Structured mixed and componentwise condition numbers of some structured matrices

Hua Xiang\textsuperscript{a,1,} \textsuperscript{1}, Yimin Wei\textsuperscript{b, c,} \textsuperscript{1}

\textsuperscript{a}Institute of Mathematics, Fudan University, Shanghai, 200433, PR China
\textsuperscript{b}School of Mathematical Sciences, Fudan University, Shanghai, 200433, PR China
\textsuperscript{c}Key Laboratory of Mathematics for Nonlinear Sciences, Fudan University, Ministry of Education, China

Received 20 May 2005; received in revised form 22 December 2005

Abstract

Structured matrices, such as Cauchy, Vandermonde, Toeplitz, Hankel, and circulant matrices, are considered in this paper. We apply a Kronecker product-based technique to deduce the structured mixed and componentwise condition numbers for the matrix inversion and for the corresponding linear systems.

© 2006 Elsevier B.V. All rights reserved.

MSC: 15A09; 65F05; 65G05

Keywords: Cauchy; Circulant; Componentwise; Condition number; Hankel; Kronecker product; Mixed; Structured matrix; Toeplitz; Vandermonde

1. Introduction and notation

The condition number is a very important concept in sensitivity analysis, and it is useful to understand the stability of algorithm [11,18]. When working with structured matrices, structured mixed and componentwise condition numbers are more meaningful than usual unstructured condition numbers. We can obtain much tighter and revealing bounds. In this paper we will deal with matrices which have special structures. The entries of such matrices depend on a small set of parameters. Clearly, the matrix inversion and the solution of linear systems are determined by these parameters [13]. Then the sensitivity of perturbation in these parameters is under consideration. Much work has been done about the structured conditioning [1,6,9,10,16,28,29], for a survey of componentwise perturbation, see [17] and the references therein. This paper focuses on the structured conditioning of Cauchy, Vandermonde, Toeplitz, Hankel and circulant matrices, with respect to the matrix inversion and the corresponding linear systems. Some results are obtained already, here we derive them via a different way.

We first summarize some results achieved. Consider the function $F(A) \mapsto X = F(A)$, where matrix $A$ is the input data of the problem. Here we utilize the vec$(\cdot)$ operator, which stacks the columns one on top of the other to form a

\hspace{1cm} $\dagger$ This project is supported by National Natural Science Foundation of China under Grant 10471027 and Shanghai Education Committee.
\hspace{1cm} * Corresponding author.
\hspace{1cm} E-mail addresses: 031018015@fudan.edu.cn (H. Xiang), ymwei@fudan.edu.cn (Y. Wei).
\hspace{1cm} 1 Present address: School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P.R. of China.

0377-0427/$-$ see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2006.02.026
large column vector, and use the vector representations to reformulate a matrix problem into vector form. Let

\[ a := \text{vec}(A), \quad (1.1) \]
\[ x := \text{vec}(X), \quad (1.2) \]

where the abbreviation ‘:=’ stands for ‘equal for definition’. Then we have the vector representation \( x = f(a) \) of the original problem \( X = F(A) \), where \( f = \text{vec} \circ F \) is a composite mapping.

Following the definition of \([9,21]\), we introduce the definition of the normwise condition number \( \kappa(F; A) \), mixed condition number \( m(F; A) \), and componentwise condition number \( c(F; A) \) as follows:

\[
\kappa(F; A) := \lim_{\varepsilon \to 0} \sup_{\|\Delta A\| \leq \varepsilon \|A\|} \frac{\|\delta x\|/\|x\|}{\|\delta a\|/\|a\|},
\]
\[
m(F; A) := \lim_{\varepsilon \to 0} \sup_{|\Delta A| \leq \varepsilon |A|} \frac{\|\delta x\|_{\infty}/\|x\|_{\infty}}{\|\delta a/a\|_{\infty}},
\]
\[
c(F; A) := \lim_{\varepsilon \to 0} \sup_{|\Delta A| \leq \varepsilon |A|} \frac{\|\delta x/x\|_{\infty}}{\|\delta a/a\|_{\infty}},
\]

where \( \delta a := \text{vec}(\Delta A), \delta x := \text{vec}(\Delta X), x, a \) are defined by (1.1) and (1.2), respectively, and \( b/a \) is the entry-wise division. Note that \( \xi/0 \) is interpreted as zero if \( \xi = 0 \) and infinity otherwise.

According to the results of \([9]\), we have

\[
\kappa(F; A) = \frac{\|f'(a)\|\|a\|}{\|f(a)\|}, \quad (1.3)
\]
\[
m(F; A) = \frac{\|f'(a)D_a\|_{\infty}}{\|f(a)\|_{\infty}}, \quad (1.4)
\]
\[
c(F; A) = \|D_{x^{-1}}f'(a)D_a\|_{\infty}, \quad (1.5)
\]

where \( D_a := \text{diag}(a), D_x := \text{diag}(x) \) in Matlab notation, and \( f' \) denotes the differential of \( f \), see also \([2,7,10,25]\).

The Frechet derivative \( f' \) is also so-called matrix derivative \([12,23,26]\). It can be explicitly expressed by Kronecker product, then we can use the expression to deduce the condition numbers for the matrix inversion and nonsingular linear systems. With this technique we can easily recover the classical results \([17,18,27,33]\). Here we will apply this Kronecker product-based technique to derive the structured mixed and componentwise condition numbers. The outline of this paper is as follows. We first consider the general cases in Section 2. In Section 3, we use this technique to deduce the structured condition numbers for Cauchy matrices. We then investigate Vandermonde matrices in Section 4, Toeplitz and Hankel matrices in Section 5, and circulant matrices in Section 6. In Section 7 we give four numerical examples to illustrate the results. Finally, some concluding remarks are made in Section 8.

Before our discussion some properties of the Kronecker product are required. The following results can be found in \([19,22,24,31]\):

\[
(A \otimes C)(B \otimes D) = (AB) \otimes (CD),
\]
\[
\|A \otimes B\| = \|A\| \|B\|,
\]
\[
|A \otimes B| = |A| \otimes |B|,
\]
\[
\text{vec}(AXB) = (B^T \otimes A) \text{vec}(X).
\]

We list some notations that will be used throughout this paper.

\[
\| \cdot \|_{\infty} \quad \text{infinity or row sum norm}
\]
\[
\| \cdot \|_{\max} \quad \|A\|_{\max} := \max_{i,j} |a_{ij}|
\]
\[
| \cdot | \quad \text{absolute value}
\]
\[
\text{vec} \quad \text{vec operator, which stacks the columns of a matrix one underneath the other}
\]
\[
\otimes \quad \text{Kronecker (or tensor) product}
\]
According to inversion and nonsingular linear systems. With this technique we can easily get the classical results [17,18,27,33].

Let's consider the linear systems $Ax = b$, where $A$ is nonsingular, and define function $\psi: (A, b) \mapsto A^{-1}b$.

We first consider the general case where $A$ is nonsingular. We will show that the differential can be explicitly expressed by Kronecker product, then we will use the expression to deduce the condition numbers for the matrix–vector product meaningful

and

$D_A := \text{diag}(\text{vec}(A))$. It is equivalent to $D_a$, where $a = \text{vec}(A)$

Using Matlab notation, $b/a := (b_{ij}/a_{ij})$. Using Matlab notation, $b/a := b/a$

$B/A$ entry-wise divide, $B/A := (b_{ij}/a_{ij})$, or $B/A$ in Matlab notation

$\Phi$ mapping concerning matrix inversion, e.g., $\Phi : A \mapsto A^{-1}$

$\varphi'$ Frechet derivative of $\varphi$, where $\varphi = \text{vec} \circ \Phi$, the vector representation of $\Phi$

$\psi$ mapping concerning linear systems solution, i.e., $\psi : (A, b) \mapsto A^{-1}b$

$\psi'$ Frechet derivative of $\psi$

2. General cases

We first consider the general case where $A$ is nonsingular matrix. We will show that the differential can be explicitly expressed by Kronecker product, then we will use the expression to deduce the condition numbers for the matrix inversion and nonsingular linear systems. With this technique we can easily get the classical results [17,18,27,33].

Case 1. Matrix inversion: Consider the function $\Phi: A \mapsto \Phi(A) = A^{-1}$. Define $\varphi(a) = \text{vec} (\Phi(A))$, where $a = \text{vec}(A)$. According to $A^{-1}A = I$, we have $(dA^{-1})A + A^{-1}dA = 0$, and so $dA^{-1} = -A^{-1}dAA^{-1}$. Therefore, we get

$$\text{vec}(dA^{-1}) = -(A^{-T} \otimes A^{-1}) \text{vec}(dA).$$

If $\varphi$ is differentiable at $a$ then we define the derivative by $\varphi'(a) = -A^{-T} \otimes A^{-1}$.

According to (1.4) and (1.5), we have [17,18]

$$m(\Phi; A) = \frac{\|\varphi'(a)D_a\|_\infty}{\|\varphi(a)\|_\infty} = \frac{\|\varphi'(a)\|_\infty}{\|\varphi(a)\|_\infty} = \frac{\|\text{vec}(A)\|_\infty}{\|\text{vec}(A^{-1})\|_\infty}$$

$$= \frac{\|A^{-T} \otimes A^{-1}\|_\infty}{\|\text{vec}(A^{-1})\|_\infty}$$

$$= \frac{\|\text{vec}(A^{-1})\|_\infty}{\|\text{vec}(A^{-1})\|_\infty}$$

and

$$c(\Phi; A) = \|D^{-1}_x \varphi'(a)D_a\|_\infty = \|D^{-1}_x \|_\infty \|\varphi'(a)\|_\infty = \|D^{-1}_x \|_\infty \|\text{vec}(A)\|_\infty$$

$$= \|D^{-1}_x \|_\infty \|A^{-T} \otimes A^{-1}\|_\infty \|\text{vec}(A)\|_\infty = \|D^{-1}_x \|_\infty \|\text{vec}(A^{-1})\|_\infty$$

$$= \frac{\|A^{-1}\|_\infty}{\|\text{vec}(A^{-1})\|_\infty}$$

which has been obtained by Rohn [27] by using the result of interval analysis.

Case 2. Nonsingular linear systems: For nonsingular linear systems, we can use this technique to obtain similar results. We now consider the linear systems $Ax = b$, where $A$ is nonsingular, and define function $\psi: (A, b) \mapsto x = \psi(A, b) = A^{-1}b$. Then we have

$$d\psi = d(A^{-1})b + A^{-1}db = -A^{-1}dAA^{-1}b + A^{-1}db$$

$$= -A^{-1}dAx + A^{-1}db = -(x^T \otimes A^{-1}) \text{vec}(dA) + A^{-1}db$$

$$= [-x^T \otimes A^{-1}, A^{-1}] \text{vec}(dA)$$
Then we get the derivative $\psi'(A, b) = [-x^T \otimes A^{-1}, A^{-1}]$. Here the input data $(A, b)$ for solving nonsingular linear systems plays the same role as the matrix $A$ in (1.4), where $A$ is the input data for matrix inversion. Using (1.4), we obtain the mixed relative condition number $m(\psi; A, b)$ for nonsingular linear systems [30].

$$m(\psi; A, b) = \frac{\|[-x^T \otimes A^{-1}, A^{-1}] \begin{bmatrix} D_A & D_b \end{bmatrix}\|_\infty}{\|x\|_\infty} = \frac{\|[-(x^T \otimes A^{-1})D_A, A^{-1}D_b]\|_\infty}{\|x\|_\infty}$$

$$= \frac{\|(x^T \otimes A^{-1})D_Ae + |A^{-1}D_b|e\|_\infty}{\|x\|_\infty} = \frac{\|(|x|^T \otimes |A^{-1}|) \operatorname{vec}(|A|) + |A^{-1}| |b|\|_\infty}{\|x\|_\infty},$$

(2.3)

where $e$ should have conformable dimension with the matrix to make the matrix–vector product meaningful. We should point out that $m(\psi; A, b)$ is the same as the componentwise condition number $\operatorname{Cond}_{[A],|b|}(A, x)$ defined in [18].

Applying (1.5), we can obtain the componentwise relative condition number $c(\psi; A, b)$ [27].

$$c(\psi; A, b) = \frac{\|D_{x}^{-1}[-x^T \otimes A^{-1}, A^{-1}] \begin{bmatrix} D_A & D_b \end{bmatrix}\|_\infty}{\|x\|_\infty} = \frac{\|D_{x}^{-1}(x^T \otimes A^{-1})D_Ae + |D_{x}^{-1}A^{-1}D_b|e\|_\infty}{\|x\|_\infty}$$

$$= \frac{\|D_{x}^{-1}(|x|^T \otimes |A^{-1}|) D_Ae + |A^{-1}D_b|e\|_\infty}{\|x\|_\infty} = \frac{\|D_{x}^{-1}(|A||x|) + |A^{-1}| |b|\|_\infty}{\|x\|_\infty}$$

$$= \frac{\|A^{-1}||A||x| + |A^{-1}| |b|\|_\infty}{\|x\|_\infty}. \quad (2.4)$$

Remarks. 1. We can also derive the normwise condition number for the matrix inversion. For example, we choose the Frobenius norm $\| \cdot \|_F$, then the corresponding normwise condition number is

$$\kappa_F(\Phi; A) = \frac{\|\phi'(a)\|_F \|a\|_F}{\|\phi(a)\|_F} = \|A^{-1}\|_F \|A\|_F.$$

2. To derive the usual normwise condition number for nonsingular linear systems, we need the partial differential $\psi_A'$ and $\psi_b'$, where $\psi_A' = -x^T \otimes A^{-1}$ and $\psi_b' = A^{-1}$. Here we use the $2$-norm, then the normwise condition number for the linear systems is [14,15,18]

$$\kappa_2(\psi; A, b) = \lim_{\epsilon \to 0} \sup_{\|\Delta A\|_2 \leq \epsilon \|A\|_2} \frac{\|\Delta A\|_2 \|A\|_2}{\|A\|_2 \|\Delta b\|_2}$$

$$= \|A^{-1}\|_2 \|A\|_2 + \frac{\|A^{-1}\|_2 \|b\|_2}{\|x\|_2}.$$ 

3. Cauchy matrices

Let $t = [t_1, t_2, \ldots, t_n]^T \in \mathbb{R}^n$ and $s = [s_1, s_2, \ldots, s_n]^T \in \mathbb{R}^n$ have pairwise distinct entries with $t_i \neq s_j$ for $1 \leq i, j \leq n$. Define $w := [t^T, s^T]^T \in \mathbb{R}^{2n}$, then the Cauchy matrix $C$ associated with $w$ is defined as

$$C(w) := \left( \frac{1}{t_i - s_j} \right)_{i,j=1}^n \in \mathbb{R}^{n \times n}.$$
Then we get the derivative
\[ \frac{\partial}{\partial w} \] where
\[ C^S := \begin{pmatrix} (1/(t_1 - s_j)^2) & \cdots & (1/(t_n - s_j)^2) \end{pmatrix}_{i,j=1} \]
\[ A_j := \text{diag}(C^S(:, j)) = \text{diag}(1/(t_1 - s_j)^2, 1/(t_2 - s_j)^2, \ldots, 1/(t_n - s_j)^2) \]
\[ E_j := \text{eye}(T), \text{i.e.}, \text{the } j\text{th column are all } 1\text{'s, and the other elements are zeros} \]
\[ H: w \mapsto \text{vec}(C\cdot(-1)), \text{where } C\cdot(-1) \text{ is the Matlab notation, i.e., } C\cdot(-1) = (t_i - s_j)_{i,j=1}^n \]
\[ G \text{ mapping a vector to a new one, whose element is the reciprocal of the original one, e.g., } G(z) = [z_1, z_2, \ldots, z_n]^T(z_i \neq 0), G(z) = z\cdot(-1) := [1/z_1, 1/z_2, \ldots, 1/z_n]^T \]

With these definitions above, we have \( \text{vec}(C) = G(Hw) \). Therefore,
\[ \text{vec}(dC) = G' H d w := K \text{ } d w, \]
where
\[ H = \begin{bmatrix} I & -E_1 \\ I & -E_2 \\ \vdots & \vdots \\ I & -E_n \end{bmatrix}, \quad G' = -\text{diag}(\text{vec}(C^S)) = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}, \quad K = G' H = \begin{bmatrix} A_1 & I & -E_1 \\ A_2 & I & -E_2 \\ \vdots & \vdots & \vdots \\ A_n & I & -E_n \end{bmatrix}. \]

Since \( A_j E_j = A_j e e^T = C^S(:, j) e e^T \), the matrix \( K \) can be easily formed.

Let \( \Phi \) be the map of matrix inversion, \( \Phi : w \mapsto \Phi(w) = C^{-1} \). The corresponding vector representation is \( \varphi : w \mapsto \text{vec} \circ \Phi(w) = \text{vec}(C^{-1}) \). Since \( \text{vec}(dC) = -(C^{-T} \otimes C^{-1}) \text{vec}(dC) = -(C^{-T} \otimes C^{-1}) K d w \), we get the derivative \( \varphi'(w) = -(C^{-T} \otimes C^{-1}) K \).

Applying (1.4), (1.5) and the above results, we obtain the mixed and componentwise structured condition numbers for the inversion of Cauchy matrix.

\[
\begin{align*}
m(\Phi; w) &= \frac{\| \varphi'(w) D w \|_\infty}{\| \varphi(w) \|_\infty} = \frac{\| (C^{-T} \otimes C^{-1}) K D w \|_\infty}{\| \varphi(w) \|_\infty} = \frac{\| (C^{-T} \otimes C^{-1}) K \|_\infty \| w \|_\infty}{\| C^{-1} \|_\text{max}}, \\
c(\Phi; w) &= \frac{\| D^{-1}_{\varphi(w)} \varphi'(w) D w \|_\infty}{\varphi(w)} = \frac{\| (C^{-T} \otimes C^{-1}) K \|_\infty \| w \|_\infty}{\varphi(w)}.
\end{align*}
\]

We then consider the Cauchy linear systems \( Cx = b \). Let \( \psi : \mathbb{R}^{2n} \times \mathbb{R}^n \mapsto \mathbb{R}^n \) be defined as \( \psi(w, b) = C^{-1} b \). So we have
\[
\begin{align*}
d\psi &= d(C^{-1}) b + C^{-1} d b = -C^{-1} d C C^{-1} b + C^{-1} d b = -(x^T \otimes C^{-1}) \text{vec}(dC) + C^{-1} d b \\
&= \left[-(x^T \otimes C^{-1}) K, C^{-1}\right] \begin{bmatrix} d w \\ d b \end{bmatrix}.
\end{align*}
\]
Then we get the derivative \( \psi'(w, b) = \left[-(x^T \otimes C^{-1}) K, C^{-1}\right] \).

A simple calculation yields
\[
\begin{align*}
\| \psi'(w, b) \begin{bmatrix} D w \\ D b \end{bmatrix} \|_\infty &= \| \left[-(x^T \otimes C^{-1}) K, C^{-1}\right] \begin{bmatrix} D w \\ D b \end{bmatrix} \|_\infty \\
&= \| \left[-(x^T \otimes C^{-1}) K D w, C^{-1} D b \right] \|_\infty \\
&= \| (x^T \otimes C^{-1} K D w \|_\infty + \| C^{-1} D b \|_\infty \\
&= \| (x^T \otimes C^{-1}) K \|_\infty \| w \| + \| C^{-1} \|_\infty \| b \|_\infty.
\end{align*}
\]
Using (1.4), (1.5) and (3.2), we have the mixed and componentwise condition number for Cauchy linear systems,

\[
m(\psi; w, b) = \frac{\|\psi(w, b) [D_w D_b]\|_\infty}{\|x\|_\infty} = \frac{\|(x^T \otimes C^{-1})K|w| + |C^{-1}|b\|_\infty}{\|x\|_\infty},
\]
\[
c(\psi; w, b) = \frac{\|D_{\psi}^{-1}\psi(w, b) [D_w D_b]\|_\infty}{\|x\|_\infty} = \frac{\|(x^T \otimes C^{-1})K|w| + |C^{-1}|b\|_\infty}{\|x\|_\infty}.
\]

In the following we will give the upper bounds of these condition numbers. We first compute

\[
|K| = \begin{bmatrix} A_1 & A_1 E_1 \\ A_2 & A_2 E_2 \\ \vdots & \vdots \\ A_n & A_n E_n \end{bmatrix}
\]
\[
|t| = \begin{bmatrix} |t| \\ |s| \end{bmatrix}
\]
\[
|s| = \begin{bmatrix} A_1(|t| + |s_1|e), A_2(|t| + |s_2|e), \ldots, A_n(|t| + |s_n|e) \\ \end{bmatrix}
\]
\[
= \text{vec}([A_1|t|, A_2|t|, \ldots, A_n|t|] + [s_1A_1e, s_2A_2e, \ldots, s_nA_ne])
\]
\[
= \text{vec}(D_{|t|}C^S + C^SD_{|s|}).
\]

And so we have,

\[
m(\Phi; w) \leq \frac{\|C^{-T} \otimes C^{-1}|K| |w|\|_\infty}{\|C^{-1}\|_{\text{max}}} = \frac{\|C^{-T} \otimes C^{-1}|\text{vec}(D_{|t|}C^S + C^SD_{|s|})|C^{-1}\|_{\text{max}}}{\|C^{-1}\|_{\text{max}}},
\]
\[
(3.3)
\]
\[
c(\Phi; w) \leq \frac{\|C^{-1}(D_{|t|}C^S + C^SD_{|s|})|C^{-1}|\|_{\text{max}}}{\|C^{-1}\|_{\text{max}}},
\]
\[
(3.4)
\]
\[
m(\psi; w, b) \leq \frac{\|x^T \otimes C^{-1}|K| |w| + |C^{-1}|b\|_\infty}{\|x\|_\infty} \leq \frac{\|x^T \otimes C^{-1}|\text{vec}(D_{|t|}C^S + C^SD_{|s|}) + |C^{-1}|b\|_\infty}{\|x\|_\infty}
\]
\[
= \frac{\|C^{-1}(D_{|t|}C^S + C^SD_{|s|})x + |C^{-1}|b\|_\infty}{\|x\|_\infty},
\]
\[
(3.5)
\]
\[
c(\psi; w, b) \leq \frac{\|C^{-1}(D_{|t|}C^S + C^SD_{|s|})x + |C^{-1}|b\|_\infty}{\|x\|_\infty}.
\]
\[
(3.6)
\]

**Remarks.**

1. For the special case when \( t \) and \( s \) have different signs, i.e., \( t_is_j \leq 0 \) for \( 1 \leq i, j \leq n \), we have \( D_{|t|}C^S + C^SD_{|s|} = |C| \). Then above upper bounds (3.3)–(3.6) are equal to the usual unstructured condition numbers (2.1)–(2.4).

2. In [8,9], Gohberg and Koltracht gave a upper bound for the structured componentwise condition number \( c(\Phi; w) \).

\[
c(\Phi; w) \leq \frac{\|w\|_\infty}{\Delta} (8n - 6),
\]
\[
(3.7)
\]

where \( \Delta = \min_{i \neq j} |w_i - w_j| \). We will compare this bound with (3.1) in the numerical example, which shows that the bound is very efficient.
4. Vandermonde matrices

A Vandermonde matrix is defined by $n$ parameters as follows:

$$
V(v) := \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & a_1 & \cdots & a_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_0 & a_1 & \cdots & a_{n-1}
\end{bmatrix} \in \mathbb{R}^{n \times n},
$$

where $v := [a_0, a_1, \ldots, a_{n-1}]^T \in \mathbb{R}^n$. Vandermonde matrices play an important role in polynomial and rational interpolation problems. Investigating the structured conditioning of $V^{-1}$ has a special relevance in the sensitivity analysis of the Lagrange functions with respect to the interpolation nodes.

We first introduce some notations for this section.

\[
\begin{align*}
\hat{V} & \quad \text{ith row of } \hat{V} \\
\hat{V}(i,:) & \quad \text{jth column of } \hat{V} \\
\Pi_{nn} & \quad \text{commutation matrix, such that } \text{vec}(A^T) = \Pi_{nn} \text{ vec}(A) \\
w & \quad w := \text{vec}(V^T) = \Pi_{nn} \text{ vec}(V) \\
E_{ii} & \quad E_{ii} = e_i e_i^T \in \mathbb{R}^{n \times n}, \text{i.e., only } (i, i) \text{ element is } 1, \text{ and others are zeros} \\
A_E & \quad A_E = [\text{vec}(E_{11}), \text{vec}(E_{22}), \ldots, \text{vec}(E_{nn})] \\
L & \quad L = n^2 \times n \text{ matrix mapping } v \text{ to } w, \text{i.e., } L : v \mapsto w = L(v) \\
L' & \quad \text{Jacobimatrix, } L' := \partial w / \partial v^T
\end{align*}
\]

According to the definition, we have $\text{vec}(V) = \Pi_{nn} \text{ vec}(V^T) = \Pi_{nn} w = \Pi_{nn}(L(v))$, and then $\text{vec}(dV) = \Pi_{nn} L' dv$. After a direct computation, we have

\[
L' = \frac{\partial w}{\partial v^T} = \begin{bmatrix}
0 \\
1 \\
\vdots \\
\text{diag}(\hat{V}(3,:)) \\
\vdots \\
\text{diag}(\hat{V}(n,:))
\end{bmatrix} \in \mathbb{R}^{n^2 \times n},
\]

\[
\Pi_{nn} L' = \begin{bmatrix}
\hat{V}(i,1) & \hat{V}(i,2) & \cdots & \hat{V}(i,n) \\
\hat{V}(2,1) & \hat{V}(2,2) & \cdots & \hat{V}(2,n) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{V}(n,1) & \hat{V}(n,2) & \cdots & \hat{V}(n,n)
\end{bmatrix} = \begin{bmatrix}
\hat{V} \\
\hat{V} \\
\vdots \\
\hat{V}
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_n
\end{bmatrix} = (I \otimes \hat{V})[\text{vec}(E_{11}), \text{vec}(E_{22}), \ldots, \text{vec}(E_{nn})] = (I \otimes \hat{V}) A_E.
\]

It is easy to verify that for any vector $z = [z_1, z_2, \ldots, z_n]^T$,

\[
A_E z = [\text{vec}(E_{11}), \text{vec}(E_{22}), \ldots, \text{vec}(E_{nn})] z = \text{vec} \left( \sum_{j=1}^{n} z_j E_{jj} \right) = \text{vec}(D_z).
\]

Let $\Phi$ be the matrix function mapping the vector $v$ to the inversion of Vandermonde matrix $V$, i.e., $\Phi : v \mapsto \Phi(v) = V^{-1}$. The vector representation of $\Phi$ is $\varphi : v \mapsto \text{vec } \circ \Phi(v) = \text{vec}(V^{-1})$. Note that

\[
\text{vec}(dV^{-1}) = -(V^{-T} \otimes V^{-1}) \text{vec}(dV) = -(V^{-T} \otimes V^{-1}) \Pi_{nn} L' dv = -(V^{-T} \otimes V^{-1} \hat{V}) A_E dv.
\]
So we obtain the matrix derivative $\varphi'(v) = -(V^T \otimes V^{-1}\dot{V})A_E$. Then we can easily derive the following expression:

$$
\|\varphi'(v)D_v\|_\infty = \|(V^T \otimes V^{-1}\dot{V})A_E D_v\|_\infty = \|(V^T \otimes V^{-1}\dot{V})A_E v\|_\infty
$$

Applying (1.4) and (1.5), we obtain the mixed and componentwise structured condition numbers for the inversion of Vandermonde matrix.

$$
m(\Phi; v) = \frac{\|\varphi'(v)D_v\|_\infty}{\|\varphi'(v)\|_\infty} = \frac{\|(V^{-1}\dot{V})D_v|V^{-1}|\|\max}{\|V^{-1}\|\max}, \quad (4.2)
$$

$$
c(\Phi; v) = \|D_{\varphi'(v)}\varphi'(v)D_v\|_\infty = \|D_{\varphi'(v)}(V^{-1}\dot{V})\|\max.
$$

We now consider the Vandermonde linear systems $Vx = b$. Let $\psi : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ be defined as $\psi(v, b) = V^{-1}b$, where $V = V(v)$ is defined by (4.1).

$$
d\psi = d(V^{-1})b + V^{-1}db = -(x^T \otimes V^{-1})vec(dV) + V^{-1}db
$$

Then we get the derivative $\psi'(v, b) = -(x^T \otimes V^{-1}\dot{V})A_E, V^{-1}\). A direct computation yields

$$
\left\|\psi'(v, b)\begin{bmatrix} D_v \\ D_b \end{bmatrix}\right\|_\infty = \left\|\begin{bmatrix} -(x^T \otimes V^{-1}\dot{V})A_E, V^{-1} \end{bmatrix}D_v - V^{-1}\right\|_\infty
$$

Applying (1.4) and (1.5), we can easily deduce the mixed and componentwise structured condition numbers for Vandermonde linear systems.

$$
m(\psi; v, b) = \left\|\psi'(v, b)\begin{bmatrix} D_v \\ D_b \end{bmatrix}\right\|_\infty = \frac{\|\psi'(v, b)\begin{bmatrix} D_v \\ D_b \end{bmatrix}\|_\infty}{\|x\|\infty}, \quad (4.4)
$$

$$
c(\psi; v, b) = \left\|\begin{bmatrix} D_{\psi'(v, b)} \psi'(v, b) \end{bmatrix}D_v - V^{-1}\right\|_\infty
$$

Remarks. 1. The results can be applied to a more general case, for example, an $n \times n$ Vandermonde-like matrix, with node $\{a_i\}_{i=0}^{n-1},$

$$
V = \begin{bmatrix}
p_0(a_0) & p_0(a_1) & \cdots & p_0(a_{n-1}) \\
p_1(a_0) & p_1(a_1) & \cdots & p_1(a_{n-1}) \\
\vdots & \vdots & \ddots & \vdots \\
p_{n-1}(a_0) & p_{n-1}(a_1) & \cdots & p_{n-1}(a_{n-1})
\end{bmatrix},
$$

where $p_k \in \mathbb{P}_k$, the class of polynomials of degree $k$. 
2. We should point out that Bartels and Higham [1] have given equivalent results for \( c(\Phi; v) \) and \( c(\psi; v, b) \). Here we obtain the similar results by a different way.

5. Toeplitz and Hankel matrices

Let \( T \) be a nonsingular Toeplitz matrices, \( T := (t_{i,j})_{i,j=0}^{n-1} = \sum_{k=-n+1}^{n-1} t_k Z_k \), where \( Z_k \) is an \( n \times n \) matrix with 1’s in the positions of \( t_k \)'s in \( T \) and zeros elsewhere. In Matlab notation, \( T = \text{toeplitz}([t_0, \ldots, t_{n-1}], [t_0, \ldots, t_{n-1}]) \).

Let \( t := [t_{n-1}, \ldots, t_0]^{\top} \), and let \( \Phi \) be the map of matrix inversion, \( \Phi : t \mapsto \Phi(t) = T^{-1} \), and \( \phi(t) := \text{vec} \circ \Phi(t) = \text{vec}(T^{-1}) \). It is easy to check that

\[
\text{vec}(dT^{-1}) = -(T^{-T} \otimes T^{-1})[\text{vec}(Z_{-n+1}), \ldots, \text{vec}(Z_{n-1})] \, dt.
\]

So the derivative can be expressed as \( \phi'(t) = -[\text{vec}(T^{-1}Z_{-n+1}T^{-1}), \ldots, \text{vec}(T^{-1}Z_{n-1}T^{-1})] \).

Applying formula (1.4), we can derive the mixed condition number for the inversion of Toeplitz matrices.

\[
m(\Phi; t) := \frac{\|\phi'(t)D_1\|_\infty}{\|\phi(t)\|_\infty} = \frac{\|\phi'(t)|t_1|\|_\infty}{\|\phi(t)\|_\infty} = \frac{\|\sum_{k=-n+1}^{n-1} |t_k| \text{vec}((T^{-1}Z_kT^{-1}))\|_\infty}{\|\phi(t)\|_\infty}.
\]

Similarly, we can obtain the componentwise condition number according to (1.5).

\[
c(\Phi; t) = \|D_{\phi'(t)}\phi'(t)D_1\|_\infty = \left\| \sum_{k=-n+1}^{n-1} |t_k| T^{-1} Z_k T^{-1} \right\|_{\max} / T^{-1}.
\]

Let us consider the corresponding linear systems \( Tx = b \). Let \( \psi : (t, b) \mapsto \psi(t, b) = T^{-1}b \). In a similar way, we get

\[
\psi'(t, b) = -[(x^T \otimes T^{-1})[\text{vec}(Z_{-n+1}), \ldots, \text{vec}(Z_{n-1})], T^{-1}]
\]

\[
= [-T^{-1}Z_{-n+1}x, \ldots, -T^{-1}Z_{n-1}x, T^{-1}],
\]

\[
m(\psi; t, b) = \left\| \psi'(t, b) \left[ \begin{array}{c} D_t \\ D_b \end{array} \right] \right\|_\infty = \left\| \psi'(t, b) \left[ \begin{array}{c} |t_1| \\ |b| \end{array} \right] \right\|_\infty.
\]

\[
= \left\| \sum_{k=-n+1}^{n-1} |t_k| T^{-1} Z_k x \right\|_{\infty} / |x|_{\infty},
\]

and

\[
c(\psi; t, b) = \left\| D_{\psi'(t, b)}^{-1} \psi'(t, b) \left[ \begin{array}{c} D_t \\ D_b \end{array} \right] \right\|_\infty = \left\| \sum_{k=-n+1}^{n-1} |t_k| T^{-1} Z_k x \right\|_{\infty} / \max(|x|_{\infty}, |b|_{\infty}).
\]

Similar technique can be applied to analyze Hankel matrices. Let \( H \) be a nonsingular Hankel matrix, \( H := (h_{j+i})_{i,j=0}^{n-1} = \sum_{k=0}^{2n-2} h_k Z_k \), where \( Z_k \) is an \( n \times n \) matrix with 1’s in the positions of \( h_k \)'s in \( H \) and zeros elsewhere.

With a slight abuse of notation, we let \( \Phi : h \mapsto \Phi(h) = H^{-1} \), and \( \psi : (h, b) \mapsto \psi(h, b) = H^{-1}b \). With similar manipulation we can easily get the following results analogous to Toeplitz matrices:

\[
m(\Phi; h) = \left\| \sum_{k=0}^{2n-2} |h_k| H^{-1} Z_k H^{-1} \right\|_{\max} / \max(H^{-1}),
\]

\[
m(\psi; h, b) = \left\| \sum_{k=0}^{2n-2} |h_k| H^{-1} Z_k x \right\|_{\max} / \max(|x|_{\infty}, |b|_{\infty}),
\]

\[
c(\Phi; h) = \left\| \sum_{k=0}^{2n-2} |h_k| H^{-1} Z_k H^{-1} \right\|_{\max},
\]

\[
c(\psi; h, b) = \left\| \sum_{k=0}^{2n-2} |h_k| H^{-1} Z_k x \right\|_{\max} / \max(|x|_{\infty}, |b|_{\infty}).
\]
Remarks. 1. If $T$ is symmetric, then $t = [t_0, \ldots, t_{n-1}]^T$, and there is only a slight modification of the condition numbers, for example, the structured condition number for matrix inversion is

$$m(\Phi; t) = \frac{\left\| \sum_{k=0}^{n-1} t_k T^{-1} Z_k T^{-1} \right\|_{\text{max}}}{\| T^{-1} \|_{\text{max}}}.$$  

2. Since $\sum_{k=-n+1}^{n-1} |t_k T^{-1} Z_k T^{-1}| \leq \sum_{k=-n+1}^{n-1} |T^{-1}| |T| |T^{-1}| = |T^{-1}| \| T \| |T^{-1}|$, and we have

$$\sum_{k=-n+1}^{n-1} |t_k T^{-1} Z_k x| \leq |T^{-1}| \| T \| \| x \|,$$

That is, the structured condition numbers for Toeplitz matrices are always less than or equal to the unstructured ones as expressed by (2.1)–(2.4). We can get the similar results for Hankel matrices.

3. From (5.1), we have

$$\| \varphi'(t) D_t \|_{\infty} = \left\| \sum_{k=-n+1}^{n-1} |t_k T^{-1} Z_k T^{-1}| \right\|_{\text{max}} = \max_{i,j=1,\ldots,n} \sum_{k=-n+1}^{n-1} |t_k| r_i Z_k c_j,$$

where $r_i$ is the $i$th row of $T^{-1}$, and $c_j$ is the $j$th column of $T^{-1}$. Thus we recover the equivalent expression given in [9].

4. We can get $\partial \Phi / \partial t_k = -T^{-1} Z_k T^{-1}$, and $\partial \Phi_{i,j} / \partial t_k = -r_i Z_k c_j$, where the minus sign is missing in [9].

6. Circulant matrices

Let $\widehat{\mathbf{C}} := [c_0, c_1, \ldots, c_{n-1}]^T$. The circulant matrix $\widehat{\mathbf{C}}$ associated with $\widehat{\mathbf{c}}$ is defined as [3]

$$\widehat{\mathbf{C}}(\widehat{\mathbf{c}}) := \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}. \quad (6.1)$$

Let $Z_1, Z_2, \ldots, Z_{n-1}$ be the pattern matrix, such that $\widehat{\mathbf{C}} = c_0 I + c_1 Z_1 + c_2 Z_2 + \cdots + c_{n-1} Z_{n-1}$. Denote by $P$ the forward-shift permutation matrix mapping $[1, 2, \ldots, n]^T$ into $[2, \ldots, n, 1]^T$. Then $Z_{\gamma} = P^\gamma (\gamma = 1, \ldots, n-1)$. We first consider the inversion of circulant matrix. Let $\Phi : \widehat{\mathbf{c}} \mapsto \Phi(\widehat{\mathbf{c}}) = \widehat{\mathbf{C}}^{-1}$, and $\varphi := \text{vec} \circ \Phi$, i.e., $\varphi(\widehat{\mathbf{c}}) = \text{vec}(\widehat{\mathbf{C}}^{-1})$.

According to the definition above, we have

$$\text{vec}(\widehat{\mathbf{C}}) = [\text{vec}(I), \text{vec}(Z_1), \ldots, \text{vec}(Z_{n-1})] \widehat{\mathbf{c}},$$

$$\text{vec}(d\widehat{\mathbf{C}}^{-1}) = -(\widehat{\mathbf{C}}^{-T} \otimes \widehat{\mathbf{C}}^{-1}) \text{vec}(d\widehat{\mathbf{C}}) = -(\widehat{\mathbf{C}}^{-T} \otimes \widehat{\mathbf{C}}^{-1})[\text{vec}(I), \text{vec}(Z_1), \ldots, \text{vec}(Z_{n-1})] d\widehat{\mathbf{c}},$$

and

$$\varphi'(\widehat{\mathbf{c}}) = -(\widehat{\mathbf{C}}^{-T} \otimes \widehat{\mathbf{C}}^{-1})[\text{vec}(I), \text{vec}(Z_1), \ldots, \text{vec}(Z_{n-1})].$$

Because $\widehat{\mathbf{C}}^{-1}, Z_i (i = 1, 2, \ldots, n-1)$ are both circulant matrices, we have $\widehat{\mathbf{C}}^{-1} Z_i = Z_i \widehat{\mathbf{C}}^{-1}$. So $\varphi'(\widehat{\mathbf{c}})$ can be rewritten as

$$\varphi'(\widehat{\mathbf{c}}) = -[\text{vec}(\widehat{\mathbf{C}}^{-1} \widehat{\mathbf{C}}^{-1}), \text{vec}(\widehat{\mathbf{C}}^{-1} Z_1 \widehat{\mathbf{C}}^{-1}), \ldots, \text{vec}(\widehat{\mathbf{C}}^{-1} Z_{n-1} \widehat{\mathbf{C}}^{-1})]$$

$$= -[\text{vec}(\widehat{\mathbf{C}}^{-2}), \text{vec}(Z_1 \widehat{\mathbf{C}}^{-2}), \ldots, \text{vec}(Z_{n-1} \widehat{\mathbf{C}}^{-2})].$$
A simple computation yields
\[|\varphi'(\hat{C})|\hat{C}| = [\text{vec}((\hat{C}^{-2})), \text{vec}((Z_1\hat{C}^{-2})), \ldots, \text{vec}((Z_{n-1}\hat{C}^{-2}))]|\hat{C}| = [\text{vec}((\hat{C}^{-2})), \text{vec}((Z_1\hat{C}^{-2})), \ldots, \text{vec}((Z_{n-1}\hat{C}^{-2}))]|\hat{C}| = \text{vec}((|c_0|I + |c_1|Z_1 + \cdots + |c_{n-1}|Z_{n-1})\hat{C}^{-2}) = \text{vec}((\hat{C}||\hat{C}^{-2})].\]

Applying (1.4) and (1.5), we obtain the mixed and componentwise condition numbers for the inversion of circulant matrix:
\[
m(\Phi; \hat{C}) = \frac{\| \varphi'(\hat{C})D_{\hat{C}} \|_\infty}{\| \varphi(\hat{C}) \|_\infty} = \frac{\| \text{vec}(\hat{C}) \|_\infty}{\| \varphi(\hat{C}) \|_\infty} = \frac{\| \text{vec}(\hat{C}) \|_\infty}{\| \text{vec}(\hat{C}^{-2}) \|_\infty}
\]
\[
c(\Phi; \hat{C}) = \left\| D_{\hat{C}}^{-1} \varphi'(\hat{C})D_{\hat{C}} \right\|_\infty = \left\| \frac{|\hat{C}|}{\hat{C}^{-1}} \right\|_\infty. \tag{6.2}
\]

We now consider the circulant linear systems \(\hat{C}x = b\). Let \(\psi : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n\) be defined as \(\psi : (\hat{C}, b) \mapsto \psi(\hat{C}, b) = \hat{C}^{-1}b\), where \(\hat{C}\) is defined by (6.1).
\[
d\psi = d(\hat{C}^{-1})b + \hat{C}^{-1}db = -\hat{C}^{-1}d\hat{C}\hat{C}^{-1}b + \hat{C}^{-1}db = -(x^T \otimes \hat{C}^{-1}) \text{vec}(d\hat{C}) + \hat{C}^{-1}db
\]
\[
= -|\hat{C}^{-1}x, \hat{C}^{-1}Z_1x, \ldots, \hat{C}^{-1}Z_{n-1}x|d\hat{C} + \hat{C}^{-1}db
\]
\[
= -|\hat{C}^{-1}x, Z_1\hat{C}^{-1}x, \ldots, Z_{n-1}\hat{C}^{-1}x|d\hat{C} + \hat{C}^{-1}db.
\]

Then we get the derivative \(\psi'(\hat{C}, b) = [-\hat{C}^{-1}x, -Z_1\hat{C}^{-1}x, \ldots, -Z_{n-1}\hat{C}^{-1}x, \hat{C}^{-1}].\)

A simple computation yields
\[
|\psi'(\hat{C}, b)| |\hat{C}| = [|\hat{C}^{-1}x||, |Z_1\hat{C}^{-1}x||, \ldots, |Z_{n-1}\hat{C}^{-1}x||]|\hat{C}| + |\hat{C}^{-1}||b|
\]
\[
= (|c_0|I + |c_1|Z_1 + \cdots + |c_{n-1}|Z_{n-1})|\hat{C}^{-1}x|| + |\hat{C}^{-1}||b|
\]
\[
= |\hat{C}||\hat{C}^{-1}x| + |\hat{C}^{-1}||b|.
\]

Applying (1.4) and (1.5), we obtain the structured mixed and componentwise condition numbers for the circulant linear systems:
\[
m(\psi; \hat{C}, b) = \left\| \frac{\psi'(\hat{C}, b)}{|x|} \right\|_\infty = \left\| \frac{\psi'(\hat{C}, b)}{|b|} \right\|_\infty
\]
\[
= \left\| \frac{|\hat{C}||\hat{C}^{-1}x| + |\hat{C}^{-1}||b|}{|x|} \right\|_\infty, \tag{6.4}
\]
\[
c(\psi; \hat{C}, b) = \left\| D_{x}^{-1} \psi'(\hat{C}, b) \right\|_\infty = \left\| \frac{|\hat{C}||\hat{C}^{-1}x| + |\hat{C}^{-1}||b|}{x} \right\|_\infty. \tag{6.5}
\]

**Remarks.** 1. Because \(\hat{C}\) and \(\hat{C}^{-1}\) commute, we have
\[
m(\Phi; \hat{C}) = \frac{\| \hat{C}||\hat{C}^{-2}||_{\max}}{\| \hat{C}^{-1}||_{\max} = \frac{\| \hat{C}^{-1}||\hat{C}||\hat{C}^{-1}||_{\max}}{\| \hat{C}^{-1}||_{\max}}},
\]
\[
c(\Phi; \hat{C}) = \frac{\| \hat{C}||\hat{C}^{-2}||_{\max}}{\| \hat{C}^{-1}||_{\max} = \frac{\| \hat{C}^{-1}||\hat{C}||\hat{C}^{-1}||_{\max}}{\| \hat{C}^{-1}||_{\max}}. \]
That is, the structured condition numbers of circulant matrix inversion are the same as the general unstructured cases (2.1) and (2.2).

2. We point out that the formula (6.4) has been obtained by Rump [29]. Here we derive it via a different method and give expressions for \( c(\psi; \hat{c}, b), m(\Phi; \hat{c}) \) and \( c(\Phi; \hat{c}) \).

3. If there is no perturbation in the right-hand side of \( \hat{C}x = b \), then the mixed condition number is \( \| \hat{C} \| | \hat{C}^{-1}x | / \| x \| \), and the componentwise condition number is \( \| \hat{C} \| | \hat{C}^{-1}x | / \| x \| \), while the unstructured mixed and componentwise condition numbers are \( \| \hat{C}^{-1} \| \| \hat{C} \| | x | / \| x \| \) and \( \| \hat{C}^{-1} \| | \hat{C} \| | x | / \| x \| \), respectively. Such structured condition numbers can be much smaller than the unstructured ones, which will be shown by the fourth numerical example in Section 7.

7. Numerical examples

In this section, we will give four examples and show the differences between structured and unstructured condition numbers. All computations were carried out in MATLAB 6.1, which has unit roundoff \( 2^{-58} \approx 1.1 \times 10^{-16} \).

**Example 1.** Condition numbers of Vandermonde matrix inversion: Consider a \( 10 \times 10 \) Vandermonde matrix \( V = V(1, 1/2, \ldots, 1/10) \). We use (2.1) and (2.2) to compute the unstructured condition numbers, and use (4.2) and (4.3) to calculate the structured ones. The results are listed in Table 1.

**Example 2.** Condition numbers of Cauchy matrix inversion: We let \( t_i = i, s_j = 1 - j \) (\( 1 \leq i, j \leq 10 \)). In this case, the Cauchy matrix is also a Hilbert matrix

\[
C = \left( \frac{1}{i + j - 1} \right)_{i,j=1}^{10}.
\]

The computation results are also shown in Table 1. Note that in this case the upper bound given by (3.7) is about 740.

**Example 3 (Rump [28]).** Condition numbers of Toeplitz linear systems. Here we consider \( Tx = b \), where \( T = \text{toeplitz}([1, -1 - \varepsilon, 1 - \varepsilon, -1 + \varepsilon]), \varepsilon = 1e-4 \). We choose \( b \) such that \( x = [1, 1, 1, 1]^T \). Because the solution \( x \) is the vector of all 1’s, the mixed and componentwise condition numbers are equal. The results are given in Table 2. We can see that the unstructured condition numbers are much larger than the structured ones.

**Example 4 (Rump [29]).** Condition numbers of Circulant linear systems: Let \( \hat{C} = \hat{C}(c_0, c_1, \ldots, c_9) \) as defined by (6.1), where \( c_0 = \cdots = c_9 = 1, c_9 = -9 + \varepsilon, \varepsilon = 1e-6 \). Consider \( Ax = b \), where \( A = \hat{C}^{-1} \), and the right-hand side \( b \) is chosen such that \( x = [1, 1, \ldots, 1]^T \in \mathbb{R}^{10} \). Here we forbid the perturbations in the right hand side, which we discuss in Remark 3 of Section 6. The computed results are listed in Table 2.

| Table 1 | Condition numbers of matrix inversion |
|---------------------|---------------------|---------------------|---------------------|---------------------|
| | Unstructured condition numbers | Structured condition numbers |
| | Mixed | Componentwise | Mixed | Componentwise |
| Example 1 | 4.25e06 | 7.06e06 | 31.4 | 47.6 |
| Example 2 | 3.13e12 | 3.65e12 | 53.9 | 81.2 |

| Table 2 | Condition numbers of linear systems |
|---------------------|---------------------|---------------------|
| | Unstructured | Structured |
| Example 3 | 1.60e09 | 1.04e09 |
| Example 4 | 1.80e07 | 1.00 |
8. Conclusion

From the above analysis we can see the structured condition numbers can be much smaller than the unstructured ones. Some results of this paper have been given by other authors via different ways, and some have not been seen in earlier literatures. We can use this Kronecker product-based technique to derive the condition numbers for the linear combination of the above matrices, and also some other structured matrices. In addition, such technique can be extended to matrix equations, block structured matrices [20], and singular structured matrices [5,4,32], which will be our future work.

Acknowledgments

The authors would like to thank Prof. L. Reichel and two referees for their useful comments on the paper. We also thank Mr. H. Diao for his helpful discussions when he visited Fudan University.

References