Fast algorithm for multiplicative noise removal

Baoli Shi\textsuperscript{a}, Lihong Huang\textsuperscript{a,c,*}, Zhi-Feng Pang\textsuperscript{b}

\textsuperscript{a}College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China
\textsuperscript{b}College of Mathematics and Information Science, Henan University, Kaifeng, Henan 475004, PR China
\textsuperscript{c}Hunan Women’s University, Changsha, Hunan 410000, PR China

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A B S T R A C T

In this work, we consider a variational restoration model for multiplicative noise removal problem. By using a maximum a posteriori estimator, we propose a strictly convex objective functional whose minimizer corresponds to the denoised image we want to recover. We incorporate the anisotropic total variation regularization in the objective functional in order to preserve the edges well. A fast alternating minimization algorithm is established to find the minimizer of the objective function efficiently. We also give the convergence of this minimization algorithm. A broad range of numerical results are given to prove the effectiveness of our proposed model.

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1. Introduction

Image denoising problem has been widely studied in the areas of image processing. Most of the literature deals with the additive noise model. But in practice, there are other types of noise such as multiplicative noise. It can also corrupt an image. In this paper, we are interested in the multiplicative noise removal problem. This problem can be expressed as follows: given a recorded image \( g \) : \( \Omega \subset \mathbb{R}^2 \to \mathbb{R} \), which is the multiplication of an original image \( u \) and a noise \( \varepsilon \):

\[
\varepsilon = u \nu.
\]

Here, \( \Omega \) denotes the image domain that is simplified a rectangle domain in usual. The images we considered are 2-dimensional matrices of size \( M \times N \). Without loss of generality, we can suppose that each value of \( u \) and \( \nu \) are positive in the noise model. Due to this degraded mechanism, nearly all the information of the original image may vanish when it is distorted by multiplicative noise. Therefore, it is important to remove multiplicative noise. The goal of restoration is to recover the true image \( u \) from the data \( g \). The problem of removing multiplicative noise occurs in many applications, such as synthetic aperture radar, ultrasound imaging and laser imaging, see \([14]\).

In literature, various variational approaches devoted to multiplicative noise removal have been proposed. The early variational approach for multiplicative noise removal is the one by Rudin et al.\textsuperscript{[14]} as used for instance in\textsuperscript{[6,9,12,16]}. By using a maximum a posteriori (MAP) estimator, Aubert and Aujol\textsuperscript{[2]} proposed a functional whose minimizer corresponds to the denoised image to be recovered. This functional is:

\[
E(u) = \int_{\Omega} |Du| + \lambda \int_{\Omega} (\log u + \frac{g}{u}) \, dx \, dy,
\]

where \( \int_{\Omega} |Du| \) denotes the total variation of \( u \) and \( \lambda \) is a regularization parameter. In their method, they considered the Gamma noise with mean one. Though the functional they proposed is not convex, they still proved the existence of the minimizer, gave a sufficient condition ensuring uniqueness and showed that a comparison principle holds. They further gave some numerical examples illustrating the capability of their model.

As a result of the drawback of the function (2) that is not convex for all \( u \), the solution for the method in [2] is likely not the global optimal solution of (2). Therefore, the quality of the denoised image may be not good. In view of this, Shi and Osher\textsuperscript{[15]} presented a convex model which adopts the fitting term in (2). They adopted inverse scale space flow as denoising technique. Moreover, Huang et al.\textsuperscript{[4]} proposed a strictly convex objective functional for multi-
The proposed model

We suppose that the multiplicative noise in each pixel follows a Gamma distribution with mean one and with its probability density function given by:

\[ f_V(v) = \begin{cases} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma)} v^\gamma e^{-v}, & v > 0, \\ 0, & \leq 0, \end{cases} \]

where \( I > 0 \) is the number of looks (in general, \( L \) is a positive integer) and \( \Gamma(\cdot) \) is a Gamma function.

According to the maximum a posteriori estimation, the restored image \( \hat{u} \) can be computed by

\[ \hat{u} = \arg \max_u f_{U|G}(u|g). \]

Applying Baye’s rule, it becomes

\[ \hat{u} = \arg \max_u \frac{f_G(u)f_U(u)}{f_G(u)} \]

By using Proposition 3.1 in [2], we get:

\[ f_{G|U}(g|u) = \frac{1}{u f_G(u)^{1/2}} e^{-u/2}. \]

Taking the logarithm transformation into account, we assume that the image prior \( f_G(u) \) is:

\[ f_G(u) = f_{U|W}(u|w)f_W(w). \]

with

\[ f_{U|W}(u|w) \propto \exp(-z_1 \log u - w_1^2/2), \]

\[ f_W(w) = \exp(-z_2 (\|w_x\|_1 + \|w_y\|_1)). \]

where \( z_1 \) and \( z_2 \) are two positive constants. Herein, we suppose that the difference between \( \log u \) and \( w \) obeys a Gaussian distribution and \( \log u \) obeys an anisotropic total variation prior. Therefore, we have

\[ f_G(u) \propto \exp(-z_1 \log u - w_1^2/2) \exp(-z_2 (\|w_x\|_1 + \|w_y\|_1)). \]

Since \( f_{U|W}(u|w) \) is a constant, (5) can be reformulated as

\[ \hat{u} = \arg \max_u f_{G|U}(g|u)f_{U|W}(u|w)f_W(w). \]

For the above problem, we take logarithm transformation in order to transform multiplication to summation. Therefore, (8) can be rewritten as the following problem:

\[ \hat{u} = \arg \min_u (-\log f_G(g|u) - \log f_{U|W}(u|w) - \log f_W(w)). \]

Using (6)–(8), we see that (9) amounts to:

\[ \hat{u} = \arg \min_u \sum_{(x,y)\in \Omega} \left( L \left( \log u(x,y) - g(x,y) / u(x,y) \right) + z_1 \log u(x,y) - w_1^2/2 \right) + z_2 (\|w_x\|_1 + \|w_y\|_1). \]

Based on the previous computation, we propose the following minimization problem by considering a new variable \( z = \log u \):

\[ \min_{z,u} \int_{\Omega} (z + g e^{-z}) \, dx \, dy + \beta_1 |z - w|_2^2 + \beta_2 (\|w_x\|_1 + \|w_y\|_1). \]

It is straightforward to check that (11) is equivalent to (9).

We will explain the proposed model. For the transformation \( z = \log u \), it is obvious that when \( u \) includes an edge, \( z \) also contains an edge at the same location, i.e. the logarithm transformation preserves image edges. In view of this, we can consider \( z \) as an image in the logarithm domain. We note that the argument \( u \) should be positive in the objective functional (2). It will affect the quality of restoration image. However, in our model (11), the argument \( z \) can be any real number, and the corresponding \( u = e^z \) is still positive. The main advantage of the first term in (11) is that its second derivative with respect to \( z \) is equal to \( ge^{-z} \), which is always greater than zero. Therefore, this term is strictly convex for all \( z \). We incorporate the fitting term \( |z - w|_2^2 \) in (11). The parameter \( \beta_1 \) measures the trade off between an image obtained by a maximum a posteriori estimation and an anisotropic total variation denoised image \( w \). The parameter \( \beta_2 \) is used to measure the amount of regularization to a denoising image \( w \). The anisotropic total variation can preserve the edges well because its diffusion is adapted to the direction of the local image features. It means that the diffusion strength along the direction which is vertical to the direction of local features is smaller than it along the direction of the local features.
2.2. The alternating minimization algorithm

In the following, we consider the discrete setting of the objective functional in (11), which can be described as follows:

$$\min_{z \in R^M} \sum_{i=1}^{M} \sum_{j=1}^{N} (z_q + g_i z_q e^{-z_v}) + \beta_1\|z - w_i\|_2^2 + \beta_2(\|\nabla_x w_i\|_1 + \|\nabla_y w_i\|_1).$$

(12)

Here, $\nabla_x$ denotes the difference operator with respect to $x$ that is given by:

$$\nabla_x w(i,j) = \begin{cases} 0, & \text{for } i = 1, \\ w(i,j) - w(i-1,j), & \text{for } i = 2, \ldots, M. \end{cases}$$

Then, $\nabla_x$ is a linear mapping from $R^M$ to $R^M$. Similarly, $\nabla_y$ is the difference operator from $R^M$ to $R^M$ with respect to $y$ which is given by:

$$\nabla_y w(i,j) = \begin{cases} 0, & \text{for } j = 1, \\ w(i,j) - w(i,j-1), & \text{for } j = 2, \ldots, N. \end{cases}$$

Starting from an initial value $w^{(0)}$, $z^{(0)}$, we propose an alternating minimization algorithm to find the minimizer of (11). This method computes a sequence of iteration:

$$z^{(1)}, w^{(1)}, z^{(2)}, w^{(2)}, \ldots, z^{(k)}, w^{(k)}, \ldots$$

such that

$$R(w^{(k-1)}) := z^{(k)} = \arg\min_{z} \sum_{i=1}^{M} \sum_{j=1}^{N} (z_q + g_i z_q e^{-z_v}) + \beta_1\|z - w^{(k-1)}\|_2^2,$$

$$S(z^{(k)}) := w^{(k)} = \arg\min_{w} \beta_1\|z^{(k)} - w\|_2^2 + \beta_2(\|\nabla_x w\|_1 + \|\nabla_y w\|_1),$$

(13)

(14)

for $k = 1, 2, \ldots$.

In fact, we can solve the following system of $MN$ decoupled nonlinear equations to find the optimal solution of (13):

$$1 - g_i e^{-z_v} + 2\beta_1 (z_q - w^{(k-1)}_q) = 0, \quad i = 1, \ldots, M, \quad j = 1, \ldots, N.$$  

Because the objective function $z_q + g_i e^{-z_v} + \beta_1 (z_q - w^{(k-1)}_q)$ is corresponding to the above system is strictly convex, the corresponding equation has a unique solution. The solution can be found by using the Newton method very efficiently. It is well-known that the Newton method is quadratic convergent. Therefore, the iterative sequence established by Newton method converges to the unique minimizer of the objective function in (13). By experiments, we find that the Newton method yields a optimal solution by running just four iterations.

For the minimization problem (14), [5] proposed a fast algorithm. Here, we use this fast algorithm to find the optimal solution of (14). This can be expressed as follows: Set $b_i^0 = 0$, $b_q^0 = 0$, and $w^0 = z^{(0)}$. For $l = 1, 2, \ldots$, let

$$b_i^l = \text{cut} \left( \nabla_x w^l + b_i^{l-1} \right);$$

$$b_q^l = \text{cut} \left( \nabla_y w^l + b_q^{l-1} \right);$$

$$w^{l+1} = z^{(k)} - \frac{\beta_2}{\beta_1} \left( \nabla_i^l b_i^l + \nabla_q^l b_q^l \right).$$

(15)

(16)

(17)

where $\nabla_i^l$ and $\nabla_q^l$ are conjugate operators of $\nabla_x$ and $\nabla_y$, respectively. More precisely, $\nabla_i^l$ and $\nabla_q^l$ are linear operator from $R^M$ to $R^M$ given by

Here, for $\lambda > 0$ and $c \in R$, we define:

$$\text{cut} \left( \frac{c}{\lambda} \right) = \begin{cases} \frac{c}{\lambda}, & \text{for } \frac{c}{\lambda} > 0, \\ 0, & \text{for } \frac{c}{\lambda} < 0. \end{cases}$$

Let $v = (v_1, \ldots, v_n)$ and $c = (c_1, \ldots, c_n)$ be two vectors in $R^n$. We write $v = \text{cut}(c, \lambda)$. If $v_l = \text{cut}(c_l, \lambda)$, then $\lim_{\lambda \to 0} w^\lambda = w^*$ where $w^*$ is the unique minimizer of (14).

In [5], they get that the results of 15 iterations are good enough by experiments. Thus, we only take 15 iterations in our experiments when solve the problem (14).

3. Convergence results of the proposed model

In this section, we show the unique existence of the minimizer of the problem (11), and study the convergence of the alternating minimization algorithm. It is obvious that the objective functional in (11) is strictly convex. Therefore, the problem (11) exists a unique minimizer $(z^*, w^*)$. We denote the functional to minimize in (11) by:

$$E(z, w) = \int_0^1 (z + g e^{-z^*}) dx dy + \beta_1\|z - w_i\|_2^2$$

$$+ \beta_2(\|\nabla_x w\|_1 + \|\nabla_y w\|_1).$$

According to (13) and (14), we get the following relationship:

$$w^k = S(z^{(k)}) = S(R(w^{(k-1)})), \quad k = 1, 2, \ldots.$$  

$$z^k = R(w^{(k-1)}) = R(S(z^{(k-1)})), \quad k = 1, 2, \ldots.$$  

For convenience, we denote $T_1(\cdot) = S(R(\cdot))$ and $T_2(\cdot) = R(S(\cdot))$, respectively.

In [11], Moreau introduced a notion of proximity operator which is a generalization of the notion of a convex projective operator. The proximal operator of a proper, convex, semi-continuous function $\phi$ is defined by

$$\text{prox}_{\phi}(y) = \arg\min_x \frac{1}{2}\|y - x\|_2^2 + \phi(x).$$

(18)

We know that the proximal operator is single-valued and continuous in [11].

**Definition 1.** A operator $P : R^p \rightarrow R^p$ is firmly non-expansive if it satisfies one of the following equivalent conditions:

(i) $\|P x - Py\|_2^2 \leq (\langle P x - Py, x - y \rangle, \quad \forall (x, y) \in R^p \times R^p$.

(ii) $\|P x - Py\|_2^2 \leq \|x - y\|_2^2 - \|I - P\|\|x - (I - P)y\|_2^2, \quad \forall (x, y) \in R^n \times R^p$. 

(19)
Definition 2. A operator $\mathcal{P} : R^n \to R^n$ is non-expansive if for any $(x,y) \in R^n \times R^n$, we have
$$\|\mathcal{P}x - \mathcal{P}y\| \leq \|x - y\|.$$ It follows immediately that a firmly non-expansive operator $\mathcal{P}$ is non-expansive. The non-expansivity of the proximal operator is proved in [3], that is the following Lemma 1.

Lemma 1. Let $\varphi$ be a proper, convex, lower semi-continuous function. Then $\text{prox}_\varphi$ is firmly non-expansive. Subsequently, it is non-expansive.

It is obvious that $\frac{\partial}{\partial w} (\|\nabla_x w\|_2 + \|\nabla_y w\|_1)$ is a proper, convex, lower semi-continuous function. Hence, the operator $\mathcal{S}$ is the proximal operator of $\frac{\partial}{\partial w} (\|\nabla_x w\|_2 + \|\nabla_y w\|_1)$. According to Lemma 1, the operator $\mathcal{S}$ is non-expansive. The next lemma indicates that the operator $\mathcal{R}$ is also non-expansive.

Lemma 2. The operator $\mathcal{R}$ is non-expansive.

Proof. The details is similar to that of Lemma 3.1 in [4].

In view of the above results, we know that $T_1(\cdot) = \mathcal{S}(\mathcal{R}(\cdot))$ and $T_2(\cdot) = \mathcal{R}(\mathcal{S}(\cdot))$ are non-expansive. □

Lemma 3. Let $z^{(k)}$ and $w^{(k)}$ be generated by (13) and (14), respectively. Then $\sum_{k=0}^{\infty} \|z^{(k+1)} - z^{(k)}\|_2^2$ and $\sum_{k=0}^{\infty} \|w^{(k+1)} - w^{(k)}\|_2^2$ converge.

Proof. Denote $E_1(z,w) = \|z - w\|_2^2$ and $E_2(w) = \|\nabla_x w\|_2 + \|\nabla_y w\|_1$. We have
$$E(z,w) = \int (z + g e^{-z}) dx dy + \beta_1 E_1(z,w) + \beta_2 E_2(w).$$

Hence,
$$E(z^{(k+1)},w^{(k)}) - E(z^{(k+1)},w^{(k+1)}) = \beta_1 (E_1(z^{(k+1)},w^{(k)}) - E_1(z^{(k+1)},w^{(k+1)})) + \beta_2 (E_2(w^{(k+1)}) - E_2(w^{(k+1)})).$$

Consider the Taylor series expansion of $E_1(z,w)$ in the second variable, i.e.
$$E_1(z^{(k+1)},w^{(k)}) = E_1(z^{(k+1)},w^{(k+1)}) + (w^{(k+1)} - w^{(k+1)})^T E_1^{(z^{(k+1)},w^{(k+1)})} (w^{(k+1)} - w^{(k+1)}).$$

Here, $x^T$ denotes the transpose of $x$. We notice that $E_1$ is quadratic about $w$. Then, $\frac{\partial^2}{\partial w^2} E_1 = I$ where $I$ is identity matrix. Moreover, since $E_2$ is a convex function, we get
$$E_2(w^{(k+1)}) \geq E_2(w^{(k+1)}) + (w^{(k+1)} - w^{(k+1)})^T \frac{\partial E_2}{\partial w} (w^{(k+1)})^{(k+1)}.$$ (21)

Combining (19)–(21), we obtain
$$E(z^{(k+1)},w^{(k)}) - E(z^{(k+1)},w^{(k+1)}) \geq (w^{(k+1)} - w^{(k+1)})^T \left( \beta_1 E_1^{(z^{(k+1)},w^{(k+1)})} + \beta_2 \frac{\partial E_2}{\partial w} (w^{(k+1)}) \right) + \beta_1 \|w^{(k+1)} - w^{(k+1)}\|_2^2.$$ (22)

We become aware of that the sub-differential of $E$ with respect to $w$ is equal to the vector sum of the sub-differential of $E_1$ and $E_2$ about $w$, t. e.
$$\frac{\partial E}{\partial w} = \beta_1 \frac{\partial E_1}{\partial w} + \beta_2 \frac{\partial E_2}{\partial w}.$$ Since $w^{(k+1)}$ is the minimizer of $E(z^{(k+1)},w)$, we have
$$\frac{\partial E}{\partial w}(z^{(k+1)},w^{(k+1)}) = 0,$$ that is
$$\beta_1 \frac{\partial E_1}{\partial w}(z^{(k+1)},w^{(k+1)}) + \beta_2 \frac{\partial E_2}{\partial w}(z^{(k+1)},w^{(k+1)}) = 0.$$

Therefore, the first term in the right hand of (22) is zero. When we solve the successive minimization problems (13) and (14), we note that $E(z^{(k)},w^{(k)}) \geq E(z^{(k+1)},w^{(k+1)})$. Hence, we get
$$E(z^{(k)},w^{(k)}) - E(z^{(k+1)},w^{(k+1)}) \geq E(z^{(k+1)},w^{(k+1)}) - E(z^{(k+1)},w^{(k+1)})$$
$$\geq \beta_1 \|w^{(k+1)} - w^{(k+1)}\|_2^2.$$ It follows that the partial sum of the sequence $\sum_{k=0}^{\infty} \|w^{(k+1)} - w^{(k+1)}\|_2^2$ is bounded. Thus, the infinite series $\sum_{k=0}^{\infty} \|w^{(k+1)} - w^{(k+1)}\|_2^2$ is convergent. □

Let $E_1(z) = \int (z + g e^{-z}) dx dy$. By considering $E(z^{(k)},w^{(k)}) - E(z^{(k+1)},w^{(k)})$ and using the similar method, we can prove that $\sum_{k=0}^{\infty} \|z^{(k+1)} - z^{(k+1)}\|_2^2$ is convergent.

Definition 3. A operator $\mathcal{P} : R^n \to R^n$ is asymptotically regular if for any $x \in R^n$, the sequence $(x^{(k+1)} - x^{(k+1)})$ tends to zero as $k \to \infty$.

Based on Lemma 3, we get the following result.

Lemma 4. For any initial value $w^{(0)}$, $z^{(0)}$ assume $z^{(k)}$ and $w^{(k)}$ are generated by (13) and (14) respectively, then $T_1$ and $T_2$ are asymptotically regular.

Proof. According to Lemma 3, we obtain
$$\lim_{k \to \infty} \|w^{(k+1)} - w^{(k+1)}\|_2^2 = 0, \quad \lim_{k \to \infty} \|z^{(k+1)} - z^{(k+1)}\|_2^2 = 0.$$ As $w^{(k)} = T_2(w^{(k-1)})$ and $z^{(k)} = T_1(z^{(k-1)})$, by the recurrence method we have
$$w^{(k)} = T_2(w^{(0)}), \quad z^{(k)} = T_1(z^{(0)}).$$

Therefore, we get
$$\lim_{k \to \infty} \|T_2^{(k-1)}(w^{(0)}) - T_2^{(k)}(w^{(0)})\|_2 = 0,$$
$$\lim_{k \to \infty} \|T_1^{(k-1)}(z^{(0)}) - T_1^{(k)}(z^{(0)})\|_2 = 0.$$ This indicates that $T_1$ and $T_2$ are asymptotically regular. □

Lemma 5. Suppose the unique minimizer of $E(z,w)$ is $(z^*,w^*)$. Then, $z^*$ and $w^*$ are the unique fixed point of $T_2$ and $T_1$, respectively.

Proof. Since $E(z,w)$ is differentiable with respect to $z$ and $w$ separately, we obtain
$$\left( \frac{\partial E}{\partial z}(z^*,w^*) \right) = 0,$$
$$\left( \frac{\partial E}{\partial w}(z^*,w^*) \right) = 0.$$ This implies that
$$z^* = \mathcal{R}(w^*) = \arg \min_w E(z,w^*),$$
$$w^* = \mathcal{S}(z^*) = \arg \min_z E(z,w^*).$$ We easily get $x^* = R(w^*) = T_2(z^*)$ and $w^* = S(z^*) = T_1(z^*)$. Therefore, $z^*$ and $w^*$ are the corresponding fixed point of $T_1$ and $T_2$, respectively.

On the other hand, we note that $E(z,w)$ is strictly convex and differentiable with respect to $z$ and $w$, respectively. Therefore, the fixed points of $T_1$ and $T_2$ are the minimizers of $E(z,w)$. By virtue of the uniqueness of the minimizer of $E(z,w)$, $T_1$ and $T_2$ exist an unique fixed point separately. That is to say that $z^*$ and $w^*$ are the unique fixed point of $T_2$ and $T_1$, respectively.

According to Theorem 1 in [13], we get the following result.
Theorem 2. For any initial value \(w^{(0)}, z^{(0)}\), assume \(z^{(k)}\) and \(w^{(k)}\) are generated by (13) and (14), respectively, then \(z^{(k)}, w^{(k)}\) converge to the corresponding fixed point of \(T_2\) and \(T_1\), i.e. \((z^{(k)}, w^{(k)})\) converges to \((z^*, w^*)\) which is the unique minimizer of \(E(z, w)\), as \(k \to \infty\).

Proof. In view of Lemmas 1, 2, 4 and 5, we know that \(T_1: \mathbb{R}^n \to \mathbb{R}^n, T_2: \mathbb{R}^n \to \mathbb{R}^n\) are non-expansive asymptotically regular mappings and have fixed point. By using Theorem 1 in [13], we get that \(z^{(k)}, w^{(k)}\) converge respectively to the fixed point of \(T_2\) and \(T_1\), i.e. \((z^{(k)}, w^{(k)})\) converges to \((z^*, w^*)\) which is the unique minimizer of \(E(z, w)\), as \(k \to \infty\).

4. Numerical experiment results

In this section, we will give some numerical results to demonstrate the performance of our proposed algorithm. We compare our results with the HNW method in [4]. In the experiments, the original image is degraded by multiplicative noise following a Gamma distribution with mean one, and the noise level is controlled by the parameter \(L\). All simulations are performed in MATLAB (R2007a) on a PC with an Intel Core(TM) 2 Duo CPU at 2.20Ghz and 1.00 GB of memory.

In this paper, we use the Peak-Signal-to-Noise Ratio (PSNR) to measure the image quality, which is given by:

\[
\text{PSNR} = 10 \cdot \log_{10} \left( \frac{m^2}{\frac{1}{M \cdot N} \sum_{i=1}^{M} \sum_{j=1}^{N} (u - \hat{u})^2} \right),
\]

where \(m\) is the maximum pixel value of the image and \(\hat{u}\) is the restored image. In addition, we also use the relative error between the restored image \(\hat{u}\) and the original image \(u\) to measure the quality of the restoration. It is given as follows:

\[
\text{ReErr} = \frac{||\hat{u} - u||_2}{||u||_2}.
\]

The restored image \(\hat{u}\) is computed by \(\exp(w^*)\) where \(w^*\) is the optimal solution of problem (14). Since \(w^*\) is the anisotropic total variation denoised image, the visual quality of \(\exp(w^*)\) is better than \(\exp(z^*)\) where \(z^*\) is the optimal solution of problem (13). Hence, we represent the restored image by \(\exp(w^*)\) in the following figures.

The stopping criterion is that the relative error between the successive iterate of the restored image should satisfy the following inequality:

\[
\frac{||u^{(k+1)} - u^{(k)}||_2}{||u^{(k)}||_2} < 10^{-4}.
\]

We note that there are two regularization parameters \(\beta_1\) and \(\beta_2\) in the proposed model. In order to reduce the computation time of looking for a good regularization parameter, we fix \(\beta_1 = 19\) in all the tests. Therefore, we just need to search for the best value of \(\beta_2\) such that the relative error between the restored image \(\hat{u}\) and the original image \(u\) is the smallest among all the tested values of \(\beta_2\).

In our experiments, we take the lena image and cameraman image for tests. The sizes of the two images are all the same which is \(256 \times 256\) pixels. The original lena image in Fig. 1(a) is distorted by a gamma noise with \(L = 33\) and \(L = 5\), see Fig. 1(b) and (c). When \(L = 33\), the PSNR and relative error between noisy image and original image are 22.3187 and 0.1748. When \(L = 5\), the PSNR and relative error are 14.1285 and

![Fig. 1. (a) Original image. (b) Noisy image with \(L = 33\). (c) Noisy image with \(L = 5\).](image1)

![Fig. 2. (a) Restored image by HNW method for \(L = 33\). (b) Restored image by our method for \(L = 33\). (c) Restored image by HNW method for \(L = 5\). (d) Restored image by our method for \(L = 5\).](image2)
The original cameraman image in Fig. 3(a) is distorted by a gamma noise with $L = 35$ and $L = 13$, see Fig. 3(b) and (c). When $L = 35$, the PSNR and relative error between noisy image and original image are 21.0223 and 0.1690. When $L = 13$, the PSNR and relative error are 16.7071 and 0.2778. We can see from Fig. 1(b), (c) and Fig. 3(b), (c) that the smaller $L$ is, the more noisy pictures are.

Besides that, we consider the peppers image with size $512 \times 512$ pixels and the Barbara image whose size is $128 \times 128$ pixels, see Fig. 5(a) and Fig. 6(a), respectively. The peppers image

![Fig. 3](image1.png)  (a) Original image. (b) Noisy image with $L = 35$. (c) Noisy image with $L = 13$.  

![Fig. 4](image2.png)  (a) Restored image by HNW method for $L = 35$. (b) Restored image by our method for $L = 35$. (c) Restored image by HNW method for $L = 13$. (d) Restored image by our method for $L = 13$.  

![Fig. 5](image3.png)  (a) Original image. (b) Noisy image with $L = 25$. (c) Restored image by HNW method. (d) Restored image by our method.

Table 1  
The computation time, the number of iterations and PSNR for different experiments.

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<th>Size</th>
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<td>ite</td>
<td>PSNR</td>
</tr>
<tr>
<td>256 x 256</td>
<td>Lena ($L = 33$)</td>
<td>339.8125</td>
<td>119</td>
<td>29.8720</td>
</tr>
<tr>
<td></td>
<td>Lena ($L = 5$)</td>
<td>460.8750</td>
<td>161</td>
<td>25.1414</td>
</tr>
<tr>
<td></td>
<td>Cameraman ($L = 35$)</td>
<td>352.2969</td>
<td>124</td>
<td>28.6591</td>
</tr>
<tr>
<td></td>
<td>Cameraman ($L = 13$)</td>
<td>391.4688</td>
<td>137</td>
<td>26.1978</td>
</tr>
<tr>
<td>512 x 512</td>
<td>Peppers ($L = 25$)</td>
<td>1812.900</td>
<td>121</td>
<td>30.9058</td>
</tr>
<tr>
<td>128 x 128</td>
<td>Barbara ($L = 18$)</td>
<td>60.6563</td>
<td>133</td>
<td>26.2579</td>
</tr>
</tbody>
</table>
is distorted by a gamma noise with $L = 25$ and the noisy image is shown in Fig. 5(b). The PSNR and relative error between noisy image and original image are 20.6088 and 0.1998. The noisy Barbara image in Fig. 6(b) is got via the distortion of the original image by a gamma noise with $L = 18$. The PSNR and relative error between noisy image and original image are 19.2562 and 0.2330.

In Table 1, we use $t (s)$ and $ite$ to denote the CPU time of computation and the number of iterations, respectively. The size of test
images emerged in second column is demonstrated in the first column. The PSNR of different experiments are also revealed in this table. From Table 1, we find that the computational time required by our method is much lesser than that of the HNW method. Precisely, the computational speed of our method is more than six times that of the HNW method. Besides that, the number of iterations required for convergence is less than that of the HNW method. Thus, our method is quite useful in real life since its computing speed is very fast. For the images with size \(256 \times 256\) pixels, the PSNR of the HNW method is a little better than that of the HNW approach. It is easy to see that the restoration results by our method are visually a little better than HNW method.

We use ReErr denotes the relative errors of the restored images and original image for different experiments and methods in Table 2. The size of test images and the optimal value of \(\beta_2\) for different experiments and methods are also given in this table. Table 2 shows that the relative errors between the restored images \(u^*\) and the original image \(u\) and the optimal value of \(\beta_2\) with respect to different models. According to the table, the relative error of HNW method is a little smaller than our method for the images with size \(256 \times 256\). However, for the image with size \(512 \times 512\) or \(128 \times 128\), the relative error of our method outperforms the HNW approach.

In order to further show the efficiency of our proposed model for multiplicative noise removal, we extend it to some real world applications such as the synthetic aperture radar (take SAR for simple) image and the ultrasound image. The original SAR image in Fig. 7(a) is contaminated by a gamma noise with \(L = 10\), see Fig. 7(b). When we take \(\beta_2 = 0.253\), the restored results of our method is optimal, which is shown in Fig. 7(c). Fig. 8(a) demonstrates the original ultrasound image. This image is distorted by a gamma noise with \(L = 22\), see Fig. 8(b). By experiments, we find that the optimal value of \(\beta_2\) is 0.110 for this image. The restored ultrasound image is shown in Fig. 8(c).

5. Conclusion

In this paper, we propose a strictly convex objective functional for multiplicative noise removal problems. A fast alternating minimization algorithm is established to solve the proposed anisotropic total variation minimization problem. Numerical experiments show that the computational speed of our method is quite fast and the restoration results is well. However, there is some undesirable staircase effect in the ramp regions. In the following, we hope to study how to reduce this undesirable effect.

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References