On the inverses of general tridiagonal matrices

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In this work, the sign distribution for all inverse elements of general tridiagonal $H$-matrices is presented. In addition, some computable upper and lower bounds for the entries of the inverses of diagonally dominant tridiagonal matrices are obtained. Based on the sign distribution, these bounds greatly improve some well-known results due to Ostrowski (1952) [23], Shivakumar and Ji (1996) [26], Nabben (1999) [21,22] and recently given by Peluso and Politi (2001) [24], Peluso and Popolizio (2008) [25] and so forth. It is also stated that the inverse of a general tridiagonal matrix may be described by $2n - 2$ parameters ($\{\theta_k\}_{k=2}^{n}$ and $\{\phi_k\}_{k=1}^{n-1}$) instead of $2n + 2$ ones as given by El-Mikkawy (2004) [3], El-Mikkawy and Karawia (2006) [4] and Huang and McColl (1997) [10]. According to these results, a new symbolic algorithm for finding the inverse of a tridiagonal matrix without imposing any restrictive conditions is presented, which improves some recent results. Finally, several applications to the preconditioning technology, the numerical solution of differential equations and the birth–death processes together with numerical tests are given.

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1. Introduction

Tridiagonal matrices arise in many areas of science and engineering, for example in parallel computing, telecommunication system analysis and in solving differential equations using finite differences [2–4]. Therefore, research about such matrices attracts the attention of many authors. A particular result is due to Gantmacher and Krein [5] who proved that the inverse of an irreducible symmetric tridiagonal matrix is a so called Green’s matrix. In [11], Ikebe further stated that the inverse of a tridiagonal matrix can be described by four vectors of real numbers (see also, Section 4.1). Subsequently, this result was generalized in several directions, for more details, see [8].

Recently, some numerical algorithms (see Section 4) have been developed in [3,4,10,11,13,18] in order to give expressions of the entries of the inverse of a general tridiagonal matrix.

However, computing the inverse by using recurrence formulas sometimes leads to overflow and underflow problems, see Higham [8, p. 303]. Therefore, for many problems, it is very useful to have upper and lower bounds for the entries (or the absolute values of the entries) of the inverse of a matrix [6,19,21]. For example, estimates for upper bounds for the inverse elements of tridiagonal matrices arising in some boundary values problems have been given by Mattheij [19]. Later on, decay rates for the entries of inverses of certain tridiagonal and band matrices were established by Demko [2] and Nabben [21], respectively.

In this paper, first we show the sign distribution for all inverse elements of general tridiagonal H-matrices. Secondly, we establish some new upper and lower bounds for the entries of the inverses of diagonally dominant tridiagonal matrices, which improve the bounds given in [13,17–19,21–26]. Finally, a new symbolic algorithm to find the inverse of a general tridiagonal matrix without imposing any restrictive conditions is presented.

This paper is organized as follows: In Section 2, we mention several known results for tridiagonal matrices, which will be used in the following sections. In Section 3, we exhibit some new bounds on inverse elements of diagonally dominant tridiagonal matrices. In particular, we obtain the sign distribution for all inverse elements of general tridiagonal H-matrices. Finally, based on these results, a new symbolic algorithm for the inverses of general tridiagonal matrices is presented in Section 4 and some applications and numerical experiments are also given in Section 5.

2. Preliminaries

At first, let us consider a real tridiagonal matrix of the form

\[
A = \text{Tridiag} \{a_i, b_i, c_i\} \triangleq \begin{bmatrix}
 b_1 & c_1 & & & \\
 a_2 & b_2 & c_2 & & \\
 & \ddots & \ddots & \ddots & \\
 & & a_{n-1} & b_{n-1} & c_{n-1} \\
 & & & a_n & b_n
\end{bmatrix}, \quad a_i c_i \neq 0, \tag{2.1}
\]

which is row diagonally dominant, i.e.,

\[|b_i| \geq |a_i| + |c_i|, \quad \text{for all } i = 1, 2, \ldots, n,\]

where \(a_1 = c_n = 0\) and \(c_1, a_n \neq 0\). Furthermore, we sometimes assume that \(|b_1| > |c_1|\) and \(|b_n| > |a_n|\).

Remark 2.1. It is natural to suppose that \(a_i c_i \neq 0\) (\(i = 2, \ldots, n-1\)) and \(c_1, a_n \neq 0\), i.e., \(A\) is irreducible. In fact, if one of the \(a_i's\) or \(c_i's\) is zero, then the problem can be easily reduced to two subproblems of smaller dimension. In addition, we only consider row diagonally dominant matrices, because similar results can be obtained for column diagonally dominant matrices by transposition.

To state some known results, the following notations and definitions are needed.

For any \(A = [a_{ij}] \in \mathbb{C}^{n \times n}\) (\(n \times n\) complex matrices) with \(a_{ij} \neq 0\) and \(i \in N \triangleq \{1, 2, \ldots, n\}\), we define
\[
\mu_i \triangleq \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}| \quad \text{and} \quad \mu_0 = \mu_{n+1} = 0.
\]

and

\[
J(A) \triangleq \{ i \in N \mid \mu_i < 1 \}.
\]

The following definitions were given in Ref. [9].

**Definition 2.1.** A complex \( n \times n \) matrix \( A = [a_{ij}] \) with \( J(A) = N \) is called row strictly diagonally dominant (SDD); if \( A \) is irreducible, \( \mu_i \leq 1 \) for all \( i \in N \) and \( J(A) \neq \emptyset \), then \( A \) is said to be irreducible diagonally dominant (IDD); \( A \) is weakly chained diagonally dominant (WCDD) if and only if \( \mu_i \leq 1 \) for all \( i \in N \) and \( J(A) \neq \emptyset \), and for \( i \notin J(A) \), there exist indices \( i_1, i_2, \ldots, i_k \) with \( a_{ir,i_{r+1}} \neq 0 \), \( 0 \leq r \leq k - 1 \), where \( i_1 = i \) and \( i_k \in J(A) \).

**Definition 2.2.** A real \( n \times n \) matrix \( A = [a_{ij}] \) with \( a_{ij} \leq 0 \) for all \( i \neq j \) is an (nonsingular) \( M \)-matrix if \( A \) is nonsingular and \( A^{-1} \geq 0 \) (i.e., \( A^{-1} \) is a nonnegative matrix).

**Definition 2.3.** A matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) is nonsingular if and only if its comparison matrix \( \langle A \rangle \) is an \( M \)-matrix, where \( \langle A \rangle = [\langle a_{ij} \rangle] \in \mathbb{R}^{n \times n} \) is defined by

\[
\langle a_{ij} \rangle = \begin{cases} 
|a_{ij}|, & \text{for } i = 1, 2, \ldots, n; \\
-|a_{ij}|, & \text{for } i \neq j, i, j = 1, 2, \ldots, n.
\end{cases}
\]

Obviously, for irreducible matrices, the following relations hold

\[
\text{SDD} \Rightarrow \text{IDD} \Rightarrow \text{WCDD} \Rightarrow \mathcal{H}.
\]

As for the entries of the inverse of a matrix, Ostrowski presented some upper and lower bounds for the entries of the inverse of an arbitrary SDD matrix in [23]. Later on, this result was generalized to WCDD matrices in [27].

**Lemma 2.1.** Let \( A \) be a SDD (or WCDD) matrix, then \( A^{-1} = [c_{ij}] \) exists and for any \( i \neq j \),
\[
|c_{ij}| \leq \mu_i |c_{ij}| \leq |c_{jj}|.
\]

and for any \( i \in N \),
\[
\frac{1}{|a_{ii}|(1 + \mu_i)} \leq |c_{ii}| \leq \frac{1}{|a_{ii}|(1 - \mu_i)}.
\]

For tridiagonal matrices, Shivakumar and Ji [26] presented the following result:

**Lemma 2.2.** For any nonsingular tridiagonal \( n \times n \) matrix \( A = \text{Tridiag}[a_i, b_i, c_i] \), \( A^{-1} = [A_{ij} / \det A] \), where the cofactors \( A_{ij} \) of \( A \) are given by
\[
A_{ij} = \begin{cases} 
(-1)^{i+j} \left( \prod_{k=i+1}^{j} a_k \right) \det A^{(1,i-1)} \det A^{(j+1,n)}, & \text{when } i \leq j; \\
(-1)^{i+j} \left( \prod_{k=j}^{i-1} c_k \right) \det A^{(1,j-1)} \det A^{(i+1,n)}, & \text{when } i > j.
\end{cases}
\]

for \( i, j = 1, 2, \ldots, n \), where \( A^{(r,s)} \) (\( s \geq r \)) is the square submatrix of order \( s - r + 1 \) of \( A \) whose diagonal entries are \( a_{rr}, a_{r+1,r+1}, \ldots, a_{ss} \). In the above, \( \det A^{(1,0)} \) and \( \det A^{(n+1,1)} \) are each defined to be one.

Though explicit formulae for the entries of the inverse of a general tridiagonal matrix can be derived from Lemma 2.2, it is very difficult to obtain them by computing \( \det A^{(i,j)} \) (\( i, j \in N \)). Therefore, some
upper and lower bounds for the entries of inverses of some tridiagonal matrices have been investigated. For example, the following bounds were given in [22,24,26], respectively.

Theorem 2.1. Let $A$ be a nonsingular row diagonally dominant tridiagonal matrix of the form (2.1), then $A^{-1} = [c_{ij}]$ exists and the following upper and lower bounds hold

$$(1) \ [26]$$
$$
\begin{align*}
|c_{ij}| \prod_{k=i+1}^{j-1} \frac{|c_k|}{|b_k|^2 + |a_k|^2} &\leq |c_{ij}| \prod_{k=i+1}^{j-1} \mu_k, \quad \text{when } i < j, \\
|c_{ij}| \prod_{k=j+1}^{j-1} \frac{|c_k|}{|b_k|^2 + |a_k|^2} &\leq |c_{ij}| \prod_{k=j+1}^{j-1} \mu_k, \quad \text{when } i > j.
\end{align*}
$$

$$(2) \ [22,24]$$
$$
\begin{align*}
|c_{ij}| \prod_{k=i+1}^{j-1} \delta_k &\leq |c_{ij}| \prod_{k=i+1}^{j-1} \tau_k, \quad \text{when } i < j, \\
|c_{ij}| \prod_{k=j+1}^{j-1} \gamma_k &\leq |c_{ij}| \prod_{k=j+1}^{j-1} \omega_k, \quad \text{when } i > j;
\end{align*}
$$

where
$$
\tau_k = \frac{|c_k|}{|b_k| - |a_k|}, \quad \omega_k = \frac{|a_k|}{|b_k| - |c_k|}, \quad \delta_k = \frac{|c_k|}{|b_k| - |a_k|}, \quad \gamma_k = \frac{|a_k|}{|b_k| - |c_k|}, \quad k \in \mathbb{N}.
$$

For the diagonal elements of $A^{-1}$, the following inequalities also hold for any $i \in \mathbb{N}$.

$$(3) \ [26]$$
$$
\begin{align*}
1 &\leq |c_{ii}| \leq \frac{1}{|b_i| + |a_i|\mu_{i-1} + |c_i|\mu_{i+1}},
\end{align*}
$$

where $\mu_0 = \mu_{n+1} = 0$.

$$(4) \ [22,24]$$
$$
\begin{align*}
1 &\leq |c_{ii}| \leq \frac{1}{|b_i| + |a_i|\tau_{i-1} + |c_i|\omega_{i+1}},
\end{align*}
$$

where $\tau_0 = \omega_{n+1} = 0$.

Decay rates for the entries of inverses of some tridiagonal matrices were given in [21].

Theorem 2.2. Let $A \in \mathbb{R}^{n \times n}$ be an irreducible tridiagonal $M$-matrix as in (2.1). If $A$ is diagonally dominant by rows and $b_1 > c_1, b_n > a_n$, then $A^{-1} = [c_{ij}]$ exists and

$$
\begin{align*}
c_{ij} &\leq \rho_1^{-1} c_{ij}, \quad \text{when } i < j; \\
c_{ij} &\leq \rho_2^{-1} c_{ij}, \quad \text{when } i > j.
\end{align*}
$$

where $\rho_1 := \max_{i \geq 2} \tau_i, \rho_2 := \max_{i \geq 2} \omega_i$.

Next, we will give some similar upper and lower bounds for all entries of $A^{-1}$, which improve the bounds mentioned above.

3. Some improvements on two-sided bounds

In this section, first we improve the result of Lemma 2.1 and obtain the sign distribution for all inverse elements of general tridiagonal $H$-matrices. Second, we derive some bounds for off-diagonal elements of the inverse of a diagonally dominant matrix as a function of the diagonal ones, which improve the known results in [13,17–19,21–26]. Finally, for the case of $M$-matrices, these bounds are further investigated.

3.1. Two general results

At first, if $\text{sign}(u)$ denotes the sign function, then we have $u = |u|\text{sign}(u)$ for any real number $u$. 
Lemma 3.1. Let $A = [a_{ij}]$ be an SDD, IDD, or WCDD matrix, then $A^{-1} = [c_{ij}]$ exists and for any $i \neq j$,

$$|c_{ij}| \leq \frac{|a_{ij}| + \sum_{k \neq i,j} |a_{ik}| \mu_k}{|a_{ii}|} |c_{jj}| \leq |c_{ij}|, \quad (3.1)$$

and for any $i \in \mathbb{N}$,

$$\frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}||\mu_j| \leq |c_{ii}| \leq \frac{1}{|a_{ii}| - \sum_{j \neq i} |a_{ij}| \mu_j}. \quad (3.2)$$

Proof. We only show (3.1) for WCDD matrices, for the other cases (SDD or IDD) can be derived similarly. Since $c_{ij}$ is the entry of the inverse of $A$, for any $i \neq j$, we then have

$$\sum_{k=1, k \neq i,j}^n a_{ik}c_{kj} + a_{ij}c_{jj} = 0,$$

i.e.,

$$a_{ij}c_{ij} = -a_{ij}c_{jj} - \sum_{k=1, k \neq i,j}^n a_{ik}c_{kj}.$$  

Therefore, by (2.2), we have

$$|a_{ii}||c_{ij}| \leq |a_{ij}||c_{jj}| + \sum_{k=1, k \neq i,j}^n |a_{ik}||c_{kj}| \leq |a_{ij}||c_{jj}| + \sum_{k=1, k \neq i,j}^n |a_{ik}| |\mu_k||c_{jj}|$$

$$\leq (|a_{ij}| + \sum_{k=1, k \neq i,j}^n |a_{ik}| |\mu_k|)|c_{jj}|.$$  

Thus, we obtain the inequality (3.1). Similarly, by (2.3), we get the inequality (3.2). The proof is completed. □

Remark 3.1. If the matrix $A$ in Lemma 3.1 is a tridiagonal matrix, then we easily imply Theorem 2.2 of [26] (i.e., the above (2.7)), therefore, this lemma is a generalization of Theorem 2.2 of [26] in the diagonally dominant case. In addition, it is worth pointing out that a special version (when $A \in \text{SDD}$) of this lemma has been mentioned by some of the authors in [14], but the above proof is simpler. For other results on diagonally dominant matrices, see [14,16].

By Definition 2.2, we know that the inverse matrix of any tridiagonal $M$-matrix is nonnegative. But for a general nonsingular tridiagonal matrix, the sign distribution of the entries of its inverse is still an open problem. In fact, for any tridiagonal $H$-matrix, we have the following more general conclusion.

Theorem 3.1. Let $A$ be a tridiagonal matrix of the form (2.1). If $A$ is a nonsingular $H$-matrix, then $A^{-1} = [c_{ij}]$ exists and

1. $\text{sign}(c_{ij}) = (-1)^{i+j}\text{sign} \left( b_j \prod_{k=j+1}^i a_k b_k \right), \quad i > j$;
2. $\text{sign}(c_{ii}) = \text{sign}(b_i), \quad i = j$;
3. $\text{sign}(c_{ij}) = (-1)^{i+j}\text{sign} \left( b_i \prod_{k=i-1}^{j-1} c_k b_k \right), \quad i < j$.

In particular, when $b_i > 0 \ (i \in \mathbb{N})$, we have that
(4) \( \text{sign}(c_{ij}) = (-1)^{(i+j)} \text{sign}\left( \prod_{k=j+1}^{i} a_k \right) \), \( i > j \);

(5) \( \text{sign}(c_{ij}) = (-1)^{(i+j)} \text{sign}\left( \prod_{k=i+1}^{j} c_k \right) \), \( i < j \).

**Proof.** We only prove (1). The other results may be derived similarly. As \( A \) is a nonsingular \( H \)-matrix, by [9, Chapter 6], there exists a positive diagonal matrix \( D \) such that \( D^{-1} A \) is a strictly row diagonally dominant matrix. Thus, according to Gerschgorin’s disc theorem [9, p. 344], we have that
\[
\text{sign}(\det A) = \text{sign}\left( \prod_{k=1}^{n} \frac{b_1 \cdots b_{k-1} (b_{i+1} \cdots b_n)}{b_1 \cdots b_k} \right).
\]

Since each principal submatrix of \( A \) is also a nonsingular \( H \)-matrix, it follows that
\[
\text{sign}(\det A_{(r,s)}) = \text{sign}(b_{r+1} \cdots b_s) \quad \text{for any } r \leq s.
\]

Now, for any \( i > j \), we have, from Lemma 2.2, that
\[
\text{sign}(c_{ij}) = \text{sign}\left( \frac{A_{ij}}{\det A} \right) = (-1)^{i+j} \text{sign}\left( \prod_{k=j+1}^{i} a_k \right) \text{sign}\left( \frac{\det A_{1j}}{\det A} \frac{\det A_{i+1,n}}{\det A} \right).
\]

Thus the proof is completed. \( \Box \)

From this theorem, one can see that the sign distribution of a tridiagonal \( H \)-matrix uniquely determines the sign pattern of its inverse, for example,
\[
\text{sign}(A) = \begin{bmatrix} - & + & + & + \\ - & + & + & + \\ + & - & - & + \\ - & + & + & - \end{bmatrix} \Rightarrow \text{sign}(A^{-1}) = \begin{bmatrix} - & + & + & - \\ - & + & + & - \\ - & + & - & + \\ - & + & - & + \end{bmatrix}.
\]

However, when the tridiagonal matrix \( A \) is not an \( H \)-matrix, this conclusion need not hold, for example,
\[
A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 1 \end{bmatrix}.
\]

**Remark 3.2.** The sign distribution of the entries of the inverse of a matrix is a very useful information in many cases such as the matrix completion problems [20] and inverse problems of matrices. In this paper, we will apply it to locate the entries of the inverses of some tridiagonal matrices and build some effective preconditioners for the matrix \( A \), see Section 5.1.

### 3.2. Upper and lower bounds for the inverse of a tridiagonal matrix

According to the above results, we now establish some upper and lower bounds for the entries of the inverses of some tridiagonal matrices in this section.
Theorem 3.2. Let $A$ be a nonsingular tri-diagonal matrix of the form (2.1). If $A$ is an SDD, IDD, or WCDD matrix, then $A^{-1} = [c_{ij}]$ exists and

1. $|c_{ij}| \prod_{k=j+1}^{i} \tilde{\mu}_k \leq |c_{ij}| \prod_{k=j+1}^{i} \tilde{\mu}_k, \ j < i,$
2. $|c_{ij}| \prod_{k=i}^{i-1} \tilde{\tau}_k \leq |c_{ij}| \prod_{k=i}^{i-1} \tilde{\tau}_k, \ j > i,$

where, for $k \in \mathbb{N},$

$$\tilde{\mu}_k = \frac{|a_k|}{|b_k| - |c_k| \mu_{k+1}}, \quad \tilde{\mu}_k = \frac{|a_k|}{|b_k| + |c_k| \mu_{k+1}},$$

$$\tilde{\tau}_k = \frac{|c_k|}{|b_k| - |a_k| \mu_{k+1}}, \quad \tilde{\tau}_k = \frac{|c_k|}{|b_k| + |a_k| \mu_{k-1}}.$$  

Proof. By Lemma 2.2, for any $i, j \in \mathbb{N}$ we have $c_{ij} = A_{j,i}/\det A$, where $A_{j,i}$ are the cofactors of $A$.

1. When $j < i$, we have, from Lemma 2.2, that

$$\frac{c_{ij}}{c_{jj}} = \frac{A_{j,i}}{A_{jj}} = \prod_{k=j+1}^{i} |a_k| \cdot \frac{\det A(i+1,n)}{\det A(i-1,n)} = \prod_{k=j+1}^{i} |a_k| \cdot \prod_{p=j+1}^{i} |a_k| \frac{\det A(9+1,n)}{\det A(p,n)}. \quad (3.3)$$

In addition, by Lemma 3.1 (see (3.2)), we get

$$\frac{1}{|b_p| - |c_p| \mu_{p+1}} \leq \frac{\det A(9+1,n)}{\det A(p,n)} \leq \frac{1}{|b_p| + |c_p| \mu_{p+1}}. \quad (3.4)$$

Now combining (3.3) and (3.4) together, the conclusion (1) follows.

2. When $j > i$, the conclusion (2) can be derived similarly by Lemmas 2.2 and 3.1.  

It is well-known that the rate of decay is an important parameter to construct sparse approximations of the inverse as preconditioners [2, 6, 21]. Note that $\tilde{\mu}_k, \tilde{\mu}_k, \tilde{\tau}_k, \tilde{\tau}_k$ are generally less than 1, so the entries of the inverse tend to zero as $|i - j|$ becomes larger for some tridiagonal matrices. The following corollary presents an elegant estimate for this problem, which improves some results of [21] (see Theorem 2.2).

Corollary 3.1. Let $A^{-1} = [c_{ij}]$ be the inverse of the matrix $A$ defined in Theorem 3.2. Then the following inequalities hold

$$c_{ij} \leq \lambda_1 1_{i-j} c_{jj}, \quad \text{when } i < j,$$

$$c_{ij} \leq \lambda_2 1_{i-j} c_{jj}, \quad \text{when } i > j,$$

where $\lambda_1 := \max_{k > i} \tilde{\tau}_k, \lambda_2 := \max_{k > j} \tilde{\mu}_k.$

Theorem 3.3. Let $A^{-1} = [c_{ij}]$ be the inverse of the matrix $A$ defined in Theorem 3.2. Then the following inequalities hold for any $i \in \mathbb{N}$:

$$\frac{1}{|b_i| + |a_i| \tilde{\tau}_{i-1} + |c_i| \tilde{\mu}_{i+1}} \leq |c_{ii}| \leq \frac{1}{|b_i| - |a_i| \tilde{\tau}_{i-1} - |c_i| \tilde{\mu}_{i+1}}, \quad (3.6)$$

where $\tilde{\tau}_0 = \tilde{\mu}_{n+1} = 0$.

Proof. Expanding $\det A$ by the ith row, we have that

$$\det A = a_i A_{i,i-1} + b_i A_{i,i} + c_i A_{i,i+1}.$$
i.e.,
\[
A_{i,i-1} + b_i \frac{A_{i,j}}{\det A} + c_i \frac{A_{i,i+1}}{\det A} = 1,
\]
where \(A_{i,0} = A_{i,n+1} = 0\).

By taking absolute values and using (3.1), we further get
\[
\left| 1 - b_i \frac{A_{i,j}}{\det A} \right| \leq (|a_i| |\tilde{\tau}_{i-1} + |c_i| |\tilde{\mu}_{i+1}|) \left| \frac{A_{i,j}}{\det A} \right|.
\]
i.e.,
\[
|1 - b_i c_i| \leq (|a_i| |\tilde{\tau}_{i-1} + |c_i| |\tilde{\mu}_{i+1}|)|c_i|,
\]
from which the conclusion (3.6) follows. \(\Box\)

Now combining Theorems 3.1, 3.2, and 3.3, we immediately obtain the following result.

**Theorem 3.4.** Let \(A^{-1} = [c_{ij}]\) be the inverse of the matrix \(A\) defined in Theorem 3.2. Then

(1)
\[
\prod_{j=i+1}^{i} \tilde{\mu}_k \left| \prod_{j=i+1}^{i} \bar{\mu}_k \right| \leq (-1)^{(i+j)} \text{sign} \left( b_i \prod_{k=i+1}^{i} a_k b_k \right) c_{ij}
\]
\[
\leq \left| b_j \right| - \left| a_j \right| \left| \tilde{\tau}_{i-1} + |c_j| |\tilde{\mu}_{i+1}| \right|, \quad j \leq i;
\]

(2)
\[
\prod_{k=i}^{i} \tilde{\tau}_k \left| \prod_{k=i}^{i} \bar{\tau}_k \right| \leq (-1)^{(i+j)} \text{sign} \left( b_j \prod_{k=i}^{i} c_k b_k \right) c_{ij}
\]
\[
\leq \left| b_j \right| - \left| a_j \right| \left| \tilde{\tau}_{i-1} + |c_j| |\tilde{\mu}_{i+1}| \right|, \quad j \geq i.
\]

Next, similar to Theorem 3.2 of [24], we may obtain sharper two-sided bounds for the diagonal elements of \(A^{-1}\), exploiting the signs of \(c_{i-1,i}\) and \(c_{i+1,i}\) (see Theorem 3.1).

**Theorem 3.5.** Let \(A\) be a nonsingular tridiagonal matrix of the form (2.1) and \(A^{-1} = [c_{ij}]\). If \(A\) is row diagonally dominant, then

\[
\frac{1}{\left| b_i \right| + p_i |a_i| + q_i |c_i|} \leq \text{sign}(b_i) |c_{ij}| \leq \frac{1}{\left| b_i \right| + f_i |a_i| + g_i |c_i|}, \quad i \in N,
\]

where
\[
p_i = \begin{cases} \tilde{\tau}_{i-1}, & \text{if } a_i b_i c_{i-1} b_{i-1} < 0, \\ -\tilde{\tau}_{i-1}, & \text{if } a_i b_i c_{i-1} b_{i-1} > 0, \end{cases}
\]
\[
f_i = \begin{cases} -\tilde{\tau}_{i-1}, & \text{if } a_i b_i c_{i-1} b_{i-1} > 0, \\ \tilde{\tau}_{i-1}, & \text{if } a_i b_i c_{i-1} b_{i-1} < 0, \end{cases}
\]
\[
q_i = \begin{cases} \tilde{\mu}_{i+1}, & \text{if } c_i b_i a_{i+1} b_{i+1} < 0, \\ -\tilde{\mu}_{i+1}, & \text{if } c_i b_i a_{i+1} b_{i+1} > 0, \end{cases}
\]
\[
g_i = \begin{cases} -\tilde{\mu}_{i+1}, & \text{if } c_i b_i a_{i+1} b_{i+1} > 0, \\ \tilde{\mu}_{i+1}, & \text{if } c_i b_i a_{i+1} b_{i+1} < 0. \end{cases}
\]

**Proof.** The proof is completely similar to that of Theorem 3.2 of [24]. \(\Box\)

**Remark 3.3.** Obviously, for diagonally dominant tridiagonal matrices, all \(\bar{\mu}_k (k \in N)\) are less than or equal to 1. Therefore, for any \(k \in N\),
\[ \tilde{\mu}_k = \frac{|a_k|}{|b_k| - |c_k|\mu_{k+1}} \leq \frac{|a_k|}{|b_k| - |c_k|} = \omega_k \leq \frac{|a_k|}{|b_k|} = \mu_k. \]

\[ \tilde{\tau}_k = \frac{|c_k|}{|b_k| - |a_k|\mu_{k-1}} \leq \frac{|c_k|}{|b_k|} = \tau_k \leq \frac{|c_k| + |a_k|}{|b_k|} = \mu_k. \]

\[ \check{\mu}_k = \frac{|a_k|}{|b_k| + |c_k|\mu_{k+1}} \geq \frac{|a_k|}{|b_k| + |c_k|} = \gamma_k \leq \frac{|a_k|}{|b_k|(1 + \mu_k)}, \]

\[ \check{\tau}_k = \frac{|c_k|}{|b_k| + |a_k|\mu_{k-1}} \geq \frac{|c_k|}{|b_k| + |a_k|} = \delta_k \leq \frac{|c_k|}{|b_k|(1 + \mu_k)}, \]

which shows that all results in this section improve those of Theorems 2.1 and 2.2. In particular, when \( A \) is an SDD matrix or \(|b_1| > |c_1|\) and \(|b_n| > |a_n|\), our results are sharper and the signs for each element of the inverse may also be obtained.

### 3.3. The case of M-matrices

If \( A \) is an M-matrix, then \( c_{ij} \geq 0 \) for any \( i, j \in N \) \((A^{-1} = [c_{ij}]\)). In this case, the lower and upper bounds for \( c_{ij} \) proved in Theorems 3.4 and 3.5 can be further improved.

**Theorem 3.6.** Let \( A \) be an M-matrix defined in Theorem 3.2, then \( A^{-1} = [c_{ij}] \) exists and

\[
\frac{1}{|b_i| - |a_i|\tilde{\tau}_{i-1} - |c_i|\tilde{\mu}_{i+1}} \leq c_{ij} \leq \frac{1}{|b_i| - |a_i|\tilde{\tau}_{i-1} - |c_i|\tilde{\mu}_{i+1}}, \quad i \in N. \tag{3.8}
\]

**Proof.** Since \( AA^{-1} = I \) (where \( I \) is the identity matrix), for any \( i \in N \), we have

\[
a_i c_{i-1,i} + b_i c_{i,i} + c_i c_{i+1,i} = 1,
\]

where \( c_{0,1} = c_{n+1,i} = 0 \). Now by Lemma 2.2, we have

\[
1 - b_i c_{i,i} = a_i c_{i-1,i} + c_i c_{i+1,i}
\]

\[
\geq \frac{a_i (-1)^{2i-1} c_{i-1} \det A^{(i+2,n)}}{\det A^{(i+1,n)}} + c_i (-1)^{2i+1} a_{i+1} \frac{\det A^{(i+2,n)}}{\det A^{(i+1,n)}} + c_i a_{i+1} \frac{\det A^{(i+2,n)}}{\det A^{(i+1,n)}} \tag{3.9}
\]

By Lemma 3.1, we get again

\[
\frac{1}{|b_{i-1}| + |a_{i-1}|\bar{\mu}_{i-2}} \leq \frac{\det A^{(i+2,n)}}{\det A^{(i+1,n)}} \leq \frac{1}{|b_{i-1}| - |a_{i-1}|\bar{\mu}_{i-2}}, \tag{3.10}
\]

\[
\frac{1}{|b_{i+1}| + |c_{i+1}|\bar{\mu}_{i+2}} \leq \frac{\det A^{(i+2,n)}}{\det A^{(i+1,n)}} \leq \frac{1}{|b_{i+1}| - |c_{i+1}|\bar{\mu}_{i+2}}. \tag{3.11}
\]

Thus from (3.9), (3.10) and (3.11), we get

\[
\frac{1}{b_i + a_i \bar{\tau}_{i-1} + c_i \bar{\mu}_{i+1}} \leq c_{ij} \leq \frac{1}{b_i + a_i \bar{\tau}_{i-1} + c_i \bar{\mu}_{i+1}}, \tag{3.12}
\]

which is equivalent to (3.8). Thus the proof is completed. \( \square \)
Remark 3.4. In fact, if $A$ is an $M$-matrix, the upper bounds in Theorem 3.6 are the same as those in (3.6), but the lower bounds are sharper than those in (3.6).

The bounds for $c_{i,j}$ from Theorem 3.5 can be improved in a similar way.

Theorem 3.7. Let $A$ be a tridiagonal $M$-matrix as in (2.1) and $A^{-1} = [c_{i,j}]$. If $A$ is row diagonally dominant, then for each $i \neq j$ we have

$$c_{i,j} = \xi_i c_{i-1,j}, \quad i = n, \ldots, j + 1; \quad j = 1, \ldots, n - 1, \quad \text{when } i > j;$$

$$c_{i,j} = \eta_i c_{i+1,j}, \quad i = 1, \ldots, j - 1; \quad j = 2, \ldots, n, \quad \text{when } i < j. \quad \text{(3.13)}$$

where

$$\xi_n = \frac{|a_n|}{b_n}, \quad \xi_i = \frac{|a_i|}{b_i+c_i \xi_{i+1}}, \quad i = n - 1, \ldots, 2;$$

$$\eta_1 = \frac{|c_1|}{b_1}, \quad \eta_i = \frac{|c_i|}{b_i+a_i \eta_{i-1}}, \quad i = 2, \ldots, n - 1. \quad \text{(3.15)}$$

Proof. Note that $AC_j = e_j$, where $C_j$ is the $j$th column of $A^{-1}$ and $e_j$ is the $j$th standard basis vector of $\mathbb{R}^n$, then for any $i, j \in N$ and $i \neq j$, we have the following conclusions:

Case 1. When $i > j$, writing the last $n - j$ equations of the system $AC_j = e_j$, with $j \leq n - 1$:

$$a_n c_{n-1,j} + b_n c_{n,j} = 0,$$

$$a_{n-1} c_{n-2,j} + b_n c_{n-1,j} + c_{n-1} c_{n,j} = 0,$$

$$\ldots$$

$$a_1 c_{i-1,j} + b_1 c_{i,j} + c_1 c_{i+1,j} = 0,$$

$$\ldots$$

$$a_{j+1} c_{j,j} + b_{j+1} c_{j+1,j} + c_{j+1} c_{j+2,j} = 0.$$ 

Since $A$ is a row diagonally dominant $M$-matrix and $a_i c_i \neq 0$ (see Remark 2.1), we have $b_i + c_i \xi_{i+1} \neq 0$, $i = n, \ldots, 2$. It follows that

$$\begin{align*}
  c_{n,j} &= -\frac{a_n}{b_n} c_{n-1,j} = \frac{|a_n|}{b_n} c_{n-1,j} = \xi_n c_{n-1,j}, \\
  c_{n-1,j} &= -\frac{a_{n-1}}{b_{n-1}+c_{n-1}} c_{n-2,j} = \frac{|a_{n-1}|}{b_{n-1}+c_{n-1}} c_{n-2,j} = \xi_{n-1} c_{n-2,j}, \\
  \ldots \\
  c_{j+1,j} &= -\frac{a_{j+1}}{b_{j+1}+c_{j+1}} c_{j,j} = \frac{|a_{j+1}|}{b_{j+1}+c_{j+1}} c_{j,j} = \xi_{j+1} c_{j+1,j},
\end{align*} \quad \text{(3.16)}$$

So

$$c_{i,j} = \xi_i c_{i-1,j}, \quad i = n, \ldots, j + 1; \quad j = 1, \ldots, n - 1.$$

Case 2. When $i < j$, similarly, writing the first $j - 1$ equations of the system $AC_j = e_j$, with $j \geq 2$, we have

$$b_1 c_{1,j} + c_1 c_{2,j} = 0,$$

$$a_2 c_{2,j} + b_2 c_{2,j} + c_2 c_{3,j} = 0,$$

$$\ldots$$

$$a_{j-1} c_{j-1,j} + b_{j-1} c_{j-1,j} + c_{j-1} c_{j,j} = 0,$$

$$\ldots$$

$$a_{j-1} c_{j-2,j} + b_{j-1} c_{j-1,j} + c_{j-1} c_{j,j} = 0.$$ 

Note that $a_i c_i \neq 0$, $b_i + a_i \eta_{i-1} \neq 0$, $i = 2, \ldots, n - 1$ and $b_1 \neq 0$. It follows that
where \( \eta \{ \) denotes the Hadamard product (elementwise product). In particular when \( A \) is a symmetric matrix, we have \( \xi_{n-k} = \eta_{1+k}, k = 0, \ldots, n - 2 \). In this case, we only need to compute \( \{ \xi_k \}_{k=2}^n \) or \( \{ \eta_k \}_{k=1}^{n-1} \) to find the inverse of \( A \). For example, let

\[
A = \begin{bmatrix}
4 & -1 & 0 & 0 \\
-1 & 4 & -1 & 0 \\
0 & -1 & 4 & -1 \\
0 & 0 & -1 & 4 \\
\end{bmatrix}
\]

Thus the proof is completed. \( \square \)

By (3.13) and (3.14), we have, for any \( i \in N \), that

\[
c_{i+1,j} = \xi_{i+1}c_{i,j} \quad \text{and} \quad c_{i-1,j} = \eta_{i-1}c_{i,j}.
\]

Since \( a_i c_{i-1,j} + b_i c_{i,j} + c_i c_{i+1,j} = 1 \) by \( AA^{-1} = I \), then we get the following corollary.

**Corollary 3.2.** Let \( A \) be an M-matrix defined in Theorem 3.7, then \( A^{-1} = \{ c_{i,j} \} \) exists and

\[
c_{i,j} = \frac{1}{b_i + a_i \eta_i - 1 + c_j \xi_{i+1}}, \quad i = 1, 2, \ldots, n,
\]

where \( \eta_0 = \xi_{n+1} = 0 \).

As can be shown by the above, for a diagonally dominant tridiagonal \( M \)-matrix \( A \) of the form (2.1), its inverse \( A^{-1} \) can be described by two sequences of real numbers \( \{ \xi_k \}_{k=2}^n \) and \( \{ \eta_k \}_{k=1}^{n-1} \) (see (3.15)), i.e.,

\[
A^{-1} = \begin{bmatrix}
c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & \cdots & c_{1,n} \\
c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & \cdots & c_{2,n} \\
c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} & \cdots & c_{3,n} \\
c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & \cdots & c_{4,n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c_{n,1} & c_{n,2} & c_{n,3} & c_{n,4} & \cdots & c_{n,n} \\
1 & \eta_1 & \eta_1 & \eta_1 & \cdots & \eta_1 \\
\xi_2 & 1 & \eta_2 & \eta_2 & \cdots & \eta_2 \\
\xi_3 & \xi_3 & 1 & \eta_3 & \cdots & \eta_3 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\xi_n & \xi_n & \xi_n & \xi_n & \cdots & 1 \\
\end{bmatrix},
\]

where ‘o’ denotes the Hadamard product (elementwise product). In particular when \( A \) is a symmetric matrix, we have \( \xi_{n-k} = \eta_{1+k}, k = 0, \ldots, n - 2 \). In this case, we only need to compute \( \{ \xi_k \}_{k=2}^n \) or \( \{ \eta_k \}_{k=1}^{n-1} \) to find the inverse of \( A \). For example, let
By (3.15) and (3.19), we have that
\[
\xi_4 = \frac{1}{4}, \quad \xi_3 = \frac{1}{4 - \xi_4}, \quad \xi_2 = \frac{1}{4 - \xi_3} = \frac{1}{15}; \\
c_{11} = \frac{1}{4 - \xi_2} = \frac{56}{209}, \quad c_{21} = \xi_2 c_{11} = \frac{15}{209}, \quad c_{31} = \xi_3 c_{21} = \frac{4}{209}, \quad c_{41} = \xi_4 c_{31} = \frac{1}{209}; \\
c_{22} = \frac{1}{4 - \xi_4 - \xi_3} = \frac{209}{60}, \quad c_{32} = \xi_3 c_{22} = \frac{16}{209}, \quad c_{42} = \xi_4 c_{32} = \frac{4}{209}; \\
c_{33} = \frac{1}{4 - \xi_3 - \xi_4} = \frac{209}{60}, \quad c_{43} = \xi_4 c_{33} = \frac{15}{209}, \quad c_{44} = \frac{1}{4 - \xi_2} = \frac{56}{209}.
\]

So
\[
A^{-1} = \frac{1}{209} \begin{bmatrix} 56 & 15 & 4 & 1 \\ 15 & 60 & 16 & 4 \\ 4 & 16 & 60 & 15 \\ 1 & 4 & 15 & 56 \end{bmatrix}.
\]

**Remark 3.5.** In fact, based on (2.6) and (2.8), Nabben also presented a refinement iterative method in [22], which yields the exact inverse after \(n-1\) steps for diagonally dominant tridiagonal \(M\)-matrices of the form (2.1). For each \(\tau\), the author applied the sequences \(\prod_{k=1}^{n-1} \tau_{k,t}\) (when \(i < j\)) or \(\prod_{k=j+1}^{n} \omega_{k,t}\) (when \(i > j\)) to approximate the inverse element \(c_{ij}\) (for more details, see Theorem 3.6 of [22]). Hence, comparing the method with Theorem 3.7 and (3.19), one can observe that our scheme is more easily computed.

### 4. A symbolic algorithm for general tridiagonal matrices

#### 4.1. Analysis of the algorithm

The main object of this section is to develop a new algorithm to find the inverse of a general tridiagonal matrix \(A\) without imposing any restrictive conditions.

As it is well known, there are many explicit formulas and algorithms for computing the inverse of a tridiagonal matrix (see [3,4,10,11,13,18]). A well-known result (see [8,11]) is that the inverse of a tridiagonal matrix \(A\) of the form (2.1) may be described by four vectors of real numbers \(u = [u_i]\), \(v = [v_i]\), \(x = [x_i]\), \(y = [y_i]\), where \(u_i v_i = x_i y_i\) for all \(i\), such that \(A^{-1} = [c_{ij}]\) and

\[
c_{ij} = \begin{cases} u_i v_j, & \text{for } i \leq j, \\ x_i y_j, & \text{for } i > j. \end{cases}
\]  

and

\[
u_i = \begin{cases} 1, & \text{for } i = 1, \\ \frac{b_i}{c_i}, & \text{for } i = 2, \ldots, n; \end{cases}
\]

\[
x_i = \begin{cases} 1, & \text{for } i = 1, \\ \frac{b_i}{c_i}, & \text{for } i = 2, \ldots, n; \end{cases}
\]

\[
y_n = \frac{1}{b_n x_n + c_n - 1}, \quad y_{n-1} = -\frac{b_n}{a_n} y_n,
\]

\[
y_i = -\frac{b_i + 1}{a_i} y_{i+1} + c_i y_{i+2}, \quad \text{for } i = n-2, \ldots, 1;
\]

\[
v_n = \frac{1}{b_n v_n + c_n - 1}, \quad v_{n-1} = -\frac{b_n}{c_n} v_n,
\]

\[
v_i = -\frac{b_i + 1}{c_i} v_{i+1} + d_i v_{i+2}, \quad \text{for } i = n-2, \ldots, 1.
\]

Subsequently, Lewis proved the following result when \(c_i \neq 0\) (\(i \in N\)) in [13].

\[
c_{ij} = \begin{cases} s_i t_j, & \text{for } i \leq j, \\ y_i y_j, & \text{for } i > j. \end{cases}
\]  

i.e.,
was mainly based on the following two \((n+1)\)-dimensional vectors \(\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)\) and \(\beta = (\beta_1, \beta_2, \ldots, \beta_{n+1})\):
\[
\alpha_i = \begin{cases} 
1, & \text{if } i = 0 \\
 b_1, & \text{if } i = 1 \\
 b_i \alpha_{i-1} - a_i c_{i-1} \alpha_{i-2}, & \text{if } i = 2, 3, \ldots, n
\end{cases}
\]
and
\[
\beta_i = \begin{cases} 
1, & \text{if } i = n + 1 \\
b_n, & \text{if } i = n \\
b_i \beta_{i+1} - a_{i+1} c_i \beta_{i+2}, & \text{if } i = n - 1, n - 2, \ldots, 1.
\end{cases}
\]

The inverse matrix \(A^{-1} = [c_{ij}]\) can be obtained by the following algorithm:
\[
c_{ij} = \begin{cases} 
(b_1 - a_2 c_1 \beta_2) \beta_2^{-1}, & \text{if } i = 1, \\
(b_n - a_0 c_{n-1} \alpha_{n-2}) \alpha_{n-1}^{-1}, & \text{if } i = n, \\
(b_1 - a_0 c_{i-1} \alpha_{i-2}) \alpha_{i-1}^{-1} - a_{i+1} c_i \beta_{i+2} \beta_{i+1}^{-1}, & \text{if } i = 2, 3, \ldots, n - 1
\end{cases}
\]
and
\[
c_{ij} = \begin{cases} 
(-1)^{i-j} \left( \prod_{k=1}^{j-i} c_{j-k} \right) \alpha_{i-1}^{-1} c_{j}, & \text{if } i < j, \\
(-1)^{i-j} \left( \prod_{k=1}^{i-j} c_{j+k} \right) \beta_{i+1} \beta_{i+1}^{-1} c_{j}, & \text{if } i > j.
\end{cases}
\]

Obviously, if the conditions \(\alpha_i \neq 0\) and \(\beta_{i+1} \neq 0 (i = 1, \ldots, n - 1)\) are not satisfied, then the above algorithm may also break down. Recently, El-Mikkawy [4] presented a symbolic technology to conquer this drawback and removed all cases \((\alpha_i = 0 \text{ and } \beta_{i+1} = 0)\) by assuming \(\alpha_i = x\) and \(\beta_{i+1} = x\) (Here, \(x\) is just a symbolic name) when \(\alpha_i = 0\) and \(\beta_{i+1} = 0\).

Next, based on the above analysis and main results in Section 3, we will further investigate this problem and present some new algorithms.

In fact, if we replace \(\xi_i, \eta_i\) with \(\theta_i\) and \(\varphi_i\) in the proof of Theorem 3.7, respectively, where
\[
\theta_n = -\frac{a_n}{b_n}, \quad \theta_i = -\frac{a_i}{b_i + c_i \theta_{i+1}}, \quad i = n - 1, \ldots, 2,
\]
\[
\varphi_1 = -\frac{c_1}{b_1}, \quad \varphi_i = -\frac{c_i}{b_i + a_i \varphi_{i-1}}, \quad i = 2, \ldots, n - 1,
\]
and
\[ b_i + c_i \theta_{i+1} \neq 0 \quad \text{and} \quad b_i + a_i \varphi_{i-1} \neq 0, \quad i \in N, \quad c_n = a_1 = 0, \]
then Theorem 3.7 and Corollary 3.2 also hold for any general tridiagonal matrix A. Thus, we may apply the following simple algorithm to find the inverse of a tridiagonal matrix of the form (2.1).

**Algorithm 4.1**

**Input.** Order \( n \) of the matrix \( A \) and the components \( a_i, b_i \) and \( c_i, i = 1, 2, \ldots, n \) \( (a_1 = c_n = 0) \).

**Output.** Inverse matrix \( A^{-1} = [c_{ij}] \).

**Step 1.** Set \( \theta_n = -\frac{a_n}{b_n}, \varphi_1 = -\frac{c_1}{b_1} \).

**Step 2.** Compute \( \theta_i, i = n-1, \ldots, 2 \) and \( \varphi_j, j = 2, \ldots, n-1 \), using \( \theta_i = \frac{-a_i}{b_i + a_i \varphi_{i-1} + c_i \theta_{i+1}} \) and \( \varphi_j = \frac{-c_j}{b_j + a_j \varphi_{j-1}} \), respectively.

**Step 3.** If \( b_i + c_i \theta_{i+1} = 0 \) or \( b_i + a_i \varphi_{i-1} = 0 \) for some \( i \in N \), then output: Failure; stop.

**Step 4.** Compute \( b_i + a_i \varphi_{i-1} + c_i \theta_{i+1}, i = 1, \ldots, n \), if \( b_i + a_i \varphi_{i-1} + c_i \theta_{i+1} = 0 \) for some \( i \in N \), then output: No inverse exists; stop. Otherwise, let \( c_{ij} = 1/(b_i + a_i \varphi_{i-1} + c_i \theta_{i+1}) \).

**Step 5.** Using \( c_{ij} = \theta_i c_{i-1,j} \), compute \( c_{ij}, j = 1, \ldots, n-1 \), from \( i = j + 1 \) to \( i = n \); Using \( c_{ij} = \varphi_i c_{i+1,j} \), compute \( c_{ij}, j = 2, \ldots, n \), from \( i = j - 1 \) to \( i = 1 \).

**Step 6.** Output the inverse matrix \( A^{-1} = [c_{ij}] \).

Obviously, this algorithm is suitable for implementation using parallel computer systems since \( \theta_i, \varphi_i \) \( (i \in N) \) and \( c_{ij} \) \( (i > j) \) and \( c_{ij} \) \( (i < j) \) may be independently computed, respectively. When matrix \( A \) of the form (2.1) is SDD or IDD, the conditions (4.10) are easily satisfied. For example, consider the following matrix \( A \) arising in spline approximation [12,22,28]:

\[
A = \begin{bmatrix}
\lambda_1 & 1 - \alpha_1 & & \\
\alpha_2 & \lambda_2 & 1 - \alpha_2 & \\
& \ddots & \ddots & \ddots \\
& & \alpha_{n-1} & \lambda_{n-1} & 1 - \alpha_{n-1} \\
& & & \alpha_n & \lambda_n
\end{bmatrix},
\]

where \( 0 < \alpha_i < 1 \) and \( \lambda_i > 1 \) for all \( i \in N \).

However, Algorithm 4.1 also has some drawbacks, for example, when the conditions (4.10) do not hold, one cannot find the inverse of matrix \( A \) by Algorithm 4.1. Similarly, to remove the conditions (4.10) in Algorithm 4.1, an improved algorithm without imposing any restrictive conditions may be obtained, basing on the symbolic idea of [3,4]:

**Algorithm 4.2 (The Matlab code is available by authors’ emails)**

**Input.** Order \( n \) of the matrix \( A \) and the components \( a_i, b_i \) and \( c_i, i = 1, 2, \ldots, n \) \( (a_1 = c_n = 0) \).

**Output.** Inverse matrix \( A^{-1} = [c_{ij}] \).

**Step 1.** Set \( \theta_n = -\frac{a_n}{b_n}, \varphi_1 = -\frac{c_1}{b_1} \). When \( b_n = 0 \) or \( b_1 = 0 \), we let \( b_n = \text{eps} \) or \( b_1 = \text{eps} \), respectively. Here the eps is the distance from 1.0 to the next larger double precision number, usually \( \text{eps} = 2^{-52} \) in Matlab 7.1.

**Step 2.** Set \( \alpha_i = b_i + c_i \theta_{i+1}, i = n-1, \ldots, 2 \) and \( \beta_j = b_j + a_j \varphi_{j-1}, j = 2, \ldots, n-1 \). Let \( \alpha_i = x \) \( (x \) is just a symbolic name) whenever \( \alpha_i = 0 \) and do the same thing if \( \beta_j = 0 \).

**Step 3.** Compute and simplify\(^1\) \( \theta_i, i = n-1, \ldots, 2 \) and \( \varphi_j, j = 2, \ldots, n-1 \), by using \( \theta_i = \frac{-a_i}{\alpha_i} \) and \( \varphi_j = \frac{-c_j}{\beta_j} \), respectively.

**Step 4.** Compute and simplify \( b_i + a_i \varphi_{i-1} + c_i \theta_{i+1}, i = 1, \ldots, n \). If \( b_i + a_i \varphi_{i-1} + c_i \theta_{i+1} = 0 \) for some \( i \in N \), then output: No inverse exists; stop. Otherwise, let \( c_{ij} = 1/(b_i + a_i \varphi_{i-1} + c_i \theta_{i+1}) \).

\(^1\) Throughout this algorithm, the word ‘simplify’ indicates that the algebraic expression under consideration should be in its simplest rational form (see [3,4]).
Table 1
Comparisons of the computational complexity for different algorithms.

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>Multiplication and division</th>
<th>Addition and subtraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>El-Mikkawy [3]</td>
<td>( \frac{1}{6} n^3 + o(n^2) )</td>
<td>( o(n^2) )</td>
</tr>
<tr>
<td>Lewis [13]</td>
<td>( \frac{1}{8} n^3 + \frac{3}{2} n^2 + o(n) )</td>
<td>( 2n - 3 )</td>
</tr>
<tr>
<td>Huang and McColl [10]</td>
<td>( 3n^2 + 2n - 4 )</td>
<td>( 4n - 2 )</td>
</tr>
<tr>
<td>El-Mikkawy and Karawia [4]</td>
<td>( 3n^2 + 2n - 4 )</td>
<td>( 4n - 2 )</td>
</tr>
<tr>
<td>Ikebe [11]</td>
<td>( n^2 + 12n - 12 )</td>
<td>( 4n - 6 )</td>
</tr>
<tr>
<td>Algorithm 4.1 or 4.2</td>
<td>( n^2 + 6n - 6 )</td>
<td>( 4n - 4 )</td>
</tr>
</tbody>
</table>

Step 5. Using \( c_{ij} = \theta_i c_{i-1,j}, \) compute and simplify \( c_{ij}, \) \( j = 1, \ldots, n - 1, \) from \( i = j + 1 \) to \( i = n; \)
Using \( c_{ij} = \phi_i c_{i+1,j}, \) compute and simplify \( c_{ij}, \) \( j = 2, \ldots, n, \) from \( i = j - 1, \ldots, 1. \)
Step 6. Substitute \( x = 0 \) in all expressions of the elements \( c_{ij}, i,j \in N, \) to obtain the actual values of these elements.
Step 7. Output the inverse matrix \( A^{-1} = [c_{ij}]. \)

Obviously, Algorithm 4.2 is different from Algorithm 4.1: In Step 1 of Algorithm 4.2, in order to allow \( \theta_n \) and \( \phi_1 \) to be computed suitably in some cases, we let \( b_n = \text{eps} \) or \( b_1 = \text{eps}, \) respectively. In addition, if the conditions (4.10) cannot be satisfied, then let \( b_i + c_i \theta_{i+1} = x \) whenever \( b_i + c_i \theta_{i+1} = 0 \) and do the same thing if \( b_i + a_i \phi_{i-1} = 0, \) such that the algorithm does not break down. Finally, when the procedure ends up, substitute \( x = 0 \) in all expressions of the elements \( c_{ij}, i,j \in N, \) to obtain the actual values of these elements.

4.2. A comparison on the computational complexity

Now, let us consider the complexity of algorithms described here. For convenience, these algorithms will be referred to as the El-Mikkawy algorithm [3], El-Mikkawy and Karawia algorithm [4], Lewis algorithm (see (4.3) or [13]), Huang and McColl algorithm (see (4.7) and (4.8) or [10]) and Ikebe algorithm (see (4.1) or [11]), respectively. For these algorithms, we calculate explicitly the computational complexity (i.e., the number of the basic arithmetic operations (addition, subtraction, multiplication and division) as follows (see Table 1), where \( n \) is the order of tridiagonal matrices.

Comparing these results, one can see that our algorithms reduce computational complexity by using less number of basic arithmetic operations. They are faster than the usual algorithms when multiplication or division takes more time than addition or subtraction.

5. Applications and numerical tests

To demonstrate the applicability of the present method, let us see the following several examples.

5.1. Applications in approximate inverses and preconditioning

Analogous to [25], all the lower and upper bounds presented above do not depend on the dimension of the matrices involved. For this reason, in this section we consider only problems of small size, but we stress that our results work well also for much larger problems.

Example 5.1 ([25]). Let us consider the matrix

\[
A = \begin{bmatrix}
-34 & -13.4 & 0.5 \\
-2.2 & 3 & 45 \\
3.3 & 21 & -2.3 \\
& 3 & 15 \\
& & 1.3 & 42
\end{bmatrix}.
\]
According to Theorems 3.4 and 3.5, we use the average of the upper and lower bounds in (3.7) and Theorem 3.4 to approximate $A^{-1} = [c_{ij}]$. We obtain, in single precision, that

$$
B = [b_{ij}] = \begin{bmatrix}
-0.0228 & -0.1029 & 0.0016 & -0.0002 & 0.0000 & -0.0000 \\
-0.0167 & 0.2611 & -0.0041 & 0.0004 & -0.0000 & 0.0000 \\
0.0012 & -0.0191 & 0.0226 & -0.0023 & 0.0001 & -0.0000 \\
0.0001 & -0.0018 & 0.0022 & -0.0454 & 0.0018 & -0.0000 \\
-0.0000 & 0.0004 & -0.0004 & 0.0091 & 0.0663 & -0.0003 \\
0.0000 & -0.0000 & 0.0000 & -0.0003 & -0.0021 & 0.0238 \\
\end{bmatrix},
$$

which approximates the inverse of $A$, where $\max_{i\neq j} |c_{ij} - b_{ij}| = 1.55 \times 10^{-4}$ and $\max_{i \in \mathbb{N}} |c_{ii} - b_{ii}| = 8.948 \times 10^{-5}$. This result is better than corresponding ones in references [13, 17–19, 21–26].

**Example 5.2 ([25]).** In [6], Benzi and Golub used the following matrix

$$
T = \begin{bmatrix}
4 & -1 & & & & \\
-1 & 4 & -1 & & & \\
& -1 & 4 & -1 & & \\
& & \ddots & \ddots & \ddots & \\
& & & -1 & 4 & -1 \\
& & & & -1 & 4 \\
\end{bmatrix},
$$

to test an approximate inverse, and they stressed that an approximate inverse of $T$ was needed in the initial step of an incomplete Cholesky factorization for the two-dimensional model problem (see [25]). Now we consider only the tridiagonal part of $T^{-1}$ (denoted by the matrix $M$), which is computed by the method of Example 5.1, as a preconditioner of $T$. Using the condition number of the matrix $MT$ to test the quality of the preconditioner $M$, we have that $\text{Cond}(MT) = 1.3543$ when the dimension of $T$ is 100, which shows that $M$ is an effective preconditioner.

5.2. Tests for the Algorithm 4.2

First, let us consider a simple numerical solution of differential equations. Suppose we have a two-point boundary value problem of the form (see [28, p. 394]):

$$
\begin{align*}
-y''(x) + \sigma(x)y(x) &= f(x), \quad 0 \leq x \leq 1, \\
y(0) &= \alpha, \\
y(1) &= \beta.
\end{align*}
$$  \tag{5.1}

where $\alpha$ and $\beta$ are given real constants, and $f(x)$ and $g(x)$ are given real-valued functions. If we discretize this problem and look only for the values of $y(kh) \triangleq y_k, \; k = 0, 1, \ldots, n + 1$, and when we use a divided difference approximation to the derivative term

$$
y''(kh) = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2},
$$  \tag{5.2}

we obtain a linear system (see [28]):

$$
Ay = w,
$$

where $y = [y_1, \ldots, y_k] \in \mathbb{R}^n$, $w = [h^2f_1 + \alpha, h^2f_2, \ldots, h^2f_{n-1}, h^2f_n + \beta]^T \in \mathbb{R}^n$ and

$$
A = \begin{bmatrix}
2 + h^2\sigma_1 & -1 & & & & \\
-1 & 2 + h^2\sigma_2 & -1 & & & \\
& \ddots & \ddots & \ddots & \ddots & \\
& & -1 & 2 + h^2\sigma_{n-1} & -1 & \\
& & & -1 & 2 + h^2\sigma_n & \end{bmatrix}.
$$  \tag{5.3}
Here, we have taken \( h = 1/(n + 1) \) for \( n \) a positive integer, \( \sigma_k = \sigma(kh) \) and \( f_k = f(kh) \).

Next, for convenience, we assume that \( \sigma(x) = x \) and \( n = 7 \). Using computer algebra systems such as MAPLE, MATHEMATICA and MATLAB to implement the improved Algorithm 4.2, we get that

\[
\begin{align*}
\theta_7 &= 0.4966, \quad \theta_6 = 0.6600, \quad \theta_5 = 0.7409, \quad \theta_4 = 0.7893, \quad \theta_3 = 0.8220, \quad \theta_2 = 0.8461, \\
\varphi_1 &= 0.4995, \quad \varphi_2 = 0.6647, \quad \varphi_3 = 0.7456, \quad \varphi_4 = 0.7923, \quad \varphi_5 = 0.8214, \quad \varphi_6 = 0.8401,
\end{align*}
\]

and

\[
A^{-1} = \begin{bmatrix}
0.8652 & 0.7320 & 0.6017 & 0.4749 & 0.3519 & 0.2322 & 0.1153 \\
0.7320 & 1.4654 & 1.2046 & 0.9508 & 0.7044 & 0.4649 & 0.2309 \\
0.6017 & 1.2046 & 1.8121 & 1.4303 & 1.0597 & 0.6994 & 0.3473 \\
0.4749 & 0.9508 & 1.4303 & 1.9183 & 1.4212 & 0.9380 & 0.4658 \\
0.3519 & 0.7044 & 1.0597 & 1.4212 & 1.7938 & 1.1840 & 0.5880 \\
0.2322 & 0.4649 & 0.6994 & 0.9380 & 1.1840 & 1.4414 & 0.7158 \\
0.1153 & 0.2309 & 0.3473 & 0.4658 & 0.5880 & 0.7158 & 0.8521
\end{bmatrix}
\]

Next, Let us consider the transition matrix\(^2\) of a birth–death process, with \( \lambda_i \) being the birth rate, and \( \mu_i \), the death rate (see \cite[p. 97]{7}):

\[
A = \begin{bmatrix}
\lambda_0 & \lambda_1 & \cdots & & & \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & & & \\
& \mu_2 & \cdots & \cdots & & \\
& & \ddots & \ddots & \ddots & \\
& & & \mu_{N-2} & -(\lambda_{N-2} + \mu_{N-2}) & \lambda_{N-2} \\
& & & & \mu_{N-1} & -\mu_{N-1}
\end{bmatrix}.
\]

Here, \( \lambda_i \geq 0 \) and \( \mu_i \geq 0 \).

However, generally speaking, the above transition matrix \( A \) is singular, so the inverse matrix does not exist. In addition, some simple experiments show that if they are directly implemented in Matlab, the outputs are often wrong. Next, we use this kind of matrices to test our Algorithm 4.2. For example, when \( N = 4 \), we apply our Matlab procedure of Algorithm 4.2 to test this problem as follows.

\begin{enumerate}
\item Step 1. Define symbolic variables \(-\text{syms } x \ y \ z \ x1 \ y1 \ z1;\)
\item Step 2. Define diagonal elements of the tridiagonal matrix \( A - a = [x, y, z]; b = [-x1, -(x + y1), -(y + z1), -z]; c = [x1, y1, z1];\)
\item Step 3. Operate the function \(-C = \text{Tri}(a, b, c).\)
\end{enumerate}

Results: "No inverse exists".

This experiment shows that Algorithm 4.2 is also effective for some symbolic matrices.

In addition, let us consider Higham test matrices (see the "gallery" function in MATLAB 7.1), which are badly conditioned. Many numerical experiments show that Algorithm 4.2 is also effective for tridiagonal matrices in the "gallery" function. For example, a \(6 \times 6\) Clement matrix (tridiagonal with zero diagonal entries)

\[
C = \begin{bmatrix}
0 & 1 & & & & \\
5 & 0 & 2 & & & \\
& 4 & 0 & 3 & & \\
& & 3 & 0 & 4 & \\
& & & 2 & 0 & 5 \\
& & & & 1 & 0
\end{bmatrix}
\]

\(^2\) This kind of matrices also arise in random walk and queuing systems, see for instance \cite{1}.
In single precision, we have, by Algorithm 4.2, that
\[
C^{-1} = \begin{bmatrix}
-0.0000 & 0.2000 & 0.0000 & -0.1333 & -0.0000 & 0.5333 \\
1.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 \\
0.0000 & -0.0000 & -0.0000 & 0.3333 & 0.0000 & -1.3333 \\
-1.3333 & 0.0000 & 0.3333 & -0.0000 & -0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 1.0000 \\
0.5333 & -0.0000 & -0.1333 & 0.0000 & 0.2000 & -0.0000
\end{bmatrix}.
\]

Finally, let us consider another simple reducible symbolic matrix [4].
\[
A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 \\
0 & -1 & 2 & p \\
0 & 0 & 0 & q
\end{bmatrix}.
\]

Using Algorithm 4.2, we have that
\[
\theta_4 = 0, \quad \theta_3 = \frac{1}{2}, \quad \theta_2 = -2; \quad \varphi_1 = -1, \quad \varphi_2 = \frac{1}{x}, \quad \varphi_3 = -\frac{px}{2x - 1}
\]
and
\[
C = \begin{bmatrix}
-1 & 2 & \frac{-1}{2x-1} & \frac{p}{q(2x-1)} \\
2 & -2 & \frac{1}{2x-1} & -\frac{p}{q(2x-1)} \\
1 & -1 & \frac{x}{2x-1} & \frac{-px}{q(2x-1)} \\
0 & 0 & 0 & \frac{1}{q}
\end{bmatrix}
\]

Set \( x = 0 \), we obtain the exact inverse of matrix \( A \):
\[
A^{-1} = \begin{bmatrix}
-1 & 2 & \frac{-p}{q} \\
2 & -2 & 1 & \frac{p}{q} \\
1 & -1 & 0 & \frac{1}{q} \\
0 & 0 & 0 & \frac{1}{q}
\end{bmatrix}.
\]

Therefore, the matrix \( A \) is nonsingular as long as \( q \) is not equal to zero, which also shows that Algorithm 4.2 is efficient for reducible symbolic tridiagonal matrices.

In conclusion, many examples show that our symbolic algorithm is competitive with the other methods for solving tridiagonal linear systems which appear in many applications and suited for implementation using parallel computer algebra systems (CAS) such as MATLAB, MAPLE and Mathematica, etc. In addition, by comparing with those of known algorithms \([3,4,10,11,13]\), it is obvious that Algorithm 4.2 is simple (it only depends on \( 2n - 2 \) parameters \( \{\theta_k\}_{k=2}^n \) and \( \{\varphi_k\}_{k=1}^{n-1} \)) and the number of required computations in our method is also less than that of earlier methods.

Finally, it is worth mentioning whether these results can be generalized to general band matrices or block tridiagonal matrices \([15]\). These problems are of interest, but it may be difficult to resolve them, which will be further investigated.

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