Generalized reflexive solutions of the matrix equation $AXB = D$ and an associated optimal approximation problem

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\textbf{Abstract}

Let $R \in \mathbb{C}^{m \times m}$ and $S \in \mathbb{C}^{n \times n}$ be nontrivial unitary involutions, i.e., $R^\dagger = R = R^{-1} \neq I_m$ and $S^\dagger = S = S^{-1} \neq I_n$. We say that $G \in \mathbb{C}^{m \times n}$ is a generalized reflexive matrix if $RGS = G$. The set of all $m \times n$ generalized reflexive matrices is denoted by $\text{GRC}_{m \times n}$. In this paper, a sufficient and necessary condition for the matrix equation $AXB = D$, where $A \in \mathbb{C}^{m \times m}, B \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times r}$, to have a solution $X \in \text{GRC}_{m \times n}$ is established, and if it exists, a representation of the solution set $S_X$ is given. An optimal approximation between a given matrix $X \in \mathbb{C}^{m \times n}$ and the affine subspace $S_X$ is discussed, an explicit formula for the unique optimal approximation solution is presented, and a numerical example is provided.

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\section{Introduction}

In this paper we shall adopt the following notation. $\mathbb{C}^{m \times n}$ denotes the set of all $m \times n$ complex matrices. $\mathbb{U}C^{m \times n}$ denotes the set of all unitary matrices in $\mathbb{C}^{m \times n}$. $A^\dagger$ and $\|A\|$ stand for the conjugate transpose and the Frobenius norm of a complex matrix $A$, respectively. For $A, B \in \mathbb{C}^{m \times n}$, we define an inner product in $\mathbb{C}^{m \times n}$: $\langle A, B \rangle = \text{trace}(B^\dagger A)$; then $\mathbb{C}^{m \times n}$ is a Hilbert space. The matrix norm $\| \cdot \|$ induced by the inner product is the Frobenius norm. $I_n$ represents the identity matrix of size $n$. For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{C}^{m \times n}, A \ast B$ represents the Hadamard product of the matrices $A$ and $B$, i.e., $A \ast B = (a_{ij}b_{ij}) \in \mathbb{C}^{m \times n}$.

Throughout this paper $R \in \mathbb{C}^{m \times m}$ and $S \in \mathbb{C}^{n \times n}$ are nontrivial unitary involutions, i.e., $R^\dagger = R = R^{-1} \neq I_m$ and $S^\dagger = S = S^{-1} \neq I_n$. We say that $G \in \mathbb{C}^{m \times n}$ is a generalized reflexive matrix [1] if $RGS = G$. Let $J_n = (j_{ik})$ represent the exchange matrix of order $n$ defined by $j_{ik} = \delta_{i, n-k+1}$ for $1 \leq i, k \leq n$, where $\delta_{i,k}$ is the Kronecker delta, i.e., $J_n$ is a matrix with ones on the cross-diagonal and zeros elsewhere. By taking $m = n, R = S = J_n$, the generalized reflexive matrices reduce to the centrosymmetric matrices which play an important role in many areas [2–6]. Therefore, centrosymmetric matrices, whose special properties have been under extensive study [2,7–12], are a special case of the generalized reflexive matrices. Chen [1] discussed applications that give rise to these matrices and considered least squares problems involving them. In the following, we denote the set of all $m \times n$ generalized reflexive matrices by $\text{GRC}_{m \times n}$.

The linear matrix equation

$$AXB = D$$

(1)

has been considered by many authors. In [13], Penrose provided a sufficient and necessary condition for the consistency of this equation and, for the consistent case, gave a representation of its general solution. Yuan [14,15], Khatri and Mitra [16] got

\textsuperscript{*} Research supported by the National Natural Science Foundation of China (No. 10271055).

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necessary and sufficient conditions for the existence of symmetric solutions and symmetric positive semidefinite solutions, as well as explicit formulae using generalized inverses. Wang and Chang [17] studied least squares symmetric solutions to the equation using the generalized singular value decomposition, and a sufficient and necessary condition for its solvability and a representation of its general solution were also established therein.

In the present paper, we will consider generalized reflexive solutions of the matrix Eq. (1), where \( A \in \mathbb{C}^{p \times m}, B \in \mathbb{C}^{n \times q} \) and \( D \in \mathbb{C}^{m \times m}, \) and an associated optimal approximation problem:

\[
\min_{X \in S_X} \|X - \tilde{X}\|,
\]

where \( \tilde{X} \) is a given matrix in \( \mathbb{C}^{m \times n} \) and \( S_X \) is the solution set of the Eq. (1).

Using the generalized singular value decomposition, we give a necessary and sufficient condition for the Eq. (1) to have a solution \( X \in \mathbb{GRC}^{m \times n} \) and construct the solution set \( S_X \) explicitly when it is nonempty. We show that there exists a unique solution to the matrix optimal approximation problem (2) if the set \( S_X \) is nonempty and present an explicit formula for the unique solution. Clearly, the results obtained are shown to include those given in [18–20] as particular cases.

2. The solution of the matrix Eq. (1)

If \( \lambda \) is an eigenvalue of \( J \in \mathbb{C}^{m \times m} \), let \( V_\lambda(J) \) denote the eigenspace of \( J \) corresponding to the eigenvalue \( \lambda \). We will say that a vector \( z \in \mathbb{C}^n \) is \( R \)-symmetric (\( R \)-skew symmetric) if \( Rz = z[Rz = -z] \); thus, \( V_\lambda(1) \) and \( V_\lambda(-1) \) are the subspaces of \( \mathbb{C}^{m \times m} \) consisting respectively of \( R \)-symmetric and \( R \)-skew symmetric vectors. Let \( r = \dim[V_\lambda(1)], s = \dim[V_\lambda(-1)] \). Since a unitary involution is diagonalizable and \( R \neq \pm I_m \), we have \( r, s \geq 1 \), and \( r + s = m \). Let \( \{p_1, \ldots, p_r\} \) and \( \{q_1, \ldots, q_s\} \) be the orthonormal bases for \( V_\lambda(1) \) and \( V_\lambda(-1) \) respectively, and define

\[
P = [p_1, \ldots, p_r] \in \mathbb{C}^{m \times r}, \quad Q = [q_1, \ldots, q_s] \in \mathbb{C}^{m \times s}.
\]

Then \( [P, Q] \) is a unitary matrix and \( R \) has the following spectral decomposition:

\[
R = [P, Q] \begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix} \begin{bmatrix} P_H & Q_H \\ Q_H & P_H \end{bmatrix}.
\]

Similarly, there are positive integers \( k \) and \( l \) such that \( k + l = n \) and the matrices \( U \in \mathbb{C}^{n \times k} \) and \( V \in \mathbb{C}^{n \times l} \) whose column vectors form the orthonormal bases for the eigenspaces \( V_\lambda(1) \) and \( V_\lambda(-1) \) respectively. Thus, \([U, V] \) is a unitary matrix and \( S \) has the spectral decomposition:

\[
S = [U, V] \begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix} \begin{bmatrix} U_H & V_H \\ V_H & U_H \end{bmatrix}.
\]

In the following \( P, Q, U, V \) are always defined by (3) and (4).

(3) and (4) yield the following characterizations of \( m \times n \) generalized reflexive matrices, which are the special cases of [21, Theorem 1].

Lemma 1. \( G \) is a generalized reflexive matrix if and only if

\[
G = [P, Q] \begin{bmatrix} G_{PU} & 0 \\ 0 & G_{QV} \end{bmatrix} \begin{bmatrix} U_H & V_H \\ V_H & U_H \end{bmatrix},
\]

where \( G_{PU} = P^HGU, G_{QV} = Q^HGV. \)

For given matrices \( A_1 \in \mathbb{C}^{p \times r} \) and \( A_2 \in \mathbb{C}^{p \times s} \), let the generalized singular value decomposition (c.f. [22–24]) of the matrix pair \([A_1, A_2]\) be

\[
A_1 = M \Sigma_1 E^H, \quad A_2 = M \Sigma_2 F^H,
\]

where \( M \in \mathbb{C}^{p \times p} \) is a nonsingular matrix, and

\[
\Sigma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
E \in \mathbb{UC}^{x \times r}, F \in \mathbb{UC}^{x \times s}, c = s + a - b - t, a = \text{rank}(A_1), t = \text{rank}([A_1, A_2]) \) and

\[
\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_b), \quad \Theta = \text{diag}(\theta_1, \ldots, \theta_b)
\]
with
\[ 1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_b > 0, \quad 0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_b < 1, \quad \lambda_i^2 + \theta_i^2 = 1 \ (i = 1, \ldots, b). \]
Likewise, for given matrices \( B_1 \in \mathbb{C}^{k_x \times q} \) and \( B_2 \in \mathbb{C}^{k_y \times q} \), suppose that the generalized singular value decomposition of the matrix pair \([B_1^H, B_2^H]\) is
\[
B_1^H = N \Omega_1 \kappa^H, \quad B_2^H = N \Omega_2 \omega^H,
\]
where \( N \in \mathbb{C}^{q \times q} \) is a nonsingular matrix, and
\[
\begin{align*}
\Omega_1 &= \begin{bmatrix} I & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & h-d \\ d-f & f & k-d \end{bmatrix}, \\
\Omega_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & f \\ k & g & h-d \end{bmatrix}.
\end{align*}
\]
\( K \in \mathbb{C}^{k_x \times k}, \ W \in \mathbb{C}^{k_y \times l}, \ g = l + d - f - h, \ d = \text{rank}(B_1), \ h = \text{rank}([B_1^H, B_2^H]) \) and
\[
\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_l), \quad \Delta = \text{diag}(\delta_1, \ldots, \delta_l)
\]
with
\[ 1 > \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_l > 0, \quad 0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_l < 1, \quad \gamma_i^2 + \delta_i^2 = 1 \ (i = 1, \ldots, l). \]

**Theorem 1.** For given matrices \( A \in \mathbb{C}^{n \times m} \), \( B \in \mathbb{C}^{n \times q} \) and \( D \in \mathbb{C}^{q \times r} \), denote \( AP, AQ, U^H B \) and \( V^H B \) by \( A_1, A_2, B_1 \) and \( B_2 \), respectively. Suppose that the GSVDs of the matrix pairs \([A_1, A_2]\) and \([B_1^H, B_2^H]\) are (6) and (7). Partition \( M^{-1} D (N^{-1})^H \) into the following form:
\[
M^{-1} D (N^{-1})^H = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \begin{bmatrix} a-b \\ b \\ t-a \\ p-t \end{bmatrix}
\]
\[
\begin{bmatrix} d-f \\ f \\ h-d \\ q-h \end{bmatrix},
\]
Then the matrix Eq. (1) has a solution \( X \in \mathbb{GRC}^{m \times n} \) if and only if
\[
D_{13} = 0, \quad D_{31} = 0, \quad D_{41} = 0, \quad D_{4i} = 0, \quad \text{for } i = 1, 2, 3, 4; j = 1, 2, 3,
\]
in which case, the solution set \( S_X \) can be expressed as
\[
S_X = \left\{ X = [P, Q] \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} \right\},
\]
where
\[
Y = E \begin{bmatrix} D_{11} \\ \Delta^{-1} D_{21} \\ \Delta^{-1} (D_{22} - \Theta Z_{22} \Delta)^{-1} Y_{13} \\ Y_{31} \end{bmatrix} K^H,
\]
\[
Z = F \begin{bmatrix} Z_{11} \\ Z_{21} \\ \Theta^{-1} D_{23} \\ Z_{31} \end{bmatrix} W^H,
\]
and \( Y_{13}, Z_{11}, Y_{31}, Z_{1k} (i = 1, 2, 3; j = 1, 2; k = 2, 3) \) are arbitrary matrices.

**Proof.** If \( X \in \mathbb{GRC}^{m \times n} \), it follows from Lemma 1 that there exist \( Y \in \mathbb{C}^{m \times k} \), \( Z \in \mathbb{C}^{n \times l} \) satisfying
\[
X = [P, Q] \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}.
\]
Therefore, the matrix equation \( AXB = D \) has a solution \( X \in \mathbb{GRC}^{m \times n} \) if and only if the equation
\[
A_1 Y B_1 + A_2 Z B_2 = D
\]
admits a solution \( (Y, Z) \), where \( Y \in \mathbb{C}^{m \times k} \) and \( Z \in \mathbb{C}^{n \times l} \). Using the GSVDs of the matrix pairs \([A_1, A_2]\) and \([B_1^H, B_2^H]\), the equation of (14) is equivalent to
\[
M \Sigma_1 E^H Y K \Omega_1^H N^H + M \Sigma_2 F^H Z W \Omega_2^H N^H = D.
\]
Write

\[ F^H YK = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} a - b \\ b \\ r - a \end{bmatrix}, \]
\[ d - f \quad f \quad k - d \]

\[ F^H ZW = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} c \\ b \\ t - a \end{bmatrix}, \]
\[ g \quad f \quad h - d \]

(16)

(17)

Inserting (8), (16) and (17) into (15), we get

\[ \begin{bmatrix} Y_{11} & Y_{12} \gamma \\ A Y_{21} & A Y_{23} + \Theta Z_{22} \Delta & \Theta Z_{23} \\ 0 & Z_{32} \Delta & Z_{33} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix}. \]

Therefore, the above equation holds if and only if

\[ Y_{11} = D_{11}, \quad Y_{12} \gamma = D_{12}, \quad A Y_{21} = D_{21}, \quad \Theta Z_{23} = D_{23}, \quad Z_{32} \Delta = D_{32}, \quad Z_{33} = D_{33}, \]
\[ A Y_{22} \gamma + \Theta Z_{22} \Delta = D_{22}, \]
\[ D_{13} = 0, \quad D_{14} = 0, \quad D_{24} = 0, \quad D_{31} = 0, \quad D_{34} = 0, \quad D_{4j} = 0, \quad j = 1, 2, 3, 4. \]

(18)

(19)

(20)

Thus, from (18)–(20) the sufficient and necessary condition (9) under which \( S_X \) is nonempty and the expression of the solution set \( S_X \) are obtained.

3. The solution of the optimal approximation problem (2)

In the preceding section we have shown that if the condition (9) is satisfied, the solution set \( S_X \) is nonempty. It is easy to verify that \( S_X \) is a closed convex set in Hilbert space \( \mathbb{C}^{m \times n} \). Therefore, for a given matrix \( \tilde{X} \in \mathbb{C}^{m \times n} \), it follows from the best approximation theorem (See Aubin [25]) that there exists a unique solution \( \tilde{X} \) in \( S_X \) such that \( \| \tilde{X} - \tilde{X} \| = \min_{X \in S_X} \| X - \tilde{X} \| \).

Now, we shall focus our attention on seeking the unique solution of the optimal approximation problem (2) in \( S_X \). For the given matrix \( \tilde{X} \in \mathbb{C}^{m \times n} \) and any matrix \( X \in S_X \), we have

\[ \| X - \tilde{X} \| = \left\| \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} - \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \right\|^2 = \| Y - \alpha \tilde{X} \|^2 + \| Z - \beta \tilde{X} \|^2 + \| \alpha \tilde{X} \|^2 + \| \beta \tilde{X} \|^2, \]

(21)

where \( Y, Z \) are given by (11) and (12). Upon substitution, we see that

\[ \| X - \tilde{X} \| = \left\| \begin{bmatrix} D_{11} & D_{12} \gamma^{-1} \\ A^{-1} D_{21} & A^{-1} (D_{22} - \Theta Z_{22} \Delta) \gamma^{-1} \\ Y_{31} & Y_{32} \end{bmatrix} - E^H p^H \tilde{X} \right\|^2 + \| p^H \tilde{X} \|^2 + \| Q^H \tilde{X} \|^2. \]

(22)

Write

\[ E^H p^H \tilde{X} = \begin{bmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} & \tilde{Y}_{13} \\ \tilde{Y}_{21} & \tilde{Y}_{22} & \tilde{Y}_{23} \\ \tilde{Y}_{31} & \tilde{Y}_{32} & \tilde{Y}_{33} \end{bmatrix} \begin{bmatrix} a - b \\ b \\ r - a \end{bmatrix}, \]
\[ d - f \quad f \quad k - d \]

\[ F^H Q^H \tilde{X} = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} & \tilde{Z}_{13} \\ \tilde{Z}_{21} & \tilde{Z}_{22} & \tilde{Z}_{23} \\ \tilde{Z}_{31} & \tilde{Z}_{32} & \tilde{Z}_{33} \end{bmatrix} \begin{bmatrix} c \\ b \\ t - a \end{bmatrix}, \]
\[ g \quad f \quad h - d \]

(23)

(24)

It follows (22)–(24) that \( \| X - \tilde{X} \| = \min \) if and only if

\[ Y_{13} = \tilde{Y}_{13}, \quad Y_{23} = \tilde{Y}_{23}, \quad Y_{31} = \tilde{Y}_{31}, \quad Y_{32} = \tilde{Y}_{32}, \quad Y_{33} = \tilde{Y}_{33}, \]
\[ Z_{11} = \tilde{Z}_{11}, \quad Z_{12} = \tilde{Z}_{12}, \quad Z_{13} = \tilde{Z}_{13}, \quad Z_{21} = \tilde{Z}_{21}, \quad Z_{31} = \tilde{Z}_{31}. \]
and
\[ f(Z_{22}) := \| \Lambda^{-1}(D_{22} - \Theta Z_{22} \Delta) \Gamma^{-1} - \tilde{Y}_{22} \| ^2 + \| Z_{22} - \bar{Z}_{22} \| ^2 = \min. \] (27)

Let \( Z_{22} = [z_{ij}] \in \mathbb{C}^{b \times f} \), \( \bar{Y}_{22} = [\bar{y}_{ij}] \in \mathbb{C}^{b \times f} \), \( D_{22} = [d_{ij}] \in \mathbb{C}^{b \times f} \) and \( \bar{Z}_{22} = [\bar{z}_{ij}] \in \mathbb{C}^{b \times f} \). From (27) we have
\[ f(Z_{22}) = \sum_{i=1}^{b} \sum_{j=1}^{f} \left( \frac{\theta \delta_j \sum_{\lambda \in \Gamma} \lambda} {\theta \delta_j \sum_{\lambda \in \Gamma} \lambda} \right) \left( \frac{\theta \delta_j \sum_{\lambda \in \Gamma} \lambda} {\theta \delta_j \sum_{\lambda \in \Gamma} \lambda} \right) + \sum_{i=1}^{b} \sum_{j=1}^{f} (z_{ij} - \bar{z}_{ij})^2. \] (28)

Clearly, \( f(Z_{22}) \) is a differentiable function of \( b \times f \) variables \( z_{ij} (i = 1, \ldots, b; j = 1, \ldots, f) \). It is easy to verify that the function \( f(Z_{22}) \) attains the smallest value at
\[ z_{ij} = \frac{1}{\theta \delta_j \sum_{\lambda \in \Gamma} \lambda} \left( \theta \delta_j \sum_{\lambda \in \Gamma} \lambda \right). \] (29)

Let \( \Phi = \left[ \frac{1}{\theta \delta_j \sum_{\lambda \in \Gamma} \lambda} \right] \in \mathbb{C}^{b \times f} \); then (29) may be expressed as
\[ Z_{22} = \Phi \ast \left( \Theta^{\dagger} \tilde{Z}_{22} \Gamma^{\dagger} + \Theta D_{22} \Delta - \Lambda \Theta \tilde{Y}_{22} \Gamma \Delta \right). \] (30)

By now, we have proved the following result.

**Theorem 2.** Given an \( m \times n \) matrix \( \tilde{X} \), assume that the solution set \( S_{X} \subseteq \text{GRC}^{m \times n} \) is nonempty; then the matrix optimal approximation problem (2) has a unique solution \( \hat{X} \in S_{X} \). Furthermore, let the partitions of \( E^{H} P^{H} \tilde{X} UK \) and \( F^{H} Q^{H} X VW \) be (23) and (24); then the unique solution \( \hat{X} \) can be expressed as
\[
\hat{X} = [P, Q] \begin{bmatrix} D_{11} & D_{12} \Gamma^{-1} & \tilde{Y}_{13} \\ \tilde{Y}_{31} & D_{32} & \tilde{Y}_{33} \end{bmatrix} K^{H} 0 \\
0 & F \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} & \tilde{Z}_{13} \\ \tilde{Z}_{21} & \tilde{Z}_{22} & \tilde{Z}_{23} \\ \tilde{Z}_{31} & D_{32} \Delta^{-1} & D_{33} \end{bmatrix} W^{H} \end{bmatrix},
\] where \( Z_{22} \) is given by (30) and \( Y_{22} = \Lambda^{-1}(D_{22} - \Theta Z_{22} \Delta) \Gamma^{-1} \).

4. A numerical example

Based on Theorems 1 and 2 we can state the following algorithm.

**Algorithm 1 (An Algorithm for Solving the Optimal Approximation Problem (2)).**

1. Input \( R, S, A, B, D, \) and \( \tilde{X} \).
2. Form the matrices \( P, Q, U \) and \( V \), respectively, by the orthonormal bases of eigenspaces \( V_{X}(1), V_{X}(-1), V_{Y}(1) \) and \( V_{Y}(-1) \).
3. Compute \( A_{1} = AP, A_{2} = AQ, B_{1} = U^{H}B \) and \( B_{2} = V^{H}B \).
4. Compute the GSVDs of the matrix pairs \( [A_{1}, A_{2}] \) and \( [B_{1}^{H}, B_{2}^{H}] \) by (6) and (7), respectively.
5. Partition matrix \( M^{-1}(N^{-1})^{H} \) as in (8) to get \( D_{ij} (i, j = 1, 2, 3, 4) \).
6. If the condition (9) is satisfied, then the solution set \( S_{X} \) is nonempty and we continue. Otherwise we stop.
7. Partition matrices \( E^{H} P^{H} \tilde{X} UK \) and \( F^{H} Q^{H} X VW \) as in (23) and (24).
8. Compute \( Z_{22} \) by (30).
9. Compute \( Y_{22} = \Lambda^{-1}(D_{22} - \Theta Z_{22} \Delta) \Gamma^{-1} \).
10. Compute \( \hat{X} \) according to (31).

**Example 1.** Let \( m = 8, n = 7, p = 4, q = 3 \). Given
\[
R = \begin{bmatrix}
-0.6537 & -0.1864 & 0.3072 & 0.2857 & -0.2857 & 0.2797 & -0.3271 & -0.3089 \\
-0.1864 & 0.8109 & 0.4180 & -0.1165 & -0.0592 & -0.2980 & -0.1142 & 0.1188 \\
0.3072 & 0.4180 & -0.4018 & 0.0647 & -0.5639 & 0.4437 & -0.1281 & -0.1844 \\
0.2857 & -0.1165 & 0.0647 & 0.1009 & -0.2308 & -0.6321 & -0.0742 & -0.6574 \\
-0.2851 & -0.0592 & -0.5639 & -0.2308 & -0.0380 & -0.3560 & -0.6063 & 0.2196 \\
0.2797 & -0.2980 & 0.4437 & -0.6321 & -0.3560 & 0.1194 & -0.2640 & 0.1610 \\
-0.3271 & -0.1142 & -0.1281 & -0.0742 & -0.6063 & -0.2640 & 0.5971 & 0.2534 \\
-0.3089 & 0.1188 & -0.1844 & -0.6574 & 0.2196 & 0.1610 & 0.2534 & -0.5348
\end{bmatrix}.
\]
It is easy to verify that the condition (9) holds. According to Algorithm 1, we obtain the unique solution of the optimal approximation problem (2) as follows:

\[
\hat{X} =
\begin{bmatrix}
0.2713 & 1.4017 & -1.8208 & 2.3942 & 1.1827 & 0.2993 & 1.2665 \\
0.1461 & 0.8696 & -1.0159 & -0.2833 & 1.7247 & -1.0629 & -1.1649 \\
-0.6349 & 1.2225 & 2.3085 & -2.1812 & -1.7698 & 0.3148 & 0.2522 \\
-0.2406 & -0.2939 & 0.5980 & 1.1620 & -0.1355 & -0.8839 & -1.0451 \\
0.8037 & -0.6536 & -0.4941 & -1.7569 & 0.7952 & 0.5531 & -0.5341 \\
0.7276 & 0.8143 & -0.2630 & 0.9589 & -0.8962 & 1.8258 & 0.9531 \\
-1.2587 & 0.3334 & 0.9202 & 1.2867 & -0.3442 & -1.9649 & 0.4833 \\
0.7508 & 0.5646 & -0.2824 & 0.4649 & 0.0695 & 0.3460 & -1.0253 \\
\end{bmatrix}
\]

Also, we can figure out

\[\|A\hat{X}B - D\| = 1.7183e - 013, \quad \|\hat{X} - \hat{X}\| = 10.5148.\]

References


