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Novel LMI-based condition on global asymptotic stability for BAM neural networks with reaction–diffusion terms and distributed delays

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A B S T R A C T

In this paper, under the assumption that the activation functions only satisfy global Lipschitz conditions, a novel LMI-based sufficient condition for global asymptotic stability of equilibrium point of a class of BAM neural networks with reaction–diffusion terms and distributed delays is obtained by using degree theory, LMI method, inequalities technique and constructing lyapunov functionals. In our results, the assumptions for boundedness and monotonicity in existing papers on the activation functions are removed.

1. Introduction

Bidirectional associative memory (BAM) neural networks model, known as an extension of the unidirectional auto-associator of Hopfield [1–3], was first introduced by Kosko [4,5]. This class of networks possess good application prospects in some fields such as pattern recognition, signal and image process, and artificial intelligence. Such applications heavily depend on the dynamical behavior of the neural networks. Thus, the study of the dynamical behavior is a very necessary step for practical design of neural networks. So far, many stability results such as global exponential stability and global asymptotic stability have been obtained for BAM neural networks with discrete time delays [6–11]. Because neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths, there will be a distribution of propagation delays. It is desired to model them by introducing continuously distributed delays [1–4]. There are generally two kinds of continuously distributed delays in the neural networks system, i.e., finitely distributed delays and infinitely distributed delays. The following neural networks with finitely distributed delays:

\[ x_i(t) = -a_i(x_i(t)) + \sum_{j=1}^{n} w_{ij} \int_{t-n}^{t} g_j(x_j(s)) \, ds \]  

and their variants have been studied in [5–9] based on linear matrix inequality method and other methods. Similarly, the following neural networks with infinitely distributed delays:

\[ x_i(t) = -a_i(x_i(t)) + \sum_{j=1}^{n} w_{ij} \int_{-\infty}^{t} \int_{t-n}^{t} K_{ij}(t-s) g_j(x_j(s)) \, ds \, dt \]  

and their variants have been studied in [2–4,10–20]. The global stability results for system (1.2) and its variants are expressed in different forms, such as M matrix form and algebraic inequality forms. If we consider dynamic behavior of neural networks which only depends on time, the model of BAM neural networks is an ordinary differential equation or a functional differential equation (see [21,22]). In the strict sense, however, diffusion effect cannot be avoided in the neural network models when electrons are moving in an asymmetric electromagnetic field. So we must consider that the space is...
varies with the time. Thus it is desired to model neural networks by introducing continuously distributed delays and reaction–diffusion terms. So, the study of stability for neural networks with distributed time delays and reaction–diffusion terms should be a focused topic of theoretical as well as practical importance. So far, some authors have discussed the stability of some one-dimensional neural networks with reaction–diffusion terms and distributed delays, for example, see [12,16,23–31] by different ways, such as M matrix method, LMI method and algebraic method.

Since two-dimensional neural networks with reaction–diffusion terms and distributed delays consider the interaction between two neural networks system, then two-dimensional neural networks with reaction–diffusion terms and distributed delays will be a greater neural network system and will have more colorful functions in pattern recognition, parallel computing, associative memory, and combinatorial optimization. Hence, the studies of stability behavior of two-dimensional neural networks with reaction–diffusion terms and distributed delays are of greater interest than the studies of stability of one-dimensional neural networks with reaction–diffusion terms and distributed delays.

Motivated by the above idea, in [23], the authors considered the following two-dimensional BAM neural networks with distributed delays and reaction–diffusion terms:

\[
\frac{\partial u_i(x, t)}{\partial t} = \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i}{\partial x_k} \right) - a_i u_i + \sum_{j=1}^{m} w_{ij} g_j(v_j) + \sum_{l=1}^{n} h_{il} f_l(u_l)
\]

\[
\frac{\partial v_j(x, t)}{\partial t} = \frac{\partial}{\partial x_k} \left( D_{jk} \frac{\partial v_j}{\partial x_k} \right) - b_j v_j + \sum_{l=1}^{n} h_{jl} f_l(u_l)
\]

\[
(1.3)
\]

for \( i = 1, 2, \ldots, n, \) \( t > 0, \) where \( x = (x_1, x_2, \ldots, x_n)^T \in \Omega \subset R^n, \) \( \Omega \) is a bounded compact set with smooth boundary \( \partial \Omega \) and \( \mu(\Omega) > 0 \) in space \( R^n; u = (u_1, u_2, \ldots, u_n)^T \in R^n, v = (v_1, v_2, \ldots, v_n)^T \in R^n, u_i(t, x) \) and \( v_j(t, x) \) are the state of the \( i \)th neurons and the \( j \)th neurons at time \( t \) and in space \( x, \) respectively; \( I_i \) and \( J_j \) denote the external inputs on the \( i \)th neurons and the \( j \)th neurons, respectively; \( a_i > 0, b_j > 0, w_{ij}, w_{ji}, h_{ij}, h_{ji} \) are constants, \( a_i \) and \( b_j \) denote the rate with which the \( i \)th neurons and the \( j \)th neurons will reset their potential to the resting state in isolation when disconnected from the networks and external inputs respectively; \( w_{ij}, w_{ji}, h_{ij}, h_{ji} \) denote the connection weights. Smooth functions \( D_{ik} = D_{ik}(x, u) \geq 0 \) and \( D_{jk} = D_{jk}(x, v) \geq 0 \) correspond to the transmission diffusion operators along the \( i \)th neurons and the \( j \)th neurons, respectively.

The boundary conditions and initial conditions of system (1.3) are given by

\[
\begin{align*}
\left. \frac{\partial u_i}{\partial n} \right|_{\partial \Omega} &= (\partial u_i/\partial x_1, \partial u_i/\partial x_2, \ldots, \partial u_i/\partial x_n)^T, \quad i = 1, 2, \ldots, n, \\
\left. \frac{\partial v_j}{\partial n} \right|_{\partial \Omega} &= (\partial v_j/\partial x_1, \partial v_j/\partial x_2, \ldots, \partial v_j/\partial x_n)^T, \quad j = 1, 2, \ldots, n
\end{align*}
\]

and

\[
\begin{align*}
u_i(s, x) &= \phi_i(s, x), \quad s \in (-\infty, 0], \quad i = 1, 2, \ldots, n, \\
v_j(s, x) &= \phi_j(s, x), \quad s \in (-\infty, 0], \quad j = 1, 2, \ldots, n.
\end{align*}
\]

In [23], by using degree theory, M-matrix theory and constructing proper Lyapunov functionals, the authors established global exponential stability of equilibrium point of system (1.3). Their global stability results are expressed in algebraic inequality form.

In [40], the authors considered the following two-dimensional BAM neural networks with time-varying delays and reaction–diffusion terms:

\[
\begin{align*}
\frac{\partial u_i(x, t)}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i}{\partial x_k} \right) - a_i u_i + \sum_{j=1}^{m} w_{ij} g_j(v_j(t - \tau_j(t, x))) + I_i, \\
\frac{\partial v_j(x, t)}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_{jk} \frac{\partial v_j}{\partial x_k} \right) - b_j v_j + \sum_{l=1}^{n} h_{jl} f_l(u_l(t - \tau_l(t, x))) + J_j,
\end{align*}
\]

\[
(1.4)
\]

In [40], by using the method of variation parameter and inequality technique, the sufficient condition in inequality form for the global exponential stability of equilibrium point of system (1.4) is obtained under the assumptions that the activation functions satisfy bounded condition and global Lipschitz condition.

In [32], the authors are concerned with the following interval general two-dimensional BAM neural networks with reaction–diffusion terms and multiple time-varying delays:

\[
\begin{align*}
\frac{\partial u_i(x, t)}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i}{\partial x_k} \right) - a_i u_i + \sum_{j=1}^{m} t_{ij} f_j(u_i(t - \tau_j(t, x))) + I_i, \\
\frac{\partial v_j(x, t)}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( C_{ik} \frac{\partial v_j}{\partial x_k} \right) - b_j v_j + \sum_{l=1}^{n} s_{ij} f_j(u_l(t - \tau_l(t, x))) + J_j,
\end{align*}
\]

\[
(1.5)
\]

The global exponential stability of equilibrium point of system (1.5) is established by using degree theory and analysis technique. The stability result is expressed in algebraic inequality form.

So far, some LMI-based global stability results have been obtained for one-dimensional neural networks with reaction–diffusion terms and time delays under the assumption that the activation functions only satisfy global Lipschitz conditions. To the best of the authors’ knowledge, no LMI-based stability results have been published for two-dimensional neural networks with reaction–diffusion terms and time delays. Motivated by the above discussion, in this paper, our purpose is to establish a novel LMI-based sufficient condition on global exponential stability for system (1.3) under the assumption that the activation functions only satisfy global Lipschitz conditions.

The paper is organized as follows. In Section 2, some lemmas are given. In Section 3, a novel LMI-based sufficient condition is derived for the existence of an equilibrium point of system (1.3). In Section 4, the novel LMI-based sufficient condition on global asymptotic stability of equilibrium point of system (1.3) is obtained. In Section 5, an example is given to show the effectiveness of our results.

2. Preliminaries

First we introduce some notations as follows:

\[
F(0, 0) = (F_1(0, 0), F_2(0, 0), \ldots, F_{2m}(0, 0), F_{2m+1}(0, 0), \ldots, F_{2n}(0, 0)),
\]

\[
r_1 = 2 \min \left\{ \max_{1 \leq i \leq n} \left[ p_i \alpha_i \right], \min_{1 \leq j \leq n} \left[ q_j \beta_j \right] \right\},
\]

\[
r_2 = 2 \max \left\{ \max_{1 \leq i \leq n} \left[ p_i |F_i(0, 0)| \right], \min_{1 \leq j \leq n} \left[ q_j |F_j(0, 0)| \right] \right\},
\]

\[
r = n \max_{1 \leq i \leq n, 1 \leq j \leq n} \left[ r_{ij}^2 \right],
\]
Lemma 3. Existence of an equilibrium point

Theorem 3.1. We assume that the following conditions hold.

(H1) There exist positive constants \( \beta_j, \alpha_i (i = 1, 2, \ldots, n) \) such that for all \( u, v \in R \):
\[
||f(u) - f(v)|| \leq \alpha_i ||u - v||.
\]

(H2) The delay kernels \( k_j, k_m^* : [0, \infty) \rightarrow [0, \infty) \) \( (i, j = 1, 2, \ldots, n) \) are real-valued non-negative continuous functions which satisfy the following conditions:
\[
\int_0^\infty k_j(s) \, ds = 1, \quad \int_0^\infty k_m^*(s) \, ds = 1.
\]

(H3) There exist positive diagonal \( n \) order matrices \( P, Q \), \( Y_1 = (1, 2, 3, 4) \), \( K, N \), positive definite3-order matrices \( M = (m_{ik}) \), \( N^* = (n_{ij}) (i, j = 1, 2, 3) \) with \( m_{ij} > m_{ij} \) \( (i, j = 1, 2, 3) \), \( n_{ij} > n_{ij} \) \( (i, j = 1, 2, 3) \), \( \beta_j > \beta_j \) \( (i, j = 1, 2, 3) \), and \( n \) order matrices \( P_2 = \text{diag}(P_{11}, P_{22}, \ldots, P_{nn}) \), \( P_3 = (p_{ij})_{n \times n} \), \( P_4 = (p_{ij})_{n \times n} \). \( P_3 = (p_{ij})_{n \times n} \) \( \text{such that} \)
\[
Q_1^* = \begin{pmatrix}
T_{11} & T_{12} & T_{13} & 0 & 0 \\
* & T_{22} & 0 & 0 & 0 \\
* & * & T_{33} & 0 & 0 \\
* & * & * & T_{44} & T_{45} \\
* & * & * & * & T_{55}
\end{pmatrix} < 0,
\]
\[
Q_2^* = \begin{pmatrix}
t_{11} & t_{12} & t_{13} & 0 & 0 \\
* & t_{22} & 0 & 0 & 0 \\
* & * & t_{33} & 0 & 0 \\
* & * & * & t_{44} & t_{45} \\
* & * & * & * & t_{55}
\end{pmatrix} < 0,
\]

where
\[
T_{11} = -PA + FY_1F + FKF + A^*I, \quad T_{12} = PW + A^*I,
\]
\[
T_{13} = PW^*, \quad T_{22} = -Y_2 + n^2A^*I,
\]
\[
T_{44} = PA + FY_3F + P_2A + P_2^2A^2, \quad T_{45} = -P_2A^*I, \quad L_1 = (r_0)_{n \times n},
\]
\[
T_{33} = -N^2A^*I, \quad T_{55} = 2mI + 2m^2I + 2m^3I + 2m^4I - Y_4,
\]
\[
T_{11} = -QB + EY_2E + ENE + B^*I, \quad T_{12} = QH + B^*I,
\]
\[
T_{13} = QH^*, \quad T_{22} = -Y_1 + n^2B^*I,
\]
\[
T_{44} = -QB + EY_4E + M_2B + M_2^2B^2, \quad T_{45} = -M_2B^*I, \quad L_2 = (l_0)_{n \times n},
\]
\[
T_{33} = -K + n^2B^*I, \quad T_{55} = 2mI + 2m^2I + 2m^3I + 2m^4I - Y_3,
\]

\[
p = \text{diag}(p_1, p_2, \ldots, p_n), \quad q = \text{diag}(q_1, q_2, \ldots, q_n).
\]
Proof. Note that if \((u^{*T}, v^{*T})^T\) is an equilibrium point of system (1.3) with \(u^{*T} = (u_1^{*T}, u_2^{*T}, \ldots, u_n^{*T}), v^{*T} = (v_1^{*T}, v_2^{*T}, \ldots, v_n^{*T})^T\), then \((u^{*T}, v^{*T})^T\) satisfies for \(i, j = 1, 2, \ldots, n\),

\[
\begin{align*}
q_i u_i^* - \sum_{j=1}^{n} (w_{ij} + w_{ji}^*) v_i^* - l_i &= 0, \\
b_i v_i^* - \sum_{j=1}^{n} (h_{ij} + h_{ji}^*) f_i(u_i^*) - j_i &= 0.
\end{align*}
\]

Define the following map associated with system (1.3):

\[
F(u, v) = \begin{pmatrix}
A u \\
B v
\end{pmatrix} - \begin{pmatrix}
W + W^* & 0 \\
0 & H + H^*
\end{pmatrix} \begin{pmatrix}
\frac{g(v)}{f(u)} \\
\frac{f(u)}{g(v)}
\end{pmatrix} + F(0, 0),
\]

where

\[
\begin{align*}
W &= (w_{ij})_{n \times n}, \quad W^* = (w_{ji}^*)_{n \times n}, \\
H &= (h_{ij})_{n \times n}, \quad H^* = (h_{ji}^*)_{n \times n}, \\
u &= (u_1, u_2, \ldots, u_n)^T, \quad v = (v_1, v_2, \ldots, v_n)^T, \\
l^* &= (l_1, l_2, \ldots, l_n)^T, \quad j^* = (j_1, j_2, \ldots, j_n)^T, \\
A &= (a_{ij} u_i, a_{ij} u_j, \ldots, a_{ij} u_n)^T, \quad B = (b_1 v_1, b_2 v_2, \ldots, b_n v_n)^T,
\end{align*}
\]

\[
f(u) = (f_1(u_1), f_2(u_2), \ldots, f_n(u_n))^T, \quad g(v) = (g_1(v_1), g_2(v_2), \ldots, g_n(v_n))^T.
\]

Obviously, the equilibrium point of system (1.3) is the solution of equation \(F(u, v) = 0\). Rewrite \(F(u, v)\) as follows:

\[
F(u, v) = \begin{pmatrix}
A u \\
B v
\end{pmatrix} - \begin{pmatrix}
W + W^* & 0 \\
0 & H + H^*
\end{pmatrix} \begin{pmatrix}
\frac{g(v)}{f(u)} \\
\frac{f(u)}{g(v)}
\end{pmatrix} + F(0, 0),
\]

where

\[
g(v) = g(v) - g(0), \quad f(u) = f(u) - f(0).
\]

We define a bounded open \(\Omega\) and a mapping \(G(\alpha, u, v)\) as follows:

\[
\Omega_\alpha = \left\{ (u^T, v^T) : \| (u^T, v^T) \| < \sqrt{\sum_{i=1}^{n} u_i^2 + \sum_{j=1}^{n} v_j^2} < \alpha^2 \right\},
\]

where \(R^2 > 2\alpha^2 \gamma / \gamma\) and

\[
G(\alpha, u, v, \alpha) = \begin{pmatrix}
A u \\
B v
\end{pmatrix} - \begin{pmatrix}
W + W^* & 0 \\
0 & H + H^*
\end{pmatrix} \begin{pmatrix}
\frac{g(v)}{f(u)} \\
\frac{f(u)}{g(v)}
\end{pmatrix} + \alpha F(0, 0),
\]

where \((u, v) \in \Omega_\alpha, \alpha \in [0, 1]\) is a parameter. Now we prove that \(G(\alpha, u, v) \neq 0\), when \(\alpha \in [0, 1], (u, v) \in \partial \Omega\). Since

\[
2(u^T P, v^T Q) \left(1 + \alpha \right) \begin{pmatrix}
A u \\
B v
\end{pmatrix} + \alpha F(0, 0)
\]

\[
= 2(1 + \alpha) \left( \sum_{i=1}^{n} p_i u_i^2 + \sum_{j=1}^{n} q_j v_j^2 \right) + 2 \sum_{i=1}^{n} p_i u_i F_i(0, 0)
\]

\[
+ 2 \sum_{j=1}^{n} q_j v_j F_j(0, 0)
\]

\[
\geq 2(1 + \alpha) \left( \sum_{i=1}^{n} p_i u_i^2 + \sum_{j=1}^{n} q_j v_j^2 \right) - 2 \sum_{i=1}^{n} p_i |u_i| |F_i(0, 0)|
\]

\[
- 2 \sum_{j=1}^{n} q_j |v_j| |F_j(0, 0)|,
\]

then we have

\[
2(u^T P, v^T Q) G(\alpha, u, v)
\]

\[
= 2(u^T P, v^T Q) \left(1 + \alpha \right) \begin{pmatrix}
A u \\
B v
\end{pmatrix} + \alpha F(0, 0) - \left( \begin{pmatrix}
W + W^* \frac{g(v)}{f(u)} \\
H + H^* \frac{f(u)}{g(v)}
\end{pmatrix} \right)
\]

\[
\geq 2(1 + \alpha) \left( \sum_{i=1}^{n} p_i u_i^2 + \sum_{j=1}^{n} q_j b_j v_j^2 \right) - 2 \sum_{i=1}^{n} p_i |u_i| |F_i(0, 0)|
\]

\[
- 2 \sum_{j=1}^{n} q_j |v_j| |F_j(0, 0)|.
\]
\[
\sum_{i=1}^{n} l_i h_j(f_i(u_i)-f_i(0))^2 \leq \sum_{i=1}^{n} |f_i(u_i)-f_i(0)|^2, \tag{3.10}
\]

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}^* W_{ij}^*(g_j(v_j)-g_j(0))^2 \leq \sum_{i=1}^{n} |g_j(v_j)-g_j(0)|^2, \tag{3.11}
\]

\[
2 \sum_{j=1}^{n} r_{gj} W_{gj}(g_j(v_j)-g_j(0)) \sum_{i=1}^{n} r_{ij}^* W_{ij}^*(g_j(v_j)-g_j(0)) \leq \left[ \sum_{j=1}^{n} r_{gj} W_{gj}(g_j(v_j)-g_j(0)) \right]^2 + \left[ \sum_{j=1}^{n} r_{ij}^* W_{ij}^*(g_j(v_j)-g_j(0)) \right]^2 \left[ \sum_{j=1}^{n} r_{gj} W_{gj}(g_j(v_j)-g_j(0)) \right]^2 \leq \gamma^r \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 + r \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2, \tag{3.12}
\]

\[
2 \sum_{i=1}^{n} l_i h_j(f_i(u_i)-f_i(0)) \sum_{j=1}^{n} r_{ij}^* W_{ij}^*(f_i(u_i)-f_i(0)) \leq \left[ \sum_{i=1}^{n} l_i h_j(f_i(u_i)-f_i(0)) \right]^2 + \left[ \sum_{j=1}^{n} r_{ij}^* W_{ij}^*(f_i(u_i)-f_i(0)) \right]^2 \left[ \sum_{i=1}^{n} l_i h_j(f_i(u_i)-f_i(0)) \right]^2 \leq \gamma^r \sum_{i=1}^{n} |f_i(u_i)-f_i(0)|^2 + l \sum_{i=1}^{n} |f_i(u_i)-f_i(0)|^2, \tag{3.13}
\]

\[
-2p_{o1}a_{oi} \sum_{j=1}^{n} r_{ij}^* W_{ij}^*(g_j(v_j)-g_j(0)) \leq p_{o1}u_{i1}^2 + p_{o1}u_{i1} \sum_{j=1}^{n} (r_{ij}^* W_{ij}^*)^2 \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 
\leq p_{o1}u_{i1}^2 + p_{o1}a_u^* \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 \leq p_{o1}u_{i1}^2 + p_{o1}a_u^* \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 \tag{3.14}
\]

and

\[
-2m_{o1}b_{v1} \sum_{j=1}^{n} l_j h_j(f_i(u_i)-f_i(0)) \leq m_{o1}b_{v1}^2 + m_{o1}b_{v1} \sum_{j=1}^{n} |f_i(u_i)-f_i(0)|^2 \leq m_{o1}b_{v1}^2 + m_{o1}b_{v1} \sum_{j=1}^{n} |f_i(u_i)-f_i(0)|^2 \tag{3.15}
\]

Substituting (3.8), (3.9), (3.12) and (3.14) into (3.6) and substituting (3.10), (3.11), (3.13) and (3.15) into (3.7), it follows that

\[
2r \sum_{j=1}^{n} (g_j(v_j)-g_j(0))^2 + (p_{o1}a_{oi} + 2a_u^*) \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 
+ (p_{o1}u_{i1}^2 + p_{o1}a_u^* \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 \tag{3.16}
\]

and

\[
2l \sum_{i=1}^{n} (f_i(u_i)-f_i(0))^2 + (m_{o1}b_{v1} + 2m_{o1}b_{v1}) \sum_{j=1}^{n} |f_i(u_i)-f_i(0)|^2 
+ (m_{o1}b_{v1} + m_{o1}b_{v1}) \sum_{j=1}^{n} l_j h_j(f_i(u_i)-f_i(0)) \geq 0 \tag{3.17}
\]

Rewriting (3.16) and (3.17) as matrix form, it follows that

\[
\begin{align*}
\|g(v)\|^2 (2m_{o1}b_{v1} + 2m_{o1}b_{v1}^2 + m_{o1}b_{v1} + m_{o1}b_{v1}^2) \|u\|^2 + \|f_i(u_i)-f_i(0)\|^2 
&\geq 0.
\end{align*}
\tag{3.18}
\]

From Lemma 5, we have by letting \(y_4 = y_5 = \cdots = y_n = 0\),

\[
(a_{ij}^2 - a_{ij}a_{ij}^2)Y_i^2 + 2(a_{ij}a_{ij} - a_{ij}a_{ij}^2)Y_iY_j + 2(a_{ij}a_{ij} - a_{ij}a_{ij}^2)Y_iY_j 
+ 2(a_{ij}a_{ij} - a_{ij}a_{ij}^2)Y_iY_j < 0.
\tag{3.19}
\]

Letting

\[
y_1 = \sum_{j=1}^{n} (g_j(v_j)-g_j(0)), \quad y_2 = \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|, \quad y_3 = u_i(t),
\]

\[
y_1 = \sum_{j=1}^{n} (f_i(u_i)-f_i(0)), \quad y_2 = \sum_{j=1}^{n} |f_i(u_i)-f_i(0)|, \quad y_3 = v_i(t),
\]

it follows that by using Lemma 5 since \(M \) and \(N^* \) are positive definite matrices

\[
(a_{ij}^2 - a_{ij}a_{ij}^2) \left\{ \sum_{j=1}^{n} (g_j(v_j)-g_j(0))^2 \right\} 
+ (a_{ij}^2 - a_{ij}a_{ij}^2) \left\{ \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 \right\} 
+ (a_{ij}^2 - a_{ij}a_{ij}^2) \left\{ \sum_{j=1}^{n} (f_i(u_i)-f_i(0))^2 \right\} 
\]

and

\[
(b_{ij}^2 - b_{ij}b_{ij}^2) \left\{ \sum_{j=1}^{n} (g_j(v_j)-g_j(0))^2 \right\} 
+ (b_{ij}^2 - b_{ij}b_{ij}^2) \left\{ \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 \right\} 
+ (b_{ij}^2 - b_{ij}b_{ij}^2) \left\{ \sum_{j=1}^{n} (f_i(u_i)-f_i(0))^2 \right\}
\tag{3.20}
\]

from which together with the conditions \(a_{ij}a_{ij} > a_{ij}^2, \quad a_{ij}a_{ij} > a_{ij}^2, \quad b_{ij}b_{ij} > b_{ij}^2, \quad b_{ij}b_{ij} > b_{ij}^2\), by using Lemma 4 we have

\[
(a_{ij}a_{ij} - a_{ij}a_{ij}^2) \sum_{j=1}^{n} (g_j(v_j)-g_j(0))^2 + n(a_{ij}a_{ij} - a_{ij}a_{ij}^2) \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 
\geq 0:
\tag{3.19}
\]

and

\[
(b_{ij}b_{ij} - b_{ij}b_{ij}^2) \sum_{j=1}^{n} (g_j(v_j)-g_j(0))^2 + n(b_{ij}b_{ij} - b_{ij}b_{ij}^2) \sum_{j=1}^{n} |g_j(v_j)-g_j(0)|^2 
\geq 0:
\tag{3.20}
\]

Rewriting (3.19) and (3.20) as matrix form, it follows that

\[
na_{ij} \|g(v)\|^2 + 2A_{ij} \|g(v)\|^2 \|u\|^2 + 2A_{ij} \|g(v)\|^2 \|v\|^2 
\geq 0.
\tag{3.21}
\]
and

\[ \nu B_f T(u) + u B_f T(u) + V_0 v^2 iv + B_0 v^2 iv + n B_f T(u) + \bar{T}(u) > 0. \]  

(3.22)

Substituting (3.2)–(3.5), (3.18), (3.21) and (3.22) into (3.1), it follows that by using Lemma 4,

\[ 2u^T P \bar{T}^2 Q G \dot{\mu}(u, v, u, v) \geq 2 \sum_{i=1}^{n} p_i a_i u_i^2 + 2 \sum_{i=1}^{n} q_i b_i v_i^2 - 2 \sum_{i=1}^{n} p_i |u_i| |F_i(u, 0)| \]

\[ - 2 \sum_{i=1}^{n} q_i |v_i| |F_i(u, 0)| + |u|^2 F_{2} F_{u} F_{2} F_{u} + \bar{T}(u) \]

\[ + \nu^2 F_{2} F_{u} F_{2} F_{u} + \bar{T}(u) \]

\[ - 2 \sum_{i=1}^{n} \frac{|u_i|}{|v_i|} F_{i}(u, 0) \geq 0. \]

Proof: Rewrite system (1.3) as the following form:

\[ \frac{d u(t, x)}{dt} = \frac{1}{k} \sum \frac{\partial \mu(t, x)}{\partial k_1} \frac{1}{k} \sum \frac{\partial \mu(t, x)}{\partial k_2} - A u(t, x) \]

\[ + W g(v(t,x)) + v^2 \int_{-\infty}^{t} K(t-s) g(v(s, x)) ds + \bar{f}, \]

\[ \frac{d v(t, x)}{dt} = \frac{1}{k} \sum \frac{\partial \mu(t, x)}{\partial k_1} \frac{1}{k} \sum \frac{\partial \mu(t, x)}{\partial k_2} - B v(t, x) + H f(u(t, x)) \]

\[ + H \int_{-\infty}^{t} K(t-s) g(v(s, x)) ds + \bar{f}, \]  

(4.1)

where

\[ A = \text{diag}(a_1, a_2, \ldots, a_n), \quad B = \text{diag}(b_1, b_2, \ldots, b_n), \]

\[ D_k = \text{diag}(D_{k1}, D_{k2}, \ldots, D_{kn}), \]

\[ f(u(t, x)) = \left( u_1(t, x), u_2(t, x), \ldots, u_n(t, x) \right)^T, \]

\[ v(t, x) = \left( v_1(t, x), v_2(t, x), \ldots, v_n(t, x) \right)^T, \]

\[ f(t, x) = \left( f_1(t, x), f_2(t, x), \ldots, f_n(t, x) \right)^T, \]

\[ g(v(t, x)) = \left( g_1(v(t, x), \ldots, g_n(v(t, x)) \right)^T, \]

\[ K(t-s) = \text{diag}(K_1(t-s), K_2(t-s), \ldots, K_n(t-s)), \]

\[ K^T(t-s) = \text{diag}(K_1(t-s), K_2(t-s), \ldots, K_n(t-s)). \]

From the proof of Theorem 3.1, we know that system (1.3) [or (4.1)] has one equilibrium point, say, \((u^*, v^*)^T\), where \(u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T, v^* = (v_1^*, v_2^*, \ldots, v_n^*)^T\). We define the following Lyapunov functional:

\[ V(t) = V_1(t) + V_2(t), \]

(4.2)

where

\[ V_1(t) = \int_{a}^{b} (u(t,x) - u^*)^2 P(u(t,x) - u^*) \]

\[ + \int_{a}^{b} (v(t,x) - v^*)^2 Q(v(t,x) - v^*) \]

\[ V_2(t) = \int_{a}^{b} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( K_1(\theta(\psi, \psi)) - v^* \right)^2 EN \int_{0}^{+\infty} \psi(\theta(\psi, \psi)) \psi \right) \]

\[ + \int_{a}^{b} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( K_2(\theta(\psi, \psi)) - u^* \right)^2 FK \int_{0}^{+\infty} \psi(\theta(\psi, \psi)) - u^* \right) \]

\[ + \int_{a}^{b} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( K_3(\theta(\psi, \psi)) - v^* \right)^2 T \int_{0}^{+\infty} \psi(\theta(\psi, \psi)) - v^* \right) \]

\[ + \int_{a}^{b} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( K_4(\theta(\psi, \psi)) - u^* \right)^2 \]

\[ + \int_{a}^{b} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( K_5(\theta(\psi, \psi)) - v^* \right)^2 \]

Computing the derivative of \(V_1(t)\) along the solution of system (4.1), we have

\[ V_1'(t) = \int_{a}^{b} \int_{-\infty}^{\infty} \left( 2u(t,x) - u^* \right)^2 P \sum_{k=1}^{n} \frac{\partial u(t,x)}{\partial k} \int_{-\infty}^{\infty} K(t-s) g(v(s, x)) \]

\[ + 2u(t,x) - u^* \right)^2 P W g(v(t,x) - v^*) \]

\[ + 2u(t,x) - u^* \right)^2 P W f(u(t,x) - f(u^*)) \]

\[ + 2u(t,x) - u^* \right)^2 Q f(u(t,x) - f(u^*)) \]

\[ + 2u(t,x) - u^* \right)^2 Q f(u(t,x) - f(u^*)) \]

(4.3)

\[ + 2u(t,x) - u^* \right)^2 Q f(u(t,x) - f(u^*)) \]

\[ + 2u(t,x) - u^* \right)^2 Q f(u(t,x) - f(u^*)) \]

\[ + 2u(t,x) - u^* \right)^2 Q f(u(t,x) - f(u^*)) \]

\[ + 2u(t,x) - u^* \right)^2 Q f(u(t,x) - f(u^*)) \]

\[ + 2u(t,x) - u^* \right)^2 Q f(u(t,x) - f(u^*)) \]

\[ + 2u(t,x) - u^* \right)^2 Q f(u(t,x) - f(u^*)) \]
From the boundary condition and in a similar manner to \([27,34]\), we have
\[
2 \int_{\Omega} \left[ \frac{\partial u(t,x)}{\partial t} + \sum_{k=1}^{r} \frac{\partial}{\partial x_k} \left(D_{\sigma_k} \frac{\partial u(t,x)}{\partial x_k} \right) \right] \, dx \leq 0, \tag{4.4}
\]
and
\[
2 \int_{\Omega} \left[ \frac{\partial v(t,x)}{\partial t} + \sum_{k=1}^{r} \frac{\partial}{\partial x_k} \left(D_{\sigma_k} \frac{\partial v(t,x)}{\partial x_k} \right) \right] \, dx \leq 0. \tag{4.5}
\]
Substituting (4.4) and (4.5) into (4.3), it follows that
\[
V_{\gamma}^{(t)}(i) \leq \int_{\Omega} \left\{ -2[u(t,x) - u^*]^3 P \left(A[u(t,x) - u^*] \right) + 2[u(t,x) - u^*]^2 \sum_{k=1}^{r} \left( D_{\sigma_k} \frac{\partial u(t,x)}{\partial x_k} \right)^2 \right\} \, dx \leq 0, \tag{4.6}
\]
From (H1), noting that \(Y_{\gamma} > 0\) for \(i = 1, 2, 3, 4\), \(K > 0\), \(N > 0\) are diagonal, we have
\[
[f(u(t,x)) - f(u^*)]^3 P[f(u(t,x)) - f(u^*)] \leq [u(t,x) - u^*]^2 F_{\gamma}(u(t,x) - u^*), \quad i = 1, 3, \tag{4.7}
\]
\[
[g(v(t,x)) - g(v^*)]^3 P[g(v(t,x)) - g(v^*)] \leq [v(t,x) - v^*]^2 F_{\gamma}(v(t,x) - v^*), \quad i = 2, 4. \tag{4.8}
\]
The derivatives of \(V_{\gamma}(t)\) are as follows:
\[
V_{\gamma}^{(t)}(t) = \int_{\Omega} \left\{ (v(t,x) - v^*)^T \left[ 3 \left[ \int_{0}^{t} K(t-s)(v(t-x)) - g(v^*) \, ds \right] \right] \right\} \, dx \leq 0. \tag{4.9}
\]
and
\[
\int_{t}^{\infty} K^*(t-s)f(u(t,s) - u^*) \, dx \leq \int_{t}^{\infty} K^*(t-s)f(u(t,s) - u^*) \, dx \leq 0. \tag{4.10}
\]
The derivatives of \(V_{\gamma}(t)\) are as follows:
\[
\int_{\Omega} \left\{ (v(t,x) - v^*)^T \left[ 3 \left[ \int_{0}^{t} K(t-s)(v(t-x)) - g(v^*) \, ds \right] \right] \right\} \, dx \leq 0. \tag{4.11}
\]
For \(i = 1, 2, \ldots, n, j = 1, 2, \ldots, n\), the following inequalities hold:
\[
\sum_{i=1}^{n} \left\{ -p_{\sigma_0} d_{\sigma_0}(u(t,x) - u^*) + \sum_{j=1}^{m} r_{\sigma_j} w_{\sigma_j}(g(v_j(t,x)) - g(v_j^*)) \right\} \right\} \, dx \leq 0. \tag{4.12}
\]
\[
\sum_{i=1}^{n} l_i h_i \left( f_i(u_i(t,x)) - f_i(u_i^*) \right)^2 \leq I \sum_{i=1}^{n} \left( f_i(u_i(t,x)) - f_i(u_i^*) \right)^2,
\]

\[
\left( \sum_{i=1}^{n} l_i h_i \int_{-\infty}^{\infty} k_i^0(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \right)^2 \leq \Gamma \left( \sum_{i=1}^{n} \int_{-\infty}^{\infty} k_i^0(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \right)^2.
\]

\[
= \Gamma \sum_{i=1}^{n} \left( \int_{-\infty}^{\infty} k^0_i(u)(f_i(u_i(t,x)) - f_i(u_i^*)) \, du \right)^2.
\]

\[
2 \sum_{j=1}^{n} r_j w_j \left( g_j(v_j(t,x)) - g_j(v_j^*) \right)^2 \sum_{i=1}^{n} r^2_j w^2_j \,
\]

\[
\times \int_{-\infty}^{\infty} k^0_j(t-s) f_j(u_j(t,x)) - f_j(u_j^*) \, ds \leq \left( \sum_{j=1}^{n} r_j^2 w^2_j \int_{-\infty}^{\infty} k^0_j(t-s) f_j(u_j(t,x)) - f_j(u_j^*) \, ds \right)^2 \leq \left( \sum_{j=1}^{n} r_j^2 w^2_j \right) \Gamma \left( \sum_{j=1}^{n} \int_{-\infty}^{\infty} k^0_j(u)(f_j(u_j(t,x)) - f_j(u_j^*)) \, du \right)^2.
\]

\[
2 \sum_{j=1}^{n} l_j h_j \left( f_j(u_j(t,x)) - f_j(u_j^*) \right)^2 \sum_{i=1}^{n} l^2_j h^2_j \,
\]

\[
\times \int_{-\infty}^{\infty} k^0_j(t-s) f_j(u_j(t,x)) - f_j(u_j^*) \, ds \leq \left( \sum_{j=1}^{n} l_j^2 h^2_j \int_{-\infty}^{\infty} k^0_j(t-s) f_j(u_j(t,x)) - f_j(u_j^*) \, ds \right)^2 \leq \left( \sum_{j=1}^{n} l_j^2 h^2_j \right) \Gamma \left( \sum_{j=1}^{n} \int_{-\infty}^{\infty} k^0_j(u)(f_j(u_j(t,x)) - f_j(u_j^*)) \, du \right)^2.
\]

\[
220 & \sum_{j=1}^{n} r_j w_j \left( g_j(v_j(t,x)) - g_j(v_j^*) \right)^2 \sum_{i=1}^{n} r^2_j w^2_j \,
\]

\[
\times \int_{-\infty}^{\infty} k^0_j(t-s) f_j(u_j(t,x)) - f_j(u_j^*) \, ds \leq \left( \sum_{j=1}^{n} r_j^2 w^2_j \int_{-\infty}^{\infty} k^0_j(t-s) f_j(u_j(t,x)) - f_j(u_j^*) \, ds \right)^2 \leq \left( \sum_{j=1}^{n} r_j^2 w^2_j \right) \Gamma \left( \sum_{j=1}^{n} \int_{-\infty}^{\infty} k^0_j(u)(f_j(u_j(t,x)) - f_j(u_j^*)) \, du \right)^2.
\]

\[
-2p_j a_j [u_j(t,x) - u_j^*] \sum_{j=1}^{n} r^2_j w^2_j \int_{-\infty}^{\infty} k^0_j(u)[g_j(v_j(t-u,x)) - g_j(v_j(x))] \, du \leq \left( \sum_{j=1}^{n} p_j a_j^2 [u_j(t,x) - u_j^*]^2 \right) \Gamma \left( \sum_{j=1}^{n} \int_{-\infty}^{\infty} k^0_j(u)(f_j(u_j(t,x)) - f_j(u_j^*)) \, du \right)^2.
\]

\[
-2 p_j a_j [u_j(t,x) - u_j^*] \sum_{j=1}^{n} r^2_j w^2_j \int_{-\infty}^{\infty} k^0_j(u)[f_j(u_j(t,u,x)) - f_j(u_j^*)] \, du \leq \left( \sum_{j=1}^{n} p_j a_j [u_j(t,x) - u_j^*]^2 \right) \Gamma \left( \sum_{j=1}^{n} \int_{-\infty}^{\infty} k^0_j(u)(f_j(u_j(t,x)) - f_j(u_j^*)) \, du \right)^2.
\]

\[
\sum_{i=1}^{n} l_i h_i \left( f_i(u_i(t,x)) - f_i(u_i^*) \right)^2 \sum_{i=1}^{n} l^2_i h^2_i \,
\]

\[
\times \int_{-\infty}^{\infty} k^0_i(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \leq \left( \sum_{i=1}^{n} l_i^2 h^2_i \int_{-\infty}^{\infty} k^0_i(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \right)^2 \leq \left( \sum_{i=1}^{n} l_i^2 h^2_i \right) \Gamma \left( \sum_{i=1}^{n} \int_{-\infty}^{\infty} k^0_i(u)(f_i(u_i(t,x)) - f_i(u_i^*)) \, du \right)^2.
\]

\[
\sum_{i=1}^{n} l_i^2 h^2_i \int_{-\infty}^{\infty} k^0_i(u)(f_i(u_i(t,x)) - f_i(u_i^*)) \, du \geq 0.
\]

\[
2 \sum_{i=1}^{n} l_i h_i \left( f_i(u_i(t,x)) - f_i(u_i^*) \right)^2 \sum_{i=1}^{n} l^2_i h^2_i \,
\]

\[
\times \int_{-\infty}^{\infty} k^0_i(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \leq \left( \sum_{i=1}^{n} l_i^2 h^2_i \int_{-\infty}^{\infty} k^0_i(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \right)^2 \leq \left( \sum_{i=1}^{n} l_i^2 h^2_i \right) \Gamma \left( \sum_{i=1}^{n} \int_{-\infty}^{\infty} k^0_i(u)(f_i(u_i(t,x)) - f_i(u_i^*)) \, du \right)^2.
\]

\[
\sum_{i=1}^{n} l_i h_i \left( f_i(u_i(t,x)) - f_i(u_i^*) \right)^2 \sum_{i=1}^{n} l^2_i h^2_i \,
\]

\[
\times \int_{-\infty}^{\infty} k^0_i(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \leq \left( \sum_{i=1}^{n} l_i^2 h^2_i \int_{-\infty}^{\infty} k^0_i(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \right)^2 \leq \left( \sum_{i=1}^{n} l_i^2 h^2_i \right) \Gamma \left( \sum_{i=1}^{n} \int_{-\infty}^{\infty} k^0_i(u)(f_i(u_i(t,x)) - f_i(u_i^*)) \, du \right)^2.
\]

\[
2 \sum_{i=1}^{n} l_i h_i \left( f_i(u_i(t,x)) - f_i(u_i^*) \right)^2 \sum_{i=1}^{n} l^2_i h^2_i \,
\]

\[
\times \int_{-\infty}^{\infty} k^0_i(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \leq \left( \sum_{i=1}^{n} l_i^2 h^2_i \int_{-\infty}^{\infty} k^0_i(t-s) f_i(u_i(t,x)) - f_i(u_i^*) \, ds \right)^2 \leq \left( \sum_{i=1}^{n} l_i^2 h^2_i \right) \Gamma \left( \sum_{i=1}^{n} \int_{-\infty}^{\infty} k^0_i(u)(f_i(u_i(t,x)) - f_i(u_i^*)) \, du \right)^2.
\]
From (4.27), it follows that

\[ \nA_2 \sum_{j=1}^{n} \left[ \left( \int_{-\infty}^{t} k_j(u_j) G_j(u_j(t-u,x)) \right) \right] + nA_3 \sum_{j=1}^{n} \left[ \left( \int_{-\infty}^{t} k_j(u_j) G_j(u_j(t-u,x)) \right) \right] + nA_4 \sum_{j=1}^{n} \sum_{i=1}^{m} k_{ij}(u_i(t-x)) < 0. \]

(4.24)

using inequality \(2ab \leq a^2 + b^2\) and Lemma 4, it follows that

\[ nA_2 \sum_{j=1}^{n} \left[ \left( \int_{-\infty}^{t} k_j(u_j) G_j(u_j(t-u,x)) \right) \right] + nA_3 \sum_{j=1}^{n} \left[ \left( \int_{-\infty}^{t} k_j(u_j) G_j(u_j(t-u,x)) \right) \right] + nA_4 \sum_{j=1}^{n} \sum_{i=1}^{m} k_{ij}(u_i(t-x)) < 0. \]

(4.25)

and

\[ nB_4 \sum_{i=1}^{n} \left[ \left( \int_{-\infty}^{t} k(u_i(t,x)) \right) \right] + nB_5 \sum_{i=1}^{n} \left[ \left( \int_{-\infty}^{t} k(u_i(t,x)) \right) \right] + nB_6 \sum_{i=1}^{n} \sum_{i=1}^{m} k_{ij}(u_i(t-x)) < 0. \]

(4.26)

Note that

\[ \int_{-\infty}^{t} k(u(t,x)) \, du = \int_{-\infty}^{t} k(t-s) g(u(s,x)) \, ds, \]

\[ \int_{-\infty}^{t} k(u(t,x)) \, du = \int_{-\infty}^{t} k(t-s) g(u(s,x)) \, ds. \]

From (4.2), (4.6), (4.7)-(4.10), (4.21), (4.22), (4.25) and (4.26), it follows that

\[ V(t) \leq \int_{D} \left( \xi Q_{15} \xi^{T} + \eta Q_{20} \eta^{T} \right) \, dx. \]

(4.27)

where

\[ \xi(t,x) = \left( u(t,x) - \xi(t), G(u(t,x)) - \xi(t) \right)^{T}, \]

\[ \eta(t,x) = \left( \eta(t,x) - \eta(t), G(\eta(t,x)) - \eta(t) \right)^{T}. \]

From (4.27), it follows that

\[ V(t) \leq 0. \]

Thus, from the lyapunov stability theory, it follows that the equilibrium point of system (1.3) is globally asymptotically stable. The completes the proof.

**Remark 1.** In our result on global stability, the two inequalities of parameters in stability result in [23,32] are replaced with \(Q_{15}^{*} < 0\) and \(Q_{20}^{*} < 0\) in Theorem 3.1 in our paper. It is easy to verify that the first inequality in [23,32] is different from \(T_{11}^{*} \) in Theorem 3.1, i.e., \(T_{11}^{*} \) cannot imply the first inequality in [23,32], the first inequality in [23,32] cannot imply \(T_{11}^{*} \) in Theorem 3.1. The second inequality in stability result in [23,32] is also different from \(T_{11}^{*} \) in Theorem 3.1 in our paper. Therefore, the two inequalities in [23,32] are different from \(Q_{15}^{*} < 0, Q_{20}^{*} < 0 \) in Theorem 3.1 in our paper. On the other hand, the conditions (ii) \( \int_{-\infty}^{t} k(t-s) g(u(s,x)) \, ds < \infty, \) (iii) \( \int_{-\infty}^{t} k(t-s) g(u(s,x)) \, ds < \infty \) in [23] are removed in our paper. Therefore, the activation functions satisfy bounded condition and global Lipschitz condition, while in our paper, the activation functions only satisfy global Lipschitz condition. Hence, the bounded condition on the activation functions in [40] is removed in our paper. Thus we establish a new LMI-based sufficient condition on global stability of the equilibrium point for BAM neural networks with reaction–diffusion and delays.

**Remark 2.** Though degree theory is applied to discuss the global stability of equilibrium point of system (1.3), the techniques used in our paper are different from those obtained in [23,32]. Moreover, the method used in our paper can be applied to discuss the global stability of other neural networks. When reaction–diffusion terms in system (1.3) become zero, system (1.3) reduces to a BAM neural network with delays. Our result on global stability is also different from those obtained in [36,37].

Compared with the results in [35,37–39,41], in our results, the boundedness in [35,37–39,41] and monotonicity in [37] on the activation functions are removed in our paper and we just require that the activation functions are only globally Lipschitz continuous. On the other hand, in LMI condition, the LMI matrices in our paper do not contain the boundedness of delays, while in [37], the LMI matrices contain the boundedness of delays.

Compared with the results in [36], the boundedness assumption on the activation functions in [36] is removed in our paper, the assumptions (ii) and (iii) on the kernels in [36] are removed in our paper. On the other hand, the first inequality in stability condition in [36] is different from \(T_{11}^{*} \) in Theorem 3.1 in our paper, the second inequality is different from \(T_{11}^{*} \) in Theorem 3.1 in our paper.

5. An example

**Example 1.** Consider the following two-dimensional BAM neural networks with distributed delays and reaction–diffusion terms:

\[ \frac{\partial u_{ij}(t,x)}{\partial t} = \sum_{k=1}^{n} \sum_{j=1}^{m} \left( D_{ik} \frac{\partial u_{ij}(t,x)}{\partial x_k} \right) - \alpha_{ij} x_{ij} + \sum_{j=1}^{n} w_{ij} g_{ij}(v_{ij}), \]

\[ \frac{\partial v_{ij}(t,x)}{\partial t} = \sum_{k=1}^{n} \sum_{j=1}^{m} \left( D_{ik} \frac{\partial v_{ij}(t,x)}{\partial x_k} \right) - \beta_{ij} v_{ij} + \sum_{j=1}^{n} h_{ij} f_{ij}(u_{ij}), \]

where

\[ \int_{-\infty}^{t} k(s) \, ds = \int_{-\infty}^{t} k(s) \, ds = 1, \]

\[ i = 1, j = 2, \quad D_{ik} = D_{kj}(t,x,u) \geq 0, \quad D_{ik} = D_{kj}(t,x,v) \geq 0, \]

\[ w_{ij} = w_{ij} = h_{ij} = h_{ij} = 1, \quad a_{ij} = b_{ij} = 2, \quad f_{ij}(y) = |y| + 1, \quad g_{ij}(x) = |x| + 1. \]

In Theorem 3.1, \( a_{ij} = b_{ij} = 1 \), it is easy to verify that the two inequalities of parameters in [23,32] are not satisfied. Therefore, the result on global stability in [23,32] cannot ensure the global
stability of equilibrium point of system (5.1). Since the activation functions in [40] are bounded, while the activation functions in system (5.1) are only Lipschitz continuous, then the result in [40] cannot ensure the global stability of equilibrium point of system (5.1). When reaction–diffusion terms in system (5.1) become zero, system (5.1) reduces to a BAM neural network with delays. Since the activation functions in system (5.1) are not bounded and monotonic, and are only globally Lipschitz continuous, then the result in [40] cannot ensure the global stability of equilibrium point of system (5.1).

However, letting \( a_{12} = a_{13} = a_{23} = a_{32} = -1, a_{31} = 6, \beta_{11} = \beta_{12} = \beta_{13} = \beta_{21} = 1, \beta_{22} = 2, \beta_{23} = \beta_{32} = -1, \beta_{33} = 6, \) then all the conditions in Theorem 3.1 in our paper can be satisfied. By using the Matlab LMI Control toolbox, it can be verified that the LMI in Theorem 3.1 is feasible and

\[
P = \begin{bmatrix} 166.681 & 0 \\ 0 & 166.681 \\ 109.2066 & 0 \\ 0 & 109.2066 \\ 538.3772 & 0 \\ 0 & 538.3772 \\ 129.2227 & 0 \\ 0 & 129.2227 \end{bmatrix}, \quad Q = \begin{bmatrix} 116.5557 & 0 \\ 0 & 116.5557 \\ 146.2646 & 0 \\ 0 & 146.2646 \\ 90.9774 & 0 \\ 0 & 90.9774 \\ 0.234 & 0 \\ 0 & 0.234 \end{bmatrix}
\]

The conditions \((H_1), (H_2)\) in Theorem 3.1 are obviously satisfied. Then all conditions in Theorem 3.1 are satisfied. By Theorem 3.1 we can conclude that system (5.1) has a unique equilibrium point, which is globally asymptotically stable. The global asymptotic stability of system (5.1) in Example 1 is shown in Fig. 1.

6. Conclusions

In this paper, under the assumption that the activation functions only satisfy global Lipschitz conditions, a novel LMI-based sufficient condition is obtained for global asymptotic stability of equilibrium point of two-dimensional BAM neural networks with reaction–diffusion terms and distributed delays by using degree theory and LMI method. In the results obtained, the assumptions for boundedness and monotonicity on the activation functions in existing papers are removed.

References


