(A_1) y^T \eta, \]

which, together with (3.14), yield

\[ (x + e \eta y) y \leq \frac{1}{2} (1 - \lambda) x x + \frac{1}{2} a_m \| 2DD^T + D^T D^T + BB^T \|_2 x x + (1 + \alpha \sigma (M)) \| I \| \| I \| \| e(b + \| \gamma^\top \| ) \| M \| \| I \| \|
\]

\[ \leq -\alpha \| x \|^2 + \alpha \| x \| + \| \beta \|, \]

(3.15)

where

\[ \alpha = \min \left\{ \frac{1}{2} a_m \| 2DD^T + D^T D^T + BB^T \|_2, \frac{1}{2} \lambda_{\text{max}}(A_1) \right\} > 0, \]

\[ \beta = (1 + \alpha \sigma (M)) \| I \|. \]

One can easily see that if \( R \) is taken sufficiently large, then we obtain

\[ (x + e \eta y)y < 0 \quad \text{for} \quad \| x \| > R, \]

which means that \( \exists \delta \in \mathbb{R} \) such as \( \| x \| > R \), i.e., \( 0 \in F(\lambda, x) \) only if \( \| x \| \leq R \), which implies that \( \Gamma \) is bounded. By now we know from Lemma 2.2 that there exists \( x^* \in \mathbb{R}^n \) such that \( x^* \in F(x^*) \). Therefore, there exists \( \gamma^* \in K[F(x^*)] \) such that

\[ 0 = -Ax^* + (D + D^T + B\gamma^*). \]

Hence, system (2.2) has at least one equilibrium point \((x^{\text{eq}}, \gamma^{\text{eq}})^\top\). The proof of Theorem 3.1 is completed. \( \square \)

### 3.2. Global asymptotic stability and convergence in finite time

The goal of this subsection is to find conditions which ensure the uniqueness and global asymptotic stability of the equilibrium point for system (2.2). Furthermore, we will derive a result on the global convergence in finite time of system (2.2).

**Theorem 3.2.** Suppose that \( f \in \mathcal{G} \) and the assumptions \((A_1) \) and \((A_2) \) hold, then system (2.2) admits one unique equilibrium point \((x^{\text{eq}}, \gamma^{\text{eq}})^\top\) that is globally asymptotically stable. Moreover, the existence interval of each solution for system (2.2) can be extended to \([0, +\infty)\).

**Proof.** Firstly, in view of the definition of negative matrix and \( 0 < \mu < 1, 0 < \tau^* < 1 \), we derive from \( A_2 < 0 \) that \( A_1 < 0 \). According to Theorem 3.1, the existence of an equilibrium point for system (2.2) can be guaranteed.

Secondly, let \((x^{\text{eq}}, \gamma^{\text{eq}})^\top\) be an equilibrium point of system (2.2) and \( \gamma^* \) be an output equilibrium point corresponding to \( x^* \). Set \( u(t) = x(t) - x^{\text{eq}}, v(t) = x(t) - x^{\text{eq}} \), then system (2.2) is transformed into

\[
\begin{align*}
\text{STM:} \quad & \frac{du(t)}{dt} = -Au(t) + D\gamma(t) + D\gamma(t - \tau(t)) + Bv(t), \\
\text{LTM:} \quad & \frac{dv(t)}{dt} = -v(t) + \gamma(t),
\end{align*}
\]

(3.16)

where

\[ \gamma(t) = \gamma(t) - \gamma^{\text{eq}}, \quad \gamma(t) \in K[F(y)], \quad f_u(u) = f_u(u) + x^*(f_u(u)). \]

Consider a Lyapunov function defined by

\[ V[u, v, \gamma(t)] = V_1(t) + V_2(t) + V_3(t), \]

where

\[ V_1(t) = \tilde{e} x^T Q x, \quad V_2(t) = tv^T u + 2\tilde{e} \sum_{i=1}^n \int_0^{u_i} f_i'(s) ds, \]

\[ V_3(t) = \tilde{e} \int_{t-t_0}^t \gamma(s) R \gamma(s) ds. \]

(3.17)

where \( \tilde{e} \) is the constant defined in (3.9) and (3.10). From Lemma 3.2, we can obtain that

\[ u^T_t + 2\tilde{e} \int_0^{u_i} f_i'(s) ds \geq \frac{1}{2} u_i^2, \]

and thus

\[ V_2(t) \geq \frac{1}{2} \| u \|^2. \]

Obviously, \( V[u, v, \gamma(t)] \) is absolutely continuous and its derivative along system (3.16) can be evaluated term by term as follows:

\[ V_1(t) = \tilde{e} - 2v^T(t)Q(v(t) + 2v^T(t)Q(t)\gamma(t)). \]

(3.19)

According to Lemma 2.2, we can see that

\[ 2v^T(t)Q(t) \leq v^T(t)Q^{-1}v(t) + \gamma^T(t)Q\gamma(t) = v^T(t)Q(v(t) + \gamma^T(t)Q\gamma(t)), \]

which, together with (3.19), yields

\[ V_1(t) \geq -\tilde{e} (v^T(t)Q(v(t) + \gamma^T(t)Q\gamma(t)). \]

(3.20)

Similar to (3.13), we derive

\[ \tilde{e} \gamma^T(t)Mu(t) \geq -2u^T(t)a_u(t), \quad \tilde{e} \gamma^T(t)M\gamma(t) \geq -2u^T(t)a_u(t), \]

then, from Lemma 2.3, we obtain that

\[ V_2(t) = (2u^T(t) + 2\tilde{e} \gamma^T(t)M)(-Au(t) + D\gamma(t) + D\gamma(t - \tau(t)) + Bv(t)) \]

\[ \leq -u^T(t)Au(t) + 2u^T(t)D\gamma(t) + 2u^T(t)D\gamma(t - \tau(t)) + 2u^T(t)Bv(t) \]

\[ + 2\tilde{e} \gamma^T(t)MD\gamma(t) + 2\tilde{e} \gamma^T(t)MD\gamma(t - \tau(t)) + 2\tilde{e} \gamma^T(t)MBv(t), \]

(3.21)

and

\[ V_3(t) = \tilde{e} \gamma^T(t)M\gamma(t) - \tilde{e} \gamma^T(t)M\gamma(t - \tau(t)) \]

\[ \leq \tilde{e} \gamma^T(t)M\gamma(t) - \tilde{e} \gamma^T(t)M\gamma(t - \tau(t)). \]

(3.22)

Put (3.20)–(3.22) together, we can deduce that

\[ V(t) \leq -u^T(t)Au(t) + 2u^T(t)D\gamma(t) + 2u^T(t)D\gamma(t - \tau(t)) + 2u^T(t)Bv(t) \]

\[ + \tilde{e} \gamma^T(t)M\gamma(t) - \tilde{e} \gamma^T(t)M\gamma(t - \tau(t)) \]

\[ - (1 - \mu) \tilde{e} \gamma^T(t)Qv(t). \]

(3.23)

We obtain from Lemma 2.2 that

\[ 2u(t)D\gamma(t) \leq \frac{1}{2} \tilde{e} \lambda_{\text{min}}(A_2)u^T(t)D^\top D^T u(t) + \frac{1}{2} \tilde{e} \lambda_{\text{min}}(A_2)\gamma^T(t)\gamma(t), \]

\[ 2u(t)D^\top D^T u(t) \leq \frac{1}{2} \tilde{e} \lambda_{\text{min}}(A_2)u^T(t)D^\top D^T u(t) \]

\[ + \tilde{e} \lambda_{\text{min}}(A_2)\gamma^T(t)\gamma(t - \tau(t)), \]

\[ 2u(t)Bv(t) \leq \frac{1}{2} \tilde{e} \lambda_{\text{min}}(A_2)u^T(t)B^T B^T u(t) + \tilde{e} \lambda_{\text{min}}(A_2)v^T(t)v(t). \]

(3.24)

Substitute (3.24) into (3.23) that

\[ V(t) \leq -u^T(t)Au(t) + \frac{1}{2} \tilde{e} \lambda_{\text{min}}(A_2)u^T(t)D^\top D^T u(t) - \frac{1}{2} \tilde{e} \lambda_{\text{min}}(A_2)\gamma^T(t)\gamma(t) \]

\[ - (1 - \mu) \tilde{e} \lambda_{\text{min}}(Q)u^T(t)Qv(t) - \frac{1}{2} \tilde{e} \lambda_{\text{min}}(A_2)\gamma^T(t)\gamma(t) \]

\[ - (1 - \mu) \tilde{e} \lambda_{\text{min}}(Q)u^T(t)Qv(t) - \frac{1}{2} \tilde{e} \lambda_{\text{min}}(A_2)\gamma^T(t)\gamma(t). \]

(3.25)
where

$$\beta = \min \left\{ e^{-\frac{1}{2} \lambda_{\min}(-A_2)} \left( |x_i(t)|^2 + |v_i(t)|^2 \right) \right\} > 0,$$

we conclude that the equilibrium point of system (2.2) is globally asymptotically stable and therefore is unique. Finally, we know from (3.17), (3.18) and (3.25) that

$$\frac{1}{2} \left( |x_i(t)|^2 + |v_i(t)|^2 \right) \leq \text{min}$$

which implies that $u(t)$ and $v(t)$ are bounded. Therefore, the existence interval of each solution for system (2.2) can be extended to $[0, +\infty)$.

In the following, based on Theorem 3.2, we give the analysis of convergence in finite time for system (2.2). To do so, we further give the following hypothesis:

$$(A_{\delta}) \quad x^*_{\delta} \text{ is a discontinuous point of } f_{\delta} \text{ for each } i = 1, 2, \ldots, n, \text{ and } f_{\delta}(x^*_{\delta}) - y^*_{\delta} < 0 < f_{\delta}(x^*_{\delta}) - y^*_{\delta}, \quad i = 1, 2, \ldots, n.$$

Therefore, if $x(t)$ is a discontinuous point of $f_{\delta}$ and follow the argument employed in the proof of Theorem 3.2, and $\lambda_{\min}(-A_2)$ converges in finite time to the equilibrium point of system (2.2), there exists a sufficiently small positive constant $\delta$ such that

$$\frac{1}{2} \left( |x_i(t)|^2 + |v_i(t)|^2 \right) \leq \text{min}$$

and

$$\frac{1}{2} \left( |x_i(t)|^2 + |v_i(t)|^2 \right) \leq \text{min}$$

Let us consider the following Lyapunov function defined by

$$V(u, v, \dot{y})(t) = V_1(t) + V_2(t) + V_3(t),$$

where

$$V_1(t) = e^{\epsilon T} V_1, \quad V_2(t) = 2 \epsilon e^{\epsilon T} \sum_{i=1}^{n} m_i \int_0^{\epsilon T} f_i(s) ds,$$

and

$$V_3(t) = \epsilon \int_{T - \epsilon T}^{T} \frac{\gamma^T (s) R \gamma(s)}{\epsilon} ds.$$

Differentiating $V_1(t)$ gives

$$V_1(t) = e^{\epsilon T} V_1(t) Q^{-1} \dot{V}(t) Q^{-1} \dot{y}(t) = e^{\epsilon T} \left( V_1(t) Q^{-1} \dot{V}(t) Q^{-1} \dot{y}(t) \right).$$

We have from Lemma 2.2 that

$$\dot{V}(t) = e^{\epsilon T} \left( V_1(t) Q^{-1} \dot{V}(t) Q^{-1} \dot{y}(t) \right),$$

which, together with (3.36), yields

$$V_1(t) \leq e^{\epsilon T} \left( V_1(t) Q^{-1} \dot{V}(t) Q^{-1} \dot{y}(t) \right).$$

In view of the monotonicity of $f_{\delta}(\cdot)$, Lemma 2.3 and $0 < \epsilon < \min\{a_1, \ldots, a_n\}$, one can obtain that

$$V_2(t) = e^{\epsilon T} \left[ 2 \epsilon e^{\epsilon T} \sum_{i=1}^{n} m_i \int_0^{\epsilon T} f_i(s) ds + 2 e^{\epsilon T} \sum_{i=1}^{n} m_i \dot{y}_i(t) \right] \leq e^{\epsilon T} \left[ 2 \epsilon e^{\epsilon T} \sum_{i=1}^{n} m_i \dot{y}_i(t) \right] \leq e^{\epsilon T} \left[ 2 \epsilon e^{\epsilon T} \sum_{i=1}^{n} m_i \dot{y}_i(t) \right],$$

and

$$\dot{V}_2(t) = e^{\epsilon T} \left( V_1(t) Q^{-1} \dot{V}(t) Q^{-1} \dot{y}(t) \right).$$

Therefore, if $t = T - \epsilon T$, then

$$\frac{1}{2} \left( |x_i(t)|^2 + |v_i(t)|^2 \right) \leq \text{min}$$

that is, $x(T) = x^*$, $S(T) = S^*$ for $t \geq T_{\min}$. The proof of Theorem 3.2 is completed.

3.3. Global exponential stability

We shall investigate the global exponential stability of system (2.2) since it is very important in designing a neural circuit, it is often desired that a neural network converges in an exponential rate to ensure fast response in the network. To achieve our goal, we need a further assumption that the activation functions are monotonically nondecreasing.

**Theorem 3.4.** Suppose that the assumptions of Theorem 3.2 and $L_i < 0$ are satisfied, then system (2.2) has one unique equilibrium point $(x^*, y^*)$ which is global exponential stability.

**Proof.** Firstly, similar to the proof of Theorem 3.2, we know that system (2.2) has at least one equilibrium point $(x^*, y^*)$.

From (3.4), we can choose a sufficiently small constant $0 < q < 1$ such that

$$0 < \epsilon q < \min\{a_1, \ldots, a_n\}$$

and

$$\left( \begin{array}{ccc} (q-1)Q & MB^T & 0 \\ MB & (MD)^T & 0 \\ 0 & (MD)^T & -(1-q)*R \end{array} \right) < 0.$$
Put (3.37)–(3.39) together, we can deduce that
\[
V(t) \leq e^{\varepsilon t}(V(t), \dot{V}(t), V(t)(t-\tau(t)))
\]
\[
\times \begin{pmatrix}
-(\varepsilon-1)Q & (MB)^T \\
MB & (e^{\varepsilon R+Q+MD} + (MD)^T) \\
0 & (MD)^T + \gamma (1-\tau^*) R
\end{pmatrix}^T
\]
which means that \( V(t) < 0 \) for any \((V(t), \dot{V}(t), V(t)(t-\tau(t))) \neq 0 \), then we get
\[
V(t) \leq V(0).
\]
Thus, one can find from (3.35) and (3.1) that
\[
V(0)e^{-\varepsilon t} \geq 2\varepsilon \sum_{i=1}^{n} m_j \int_{0}^{t} f_i(s) ds
\]
\[
\geq -2\varepsilon \sum_{i=1}^{n} m_j L_j \int_{0}^{t} \text{e}^{-\varepsilon s} s ds
\]
\[
\geq \lambda_{\text{min}}(-LM)e^{-\varepsilon t},
\]
and
\[
\lambda_{\text{min}}(Q)V(t)\leq V(t)(t)QV(t) \leq V(0)e^{-\varepsilon t}.
\]
That is
\[
\|u(t)\| \leq \frac{V(0)}{\lambda_{\text{min}}(-LM)e^{-\varepsilon t}}, \quad \|v(t)\| \leq \frac{V(0)}{\lambda_{\text{min}}(Q)e^{-\varepsilon t}}.
\]
and one can deduce from (3.43) that
\[
\|u(t)\| + \|v(t)\| \leq \sqrt{2 \left( \frac{V(0)}{\lambda_{\text{min}}(-LM)e^{-\varepsilon t}} + \frac{V(0)}{\lambda_{\text{min}}(Q)e^{-\varepsilon t}} \right)} e^{-\varepsilon/2}.\]
This proves that the equilibrium point of system (2.2) is global exponential stability. \( \square \)

**Remark 3.2.** It is worth pointing out that various competitive neural network model with or without constant time delay were investigated by many authors [16–19,23,25]. To the best of our knowledge, however, the case with time varying delay for this model is seldom considered in the literature, which implies that the results obtained in this paper are new. On the other hand, as pointed out in [26,27,31], the property of global convergence in finite time is even more desirable when the minimum must be computed in real time and such a property cannot be displayed by smooth dynamical systems, therefore, the obtained results in the present paper complement previously known results.

### 4. Examples and numerical simulations

In this section, we give two examples to illustrate the results obtained in the previous sections.

**Example 4.1.** We consider a two-neuron competitive neural as follows:
\[
\begin{align*}
\text{STM}: \quad & \frac{d}{dt}x_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} D_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} D_{ij} f_j(x_j(t-\tau(t))) + B_i S_i(t), \\
\text{LTM}: \quad & \frac{d}{dt}S_i(t) = -S_i(t) + f_i(x_i(t)), \quad i = 1, 2,
\end{align*}
\]
where
\[
A = \begin{pmatrix} 2.2 & 0 \\ 0 & 2.2 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0.3 \\ 0.3 & -1 \end{pmatrix}, \quad D^T = \begin{pmatrix} -1.2 & 0.5 \\ 0.5 & -1.2 \end{pmatrix},
\]
and \( \tau(t) = 0.1 \), the parameters \( \varepsilon = 1 \), the discontinuous activation is given by
\[
f_i(s) = \begin{cases} 
0.01s + 0.05 & \text{if } s \geq 0, \\
-0.01s - 0.05 & \text{if } s < 0, \end{cases} \quad i = 1, 2.
\]

It is obvious that \( f(s) = (f_1(x_1), f_2(x_2))' \) is discontinuous, unbounded and nonmonotonic with \( L_i = 0.01 \) (\( i = 1, 2 \)), see Fig. 1.

![Fig. 1. Discontinuous activation function for system (4.1) (left) and system (4.3) (right), respectively.](image)

**Fig. 2.** Time-domain behavior of the state variables \( x_1, x_2, S_1, \) and \( S_2 \) for system (4.1).
admits one unique equilibrium point which is globally asymptotically stable. Consequently, all the conditions in Theorem 3.4 hold, we see Fig. 1.

A straightforward calculation shows that

\[
A = \begin{pmatrix}
-0.9 & 0 & -0.1 & 0 & 0 & 0 \\
0 & -0.9 & 0 & 0.3 & 0 & 0 \\
0.1 & 0 & -1.4 & 0.6 & -1.2 & 0.5 \\
0 & 0.3 & 0.6 & -1.4 & 0.5 & -1.2 \\
0 & 0 & -1.2 & 0.5 & -1.6 & 0 \\
0 & 0 & 0.5 & -1.2 & 0 & -1.6 \\
\end{pmatrix},
\]

and

\[
\lambda_{\min}(-A) = 0.05897, \quad \|2DD^T + D'D^T + BB^T\|_2 = 6.32033.
\]

Consequently, all the conditions in Theorem 3.2 hold, system (4.1) admits one unique equilibrium point which is globally asymptotically stable, and this fact is supported by the numerical simulations in Figs. 2 and 3 with the initial conditions \( \phi(t) = (1, 2.5)^T \) for \( t \in [-0.1, 0] \), and \( \psi(t) = (-2.5, 4)^T \) for \( t \in [-0.1, 0] \).

**Example 4.2.** Consider the following two-dimensional discontinuous competitive neural networks:

\[
\begin{aligned}
\text{STM: } & \dot{x}_1(t) = -4x_1(t) - 2.15f_i(x_1(t)) + 0.3f_j(x_2(t)) - 0.3f_j(x_1(t - \tau(t))) - 0.1f_j(x_2(t - \tau(t))) - 0.85x_1(t), \\
& \dot{x}_2(t) = -4x_2(t) + 0.3f_i(x_1(t)) - 2.05f_j(x_2(t)) - 0.4f_j(x_1(t - \tau(t))) - 0.2f_j(x_2(t - \tau(t))) + 0.15x_2(t), \\
\text{LTM: } & \dot{S}_1(t) = -S_1(t) + f_j(x_1(t)), \\
& \dot{S}_2(t) = -S_2(t) + f_j(x_2(t)),
\end{aligned}
\]

(4.3)

where \( \tau(t) = 0.2 \) and the parameters \( \epsilon = 1/2 \), the discontinuous activation function defined by

\[
f_i(s) = \begin{cases} 
0.002s + 0.005 & \text{if } s \geq 0, \\
0.002s - 0.005 & \text{if } s < 0,
\end{cases} \quad i = 1, 2.
\]

(4.4)

It is obvious that \( f(x) = (f_1(x_1), f_2(x_2))^T \) is discontinuous, unbounded and monotonic nondecreasing with \( L_i = 0.002 \) \( (i = 1, 2) \), see Fig. 1.

Choose \( \mu = 0.5 \), \( R = \begin{pmatrix} 1.6 & 0 \\ 0 & 1.6 \end{pmatrix} \), \( Q = \begin{pmatrix} 1.8 & 0 \\ 0 & 1.8 \end{pmatrix} \), \( M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

A straightforward calculation shows that

\[
A_{\mu} = \begin{pmatrix}
-0.9 & 0 & -0.1 & 0 & 0 & 0 \\
0 & -0.9 & 0 & 0.3 & 0 & 0 \\
0 & 0 & -1.4 & 0.6 & -1.2 & 0.5 \\
0 & 0 & 0.3 & -1.4 & 0.5 & -1.2 \\
0 & 0 & -1.2 & 0.5 & -1.6 & 0 \\
0 & 0 & 0.5 & -1.2 & 0 & -1.6 \\
\end{pmatrix},
\]

and

\[
\lambda_{\min}(-A_{\mu}) = 0.05897, \quad \|2DD^T + D'D^T + BB^T\|_2 = 6.32033.
\]

Choose \( \mu = 0.6 \), we can easily calculate that

\[
A_{\mu} = \begin{pmatrix}
-0.9 & 0 & -0.1 & 0 & 0 & 0 \\
0 & -0.9 & 0 & 0.3 & 0 & 0 \\
0 & 0 & -1.4 & 0.6 & -0.3 & 0.1 \\
0 & 0 & 0.3 & -1.4 & 0.5 & -0.4 \\
0 & 0 & -1.2 & 0.5 & -1.6 & 0 \\
0 & 0 & 0.5 & -1.2 & 0 & -1.6 \\
\end{pmatrix},
\]

and

\[
\lambda_{\min}(-A_{\mu}) = 0.01771, \quad \|2DD^T + D'D^T + BB^T\|_2 = 11.96114.
\]

Consequently, all the conditions in Theorem 3.4 hold, we see that system (4.3) has an equilibrium point which is globally exponentially stable. The simulation results are given in Figs. 4 and 5 with the initial conditions \( \phi(t) = (-1.2, 5)^T \) for \( t \in [-0.2, 0] \), and \( \psi(t) = (1.5, -2)^T \) for \( t \in [-0.2, 0] \) and these numerical simulations clearly verify the effectiveness and feasibility of the theoretical analysis.

**Remark 4.1.** In [17,19,25], the dynamics of competitive neural networks are studied, one can observe that the activation functions are required to satisfy boundedness and differentiability [17], monotonic nondecreasing [19], global Lipschitz condition [19,25], respectively. However, it can be easily seen in the present paper that we successfully remove the boundedness and monotonic nondecreasing properties. Moreover, the activation functions in

![Fig. 3. Three-dimensional trajectory of the state variables \( x_1, x_2 \) and \( S_1, S_2 \) for system (4.1).](image)

![Fig. 4. Time-domain behavior of the state variables \( x_1, x_2, S_1, S_2 \) for system (4.3).](image)
our results are discontinuous, which are more relaxing than the globally Lipschitz condition or differentiability in [17,19,25]. Therefore, the criteria here improve and extend previously known results to some extent.

5. Conclusions

In this paper, a class of delayed competitive neural networks with different time scales and discontinuous activations has been investigated. Employing the Leray–Schauder alternative theorem in a multivalued analysis, linear matrix inequality technique and generalized Lyapunov–like method, we performed a thorough analysis in existence, uniqueness and global stability and the global convergence in finite time of equilibrium point, these results obtained in this paper improve and extend previously known results of the existing references. Finally, two numerical examples have been finally provided to illustrate the performance and effectiveness of the developed approach.

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References


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