Slacks-based measures of efficiency in imprecise data envelopment analysis: An approach based on data envelopment analysis with double frontiers

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1. Introduction

Data envelopment analysis is a non-parametric technique for evaluating the relative efficiency of decision-making units (DMUs) that use multiple inputs to produce multiple outputs. Traditional DEA models assume that all input and output data are known exactly. In many situations, however, some inputs and/or outputs take imprecise data. In this paper, we present optimistic and pessimistic perspectives for obtaining an efficiency evaluation for the DMU under consideration with imprecise data. Additionally, slacks-based measures of efficiency are used for direct assessment of efficiency in the presence of imprecise data with slack values. Finally, the geometric average of the two efficiency values is used to determine the DMU with the best performance. A ranking approach based on degree of preference is used for ranking the efficiency intervals of the DMUs. Two numerical examples are used to show the application of the proposed DEA approach.

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DEA models were able to work with interval data and fuzzy data, but their models had some fundamental drawbacks. Their proposed DEA models make use of only one input and one output for determining the lower bound of the efficiency interval of each DMU regardless of the number of inputs and outputs, which leads to the loss of a great deal of input and output information about the DMU under consideration. To overcome these problems, several authors have developed models to measure the efficiency interval of the DMUs in a logical way (including Azizi, 2011, 2014a; Azizi & Fathi Ajjuru, 2010; Azizi & Jahed, 2011; Azizi & Wang, 2013; Wang & Yang, 2007; Wang, Yi, & Wang, 2008). Wang and Luo (2006) combined TOPSIS method of multiple attribute decision making with DEA. They introduced two virtual DMUs, called the ideal DMU and the anti-ideal DMU, and presented two DEA models for computation of the optimistic and pessimistic efficiencies. By integrating these two distinct efficiencies, they managed to rank the DMUs by using an index called the relative closeness to the ideal DMU. Wang, Chin, and Yang (2007) proposed a geometric average efficiency measure for assessing the overall performance of each DMU. The geometric average efficiency integrates both optimistic and pessimistic efficiencies of each DMU and so is more comprehensive than either optimistic or pessimistic perspective. Wang and Chin (2009) proposed an overall performance measure for ranking advanced manufacturing technologies. Their proposed DEA approach considers both optimistic and pessimistic efficiencies of the advanced manufacturing technologies simultaneously. Amirteimoori (2007) introduced an efficiency measure using two ideal and anti-ideal indices that are formed based on efficient and inefficient production frontiers. The rationale for these two indices is to maximize the weighted distance of a particular DMU relative to the efficient and inefficient production frontiers. Amirteimoori, Kordrostami, and Rezaatbar (2006) improved the cost efficiency interval of a DMU by adjusting its given inputs and outputs using the optimistic and pessimistic perspectives.

In this paper, we focus on how to use imprecise data (i.e., a mixture of ordinal and interval data) to measure the performance of the DMUs from both optimistic and pessimistic perspectives. Numerous articles have been published on imprecise DEA (IDEA) (Amirteimoori & Kordrostami, 2005; Azizi, 2013b; Azizi & Ganjeh Ajirlu, 2011; Cook & Zhu, 2006; Cooper, Park, & Yu, 2001a; Despotis & Smirlis, 2002; Emrouznejad, Rostamy-Malkhalifeh, Hatami-Marbini, & Tavana, 2012; Fang & Hechgen, 2013; Kao, 2006; Kao & Liu, 2000; Kim, Park, & Park, 1999; Lee, Park, & Kim, 2002; Park, 2007; Smirlis, Maragos, & Despotis, 2006; Wang, Greatbanks, & Yang, 2005; Wang, Luo, & Liang, 2009; Zhu, 2003). Most of them, however, are radial models. Radial IDEA models have a fundamental problem, in that slack values must be computed indirectly through a complicated procedure. For direct assessment of the efficiency interval in the presence of imprecise data with slack values, slacks-based measures (SBM) can be used. It should be noted that Hosseinzadeh Lotfi, Jahanshahloo, and Esmaeili (2007) have proposed SBM models for measuring DMU performances in the presence of interval data only from the optimistic perspective. In their lower-bound and upper-bound SBM models, different sets of constraints are used for measuring efficiencies of different DMUs, and even the constraint sets used for measuring the lower bound and upper bound of the efficiency of each DMU are different. The main disadvantage of using different constraint sets for measuring DMU efficiencies is that it is not possible to compare the efficiencies, because different production frontiers have been used in the process of efficiency measurement. For this reason, the main objective of the present paper is to formulate new SBM models in IDEA from both optimistic and pessimistic perspectives. Finally, we present an index for combining the efficiencies obtained from the optimistic and pessimistic perspectives, so that we can measure the overall performance of the DMUs in a logical manner.

And for ranking and comparing the efficiency intervals of the DMUs, we will use a ranking approach based on degree of preference (Wang, Yang, & Xu, 2005a, 2005b). Two numerical examples will be used to illustrate the applications of the proposed approach.

This paper is organized as follows. In Section 2, we present SBM models for measuring the optimistic and pessimistic efficiencies of the DMUs. In Section 3, we develop SBM models for dealing with interval data. Section 4 proposes overall performance measures and Section 5 introduces an approach for ranking efficiency intervals. Two empirical examples of performance evaluation are presented in Section 6. Section 7 concludes the paper.

2. SBM models for measuring the optimistic and pessimistic efficiencies with crisp data

2.1. SBM model for measuring the optimistic efficiency with crisp data

Assume that there are n DMUs to be evaluated, each consisting of m inputs and s outputs. \( x_{ij} \) \((i = 1, \ldots, m)\) and \( y_{jr} \) \((r = 1, \ldots, s)\) denote the input and output values of DMU \( j \) \((j = 1, \ldots, n)\), all of which are known and non-negative. We assume that \( X_{j} = (x_{1j}, \ldots, x_{mj}) \geq 0 \), \( X_{j} = 0 \) \((j = 1, \ldots, n)\) and \( Y_{j} = (y_{1j}, \ldots, y_{sj}) \geq 0 \), \( Y_{j} = 0 \) \((j = 1, \ldots, n)\). Then, the production possibility set is defined as follows:

\[
T = \left\{ (X, Y) \mid \sum_{j=1}^{n} x_{ij} \leq X_{i}, \sum_{j=1}^{n} y_{jr} \geq Y_{r}, \lambda_{j} \geq 0, j = 1, \ldots, n \right\}
\]

(1)

\( T \) is a closed and convex set. Boundary points of \( T \) are defined as the efficient production frontier.

For direct assessment of the efficiency with slack values, an efficiency SBM was proposed by Tone (2001). According to the concept of the efficient production frontier, the SBM model is defined as follows:

\[
\begin{align*}
\min \quad & \rho = \frac{1}{1+\lambda} \sum_{i=1}^{m} S_{i}^{+}/x_{0i} - \frac{1}{1+\lambda} \sum_{i=1}^{m} S_{i}^{-}/y_{0i} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} S_{i}^{+} = x_{0i}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} y_{jr} S_{r}^{-} = y_{0r}, \quad r = 1, \ldots, s, \\
& \lambda_{j} \geq 0, \quad S_{i}^{-} \geq 0, \quad S_{i}^{+} \geq 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, \quad j = 1, \ldots, n.
\end{align*}
\]

(2)

where \( DMU_{j} \) denotes the DMU under evaluation. \( S_{i}^{-} \) \((i = 1, \ldots, m)\) and \( S_{r}^{+} \) \((r = 1, \ldots, s)\) are called slacks. If \( x_{0i} = 0 \), then the term \( S_{i}^{-}/x_{0i} \) is eliminated. If \( y_{0r} = 0 \), then it is replaced by a very small number, so that the term \( S_{r}^{+}/y_{0r} \) has compensatory effect. Using scale transformation, model (2) can be converted into the following linear programming (LP) model:

\[
\begin{align*}
\min \quad & \tau = \frac{1}{m} \sum_{r=1}^{m} S_{r}^{+}/x_{0r} - \frac{1}{m} \sum_{r=1}^{m} S_{r}^{-}/y_{0r} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} S_{i}^{+} = x_{0i}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} y_{jr} S_{r}^{-} = y_{0r}, \quad r = 1, \ldots, s, \\
& \frac{1}{m} \sum_{r=1}^{m} S_{r}^{+}/y_{0r} = t > 0, \\
& \lambda_{j} \geq 0, \quad S_{i}^{-} \geq 0, \quad S_{i}^{+} \geq 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, \quad j = 1, \ldots, n, \quad t > 0.
\end{align*}
\]

An important property of efficiency SBM is that \( \rho \) is independent of the measurement unit used for inputs and outputs, and is monotonically decreasing in each input and output slack. Therefore, the
larger value of $\rho$, the better performance of the DMU. When the optimal value of $\rho$ occurs, i.e. $\rho^* = 1$, the respective DMU is called optimistic efficient; otherwise it is called optimistic non-efficient.

### 2.2. SBM model for measuring the pessimistic efficiency with crisp data

If the classic technology with constant returns to scale is used, then the inefficient production possibility set is defined as follows (Azizi & Ganjeh Ajirlu, 2011; Liu & Chen, 2009; Parkan, 2000):

$$\bar{T} = \left\{ (X, Y) \mid \sum_{j=1}^{n} \lambda_j X_j \geq X, \sum_{j=1}^{n} \lambda_j Y_j \leq Y, \lambda_j \geq 0, j = 1, \ldots, n \right\}$$  \hspace{1cm} (4)

$\bar{T}$ is a closed and convex set. Boundary points of $\bar{T}$ are defined as the inefficient production frontier. The pessimistic efficiency of each unit is generally obtained by comparing it with a point on the boundary of the inefficient production possibility set.

The following SBM model is used for measuring the pessimistic efficiency of the DMUs with slack values (Liu & Chen, 2009; Parkan, 2000):

$$\begin{align*}
\max & \quad \phi \equiv \frac{1}{1-\rho^*} \frac{\sum_{j=1}^{n} S_j^{+}}{s_j^{-}} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_j - s_j^{+} = x_a, \ i = 1, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_j + s_j^{-} = y_a, \ r = 1, \ldots, s, \\
& \quad \lambda_j \geq 0, s_j^{+} \geq 0, s_j^{-} \geq 0, \ i = 1, \ldots, m, \ r = 1, \ldots, s, \ j = 1, \ldots, n.
\end{align*}$$  \hspace{1cm} (5)

Again, if $x_a = 0$, the term $s_j^{+}/s_a$ is eliminated. If $y_a = 0$, the term $s_j^{-}/y_a$ is eliminated. The numerator of the objective function is always greater than or equal to unity, while its denominator is less than or equal to unity. Therefore, $\phi \geq 1$. $\phi$ is independent of the measurement unit used for inputs and outputs and is monotonically increasing in each input or output slack. In order to solve model (5), it can be converted to the following LP model:

$$\begin{align*}
\max & \quad \phi = t + \frac{1}{t} \sum_{i=1}^{m} S_i^{+}/x_{a}^{+} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_j - s_j^{+} = x_a, \ i = 1, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_j + s_j^{-} = y_a, \ r = 1, \ldots, s, \\
& \quad 1 = t - \frac{1}{t} \sum_{i=1}^{m} S_i^{-}/y_{a}^{-}, \\
& \quad \lambda_j \geq 0, s_j^{+} \geq 0, s_j^{-} \geq 0, \ i = 1, \ldots, m, \ r = 1, \ldots, s, \ j = 1, \ldots, n, \ t > 0.
\end{align*}$$  \hspace{1cm} (6)

If $\phi^* = 1$, DMU$_i$ is said to be pessimistic inefficient; otherwise, it is said to be pessimistic non-inefficient. Evidently, a pessimistic inefficient DMU is not necessarily optimistic efficient. The pessimistic inefficient units collectively form an inefficient production frontier. Fig. 1 shows the efficient and inefficient production frontiers for a dataset with one input and one output. These frontiers are built from the observed data.

In summary, DMUs can be divided into three categories: optimistic efficient, pessimistic inefficient, and indeterminate. An indeterminate DMU is neither optimistic efficient nor pessimistic inefficient, and is always circumscribed between the efficient and inefficient production frontiers. As an example, in Fig. 1, DMU$_{K}$ is optimistic efficient relative to the efficient production frontier and pessimistic non-inefficient relative to the inefficient production frontier; DMU$_{L}$ is optimistic non-efficient relative to the efficient production frontier and pessimistic inefficient relative to the inefficient production frontier; and DMU$_{M}$ is optimistic non-efficient relative to the efficient production frontier and pessimistic non-efficient relative to the inefficient production frontier, and, hence, is indeterminate.

### 3. SBM models for measuring the optimistic and pessimistic efficiencies with interval data

#### 3.1. SBM models for measuring the optimistic efficiency with interval data

In interval DEA, it is assumed that some input values $x_j$ and output values $y_j$ are not known exactly. It is only known that they lie within the upper and lower bounds of the intervals $[x_j^L, x_j^U]$ and $[y_j^L, y_j^U]$, and each DMU has at least one positive lower bound for input and one positive lower bound for output. To deal with such an uncertain situation, we present the following LP models for obtaining the upper and lower bounds of efficiency with slack values. These models measure the optimistic efficiencies of the DMUs.

$$\begin{align*}
\min & \quad \tau_0^{-} = t - \frac{1}{t} \sum_{i=1}^{m} S_i^{+}/x_{a}^{+} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_j^L - s_j^{+} = x_a, \ i = 1, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_j^L + s_j^{-} = y_a, \ r = 1, \ldots, s, \\
& \quad 1 = t + \frac{1}{t} \sum_{i=1}^{m} S_i^{-}/y_{a}^{-}, \\
& \quad \lambda_j \geq 0, s_j^{+} \geq 0, s_j^{-} \geq 0, \ i = 1, \ldots, m, \ r = 1, \ldots, s, \ j = 1, \ldots, n, \ t > 0.
\end{align*}$$  \hspace{1cm} (7)

$$\begin{align*}
\min & \quad \tau_0^{+} = t - \frac{1}{t} \sum_{i=1}^{m} S_i^{+}/x_{a}^{+} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_j^U - s_j^{+} = x_a, \ i = 1, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_j^U + s_j^{-} = y_a, \ r = 1, \ldots, s, \\
& \quad 1 = t + \frac{1}{t} \sum_{i=1}^{m} S_i^{-}/y_{a}^{-}, \\
& \quad \lambda_j \geq 0, s_j^{+} \geq 0, s_j^{-} \geq 0, \ i = 1, \ldots, m, \ r = 1, \ldots, s, \ j = 1, \ldots, n, \ t > 0.
\end{align*}$$  \hspace{1cm} (8)
\( \tau_0^i \) is the optimistic efficiency under the most favorable conditions and \( \tau_1^i \) is the optimistic efficiency under the most unfavorable conditions for DMU_0. They form the optimistic efficiency interval \([\tau_0^i, \tau_1^i]\). If \( \tau_0^i = 1 \), then DMU_0 is called optimistic efficient; otherwise, it is called optimistic non-efficient.

In Appendix A, we will show that \( \rho \) is independent from the measurement unit used for input and output data and is monotonically descending in each input and output slack.

### 3.2. SBM models for measuring the pessimistic efficiency with interval data

Now we present models that are used for determining the pessimistic efficiency interval of a DMU in contrast to its optimistic efficiency interval, which is determined in LPs (7) and (8):

\[
\text{max } \phi_b^0 = t + \frac{1}{2} \sum_{i=1}^{n} \left( S_i / x_{b0}^i \right) \quad \text{s.t.} \quad \sum_{j=1}^{m} \lambda_j x_{bj} - S_i = b \lambda_{b0}, \quad i = 1, \ldots, m, \\
\sum_{j=1}^{m} \lambda_j x_{bj} + S_i = b \lambda_{b0}, \quad r = 1, \ldots, s, \\
1 = t - \frac{1}{2} \sum_{r=1}^{s} S_i / y_{br}, \\
\lambda_j \geq 0, S_i \geq 0, S_r \geq 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, \quad j = 1, \ldots, n, \quad t > 0.
\]

(9)

\[
\text{max } \phi_b^1 = t + \frac{1}{2} \sum_{i=1}^{n} \left( S_i / x_{b0}^i \right) \quad \text{s.t.} \quad \sum_{j=1}^{m} \lambda_j x_{bj} - S_i = b \lambda_{b0}, \quad i = 1, \ldots, m, \\
\sum_{j=1}^{m} \lambda_j x_{bj} + S_i = b \lambda_{b0}, \quad r = 1, \ldots, s, \\
1 = t - \frac{1}{2} \sum_{r=1}^{s} S_i / y_{br}, \\
\lambda_j \geq 0, S_i \geq 0, S_r \geq 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, \quad j = 1, \ldots, n, \quad t > 0.
\]

(10)

In models (9) and (10), \( \phi_b^0 \) is the pessimistic efficiency under the most unfavorable conditions and \( \phi_b^1 \) is the pessimistic efficiency under the most favorable conditions for DMU_0. They form the pessimistic efficiency interval \([\phi_b^0, \phi_b^1]\) for DMU_0. When \( \phi_b^0 = 1 \), DMU_0 is called pessimistic inefficient. Otherwise, it is called pessimistic non-efficient. Here, too, \( \phi \) is unit-invariant for interval input and output data and is monotonically ascending in each input and output slack. Also, it preserves the constant returns to scale property.

### 4. Overall performance measures

The optimistic and pessimistic efficiencies are measures from different points of view, leading to two different evaluations for the DMUs. Consequently, an overall performance measure is needed to produce an overall score for each DMU. In this paper, we use the geometric average measure proposed by Wang et al. (2007) for scoring DMUs with crisp data. This measure is defined as follows:

\[
\phi_j = \sqrt{\phi_j^* \cdot \phi_j^*}, \quad j = 1, \ldots, n
\]

(11)

where \( \tau_j^* \) and \( \phi_j^* \) are the optimistic and pessimistic efficiencies of DMU_j, respectively. It is evident that the geometric average measure defined in (11) takes into account both efficiencies simultaneously.

Let \( \tau_j^* = [\tau_j^*_{-1}, \tau_j^*_{+1}] \) and \( \phi_j^* = [\phi_j^*_{-1}, \phi_j^*_{+1}] \) be the optimistic and pessimistic efficiency intervals of DMU_j, respectively. Based on the rules of operations on interval numbers, we have (Moore & Bierbaum, 1979):

\[
\phi_j = \sqrt{[\tau_j^*_{-1}, \tau_j^*_{+1}] \cdot [\phi_j^*_{-1}, \phi_j^*_{+1}]} = \sqrt{[\tau_j^*_{-1} \cdot \phi_j^*_{-1}, \tau_j^*_{+1} \cdot \phi_j^*_{+1}]} = \left[ \sqrt{\tau_j^*_{-1} \cdot \phi_j^*_{-1}}, \sqrt{\tau_j^*_{+1} \cdot \phi_j^*_{+1}} \right], \quad j = 1, \ldots, n.
\]

(12)

Obviously, \( \phi_j(j = 1, \ldots, n) \) should also be an interval number, which we show by \([\phi_j^*_{-1}, \phi_j^*_{+1}](j = 1, \ldots, n)\). Then, we have:

\[
\phi_j^* = \sqrt{\tau_j^*_{-1} \cdot \phi_j^*_{-1}}, \quad j = 1, \ldots, n,
\]

\[
\phi_j^* = \sqrt{\tau_j^*_{+1} \cdot \phi_j^*_{+1}}, \quad j = 1, \ldots, n.
\]

(13)

The optimistic and pessimistic efficiencies measure the performance of n DMUs at two extreme cases, i.e., under the best and the worst conditions (Wang et al., 2007). Theoretically, these two efficiencies should be taken into account simultaneously to obtain an overall assessment of the performance of each of n DMUs (Wang & Lan, 2011, 2013).

### 5. A preference degree approach for comparing and ranking interval numbers

Since the efficiency score of each DMU is determined as an interval, we need a simple, yet practical, approach for comparing and ranking interval numbers. A number of approaches have been developed for ranking interval numbers in the past, but all of them have shortcomings (Ishibuchi & Tanaka, 1990; Salo & Hämmäläinen, 1995; Sengupta & Pal, 2000; Song, Liang, & Qian, 2012). Specifically, when some interval numbers have identical centers but different widths, all these approaches fail to distinguish those numbers. We use the preference-based approach developed by Wang et al. (2005a) for comparing and ranking the interval efficiencies of the DMUs. This approach is summarized below.

Let \( a = [a^l, a^u] \) and \( b = [b^l, b^u] \) be two interval numbers. The degree of preference of an interval number refers to the degree of the interval number being greater than another interval number. Accordingly, the following definitions and properties are in order:

**Definition 1.** The degree of preference of \( a \) over \( b \) (or \( a > b \)) is defined as follows:

\[
P(a > b) = \frac{\max(0, a^u - b^l) - \max(0, a^l - b^u)}{(a^u - a^l) + (b^u - b^l)}
\]

(14)

The degree of preference of \( b \) over \( a \) (or \( b > a \)) is defined similarly, i.e.:

\[
P(b > a) = \frac{\max(0, b^u - a^l) - \max(0, b^l - a^u)}{(a^u - a^l) + (b^u - b^l)}
\]

(15)

It is obvious that \( P(a > b) + P(b > a) = 1 \) and we have \( P(a > b) = P(b > a) = 0.5 \) when \( a = b \), i.e. \( a^l = b^l \) and \( a^u = b^u \).

**Definition 2.** If \( P(a > b) > P(b > a) \), it is said that \( a \) is superior to \( b \) to the degree \( P(a > b) \), and this is denoted by \( a \succ b \); if \( P(a > b) = P(b > a) = 0.5 \), it is said that \( a \) is indifferent to \( b \), and is denoted by \( a \sim b \); and if \( P(b > a) > P(a > b) \), then it is said that \( a \sim b \).

**Property 1.** \( P(a > b) = 1 \), if and only if \( a^l \prec b^u \).
Property 2. If \( a^l > b^l \) and \( a^u > b^u \), then \( P(a > b) \geq 0.5 \) and \( P(b > a) < 0.5 \).

Property 3. If \( b \) is inside \( a \), i.e., \( a^l < b^l \) and \( a^u > b^u \), then \( P(a > b) \geq 0.5 \) if and only if \( \frac{a^l - b^l}{2} \geq \frac{b^u - a^u}{2} \).

Property 4. If \( P(a > b) \geq 0.5 \) and \( P(b > c) \geq 0.5 \), then \( P(a > c) \geq 0.5 \).

The four properties above are very useful when comparing interval numbers. Property 1 shows that if two interval numbers do not overlap, then the interval at the high end will be 100% superior to the interval at the low end. Property 2 is similar to the comparison rule for interval numbers. Property 3 shows the comparison of two interval numbers when one interval number lies inside the other. Property 4 shows that the preference relations are transitive. Using transitivity, a complete ranking of the interval numbers can be achieved. The ranking process is explained below:

Step 1. Calculate the matrix of degrees of preference:

\[
M_p = \begin{bmatrix}
0_l & 0_2 & \ldots & 0_n \\
- & p_{12} & \ldots & p_{1n} \\
& \vdots & \ddots & \vdots \\
& & \vdots & - \\
0_a & p_{n1} & p_{n2} & \ldots & -
\end{bmatrix}
\]  \hspace{1cm} (16)

where

\[
p_{ij} = P(0_i > 0_j) = \frac{\max(0_i, 0_j^l - 0_j^u) - \max(0_j, 0_i^l - 0_i^u)}{(0_j^l - 0_j^u) + (0_i^l - 0_i^u)},
\]

\(i, j = 1, \ldots, n; \; i \neq j\).  \hspace{1cm} (17)

Step 2. Find a row in the matrix of degrees of preference in which all elements except the diagonal element are greater than or equal to 0.5. If this row corresponds with \(0_i\), then \(0_i\) is the most preferred interval number.

Step 3. Remove row \(i\) and column \(i\) (and therefore \(0_i\)) from the matrix. In the reduced matrix, if \(0_i\) is the most preferred interval number among the remaining intervals, then \(0_i\) receives the second place and is shown as \(0_i > 0_j\) if \(p_{ij} > 0.5\), or as \(0_i \sim 0_j\) if \(p_{ij} = 0.5\).

Step 4. For further analysis, remove row \(j\) and column \(j\) from the reduced matrix and continue the process until the remaining intervals are ranked.

We will illustrate this ranking process in the numerical examples of the next section.

**Theorem 1.** There is a row in the matrix of degrees of preference in which all elements except the diagonal element are greater than or equal to 0.5.

**Proof.** Let

\[x_i = \min_{j \neq i}(p_{ij}), \; \quad i = 1, \ldots, n,\]

Also, let

\[x_q = \max_{i \leq q}(x_i).\]

We will prove that \(x_q \geq 0.5\). Since \(x_q = \min_{i \leq q}(p_{ij})\), then we have:

\[p_{qj} \geq x_q, \; \quad j = 1, \ldots, n.\]

On the other hand, \(p_{qj} + p_{qj} = 1, j = 1, \ldots, n.\) Therefore, we have:

\[p_{qj} = 1 - p_{qj} \leq 1 - x_q < 1 - 0.5 = 0.5, \; \quad j = 1, \ldots, n.\]

Consequently, \(p_{qj} > 0.5, \; j = 1, \ldots, n,\) i.e., \(x_q = \min_{i \leq q}(p_{ij}) > 0.5.\) This completes the proof. \(\square\)

**6. Illustrative examples**

In this section, we study two performance evaluation problems using the SBM models developed in this paper. One of the examples uses interval data; the other one uses a mixture of exact data, interval data, and ordinal preference information. These examples can be evaluated using the proposed SBM models.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Efficiency intervals for the 24 commercial banks.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commercial bank (DMU)</td>
<td>Optimistic efficiency interval ((\varphi_i^l, \varphi_i^u))</td>
</tr>
<tr>
<td>1</td>
<td>[0.5604, 0.8069]</td>
</tr>
<tr>
<td>2</td>
<td>[0.6197, 0.8373]</td>
</tr>
<tr>
<td>3</td>
<td>[0.6491, 0.9099]</td>
</tr>
<tr>
<td>4</td>
<td>[0.6803, 1.0000]</td>
</tr>
<tr>
<td>5</td>
<td>[0.4817, 0.6872]</td>
</tr>
<tr>
<td>6</td>
<td>[0.7791, 1.0000]</td>
</tr>
<tr>
<td>7</td>
<td>[0.2389, 0.3053]</td>
</tr>
<tr>
<td>8</td>
<td>[0.1434, 0.1745]</td>
</tr>
<tr>
<td>9</td>
<td>[0.1799, 0.2241]</td>
</tr>
<tr>
<td>10</td>
<td>[0.5480, 1.0000]</td>
</tr>
<tr>
<td>11</td>
<td>[0.4721, 0.6861]</td>
</tr>
<tr>
<td>12</td>
<td>[0.3781, 1.0000]</td>
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<tr>
<td>13</td>
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<td>14</td>
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<td>15</td>
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<tr>
<td>16</td>
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<tr>
<td>17</td>
<td>[0.2201, 1.0000]</td>
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<tr>
<td>18</td>
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<tr>
<td>19</td>
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<tr>
<td>20</td>
<td>[0.8006, 1.0000]</td>
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<tr>
<td>21</td>
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<td>22</td>
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</tr>
<tr>
<td>23</td>
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<tr>
<td>24</td>
<td>[0.5000, 1.0000]</td>
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</table>
Table 2
Rankings of the 24 commercial banks.

<table>
<thead>
<tr>
<th>Commercial bank (DMU)</th>
<th>Rank according to Optimistic efficiency interval</th>
<th>Pessimistic efficiency interval</th>
<th>Overall efficiency interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
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<td>8</td>
<td>7</td>
</tr>
<tr>
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<td>5</td>
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<td>5</td>
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<td>3</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>21</td>
<td>20</td>
</tr>
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<td>24</td>
<td>22</td>
<td>22</td>
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<td>24</td>
<td>23</td>
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<td>10</td>
<td>9</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>17</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>18</td>
<td>11</td>
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<td>21</td>
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<td>21</td>
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<tr>
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<td>10</td>
<td>9</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>21</td>
<td>19</td>
<td>18</td>
<td>19</td>
</tr>
<tr>
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<td>6</td>
<td>13</td>
<td>10</td>
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<tr>
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<td>23</td>
<td>22</td>
</tr>
<tr>
<td>24</td>
<td>11</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 3
Data for 8 TELECOM branches.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>1</td>
<td>124</td>
<td>18.22</td>
</tr>
<tr>
<td>2</td>
<td>95</td>
<td>9.23</td>
</tr>
<tr>
<td>3</td>
<td>92</td>
<td>8.07</td>
</tr>
<tr>
<td>4</td>
<td>61</td>
<td>5.62</td>
</tr>
<tr>
<td>5</td>
<td>63</td>
<td>5.33</td>
</tr>
<tr>
<td>6</td>
<td>50</td>
<td>3.53</td>
</tr>
<tr>
<td>7</td>
<td>40</td>
<td>3.50</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>1.17</td>
</tr>
</tbody>
</table>

Example 1. Performance evaluation of 24 commercial banks in Taiwan (Kao & Liu, 2004).

Twenty-four commercial banks in Taiwan (DMUs) are evaluated for three inputs and three outputs described below.

$y_1$: Total loans
$y_2$: Interest income
$y_3$: Non-interest income

Total deposits consist of current accounts and long-term deposits. Interest costs include the costs of deposits and other borrowed funds. Non-interest costs include service costs and commissions, general management costs, salaries, and other expenses. These inputs represent the costs of personnel, management, equipment, and purchased funds from banking operations and are the sources of loanable funds for investment (Kao & Liu, 2004).

Table 4
Converted ordinal data and efficiency intervals for the 8 TELECOM branches.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Converted ordinal data of management level</th>
<th>Optimistic efficiency interval</th>
<th>Pessimistic efficiency interval</th>
<th>Overall efficiency interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0.1405, 0.6355]</td>
<td>[0.7252, 1.0000]</td>
<td>[0.1000, 2.2180]</td>
<td>[0.8516, 1.4893]</td>
</tr>
<tr>
<td>2</td>
<td>[0.1120, 0.5066]</td>
<td>[0.7261, 1.0000]</td>
<td>[1.4755, 3.9566]</td>
<td>[1.0351, 1.9891]</td>
</tr>
<tr>
<td>3</td>
<td>[0.1762, 0.7972]</td>
<td>[0.1622, 0.4329]</td>
<td>[0.4028, 0.9805]</td>
<td>[0.4947, 1.0477]</td>
</tr>
<tr>
<td>4</td>
<td>[0.2211, 1.0000]</td>
<td>[0.2448, 0.4463]</td>
<td>[0.4947, 1.0477]</td>
<td>[0.4947, 1.0477]</td>
</tr>
<tr>
<td>5</td>
<td>[0.1974, 0.8929]</td>
<td>[0.2052, 0.3890]</td>
<td>[0.4947, 1.0477]</td>
<td>[0.4947, 1.0477]</td>
</tr>
<tr>
<td>6</td>
<td>[0.1254, 0.5674]</td>
<td>[0.3068, 0.6835]</td>
<td>[1.5901, 3.6739]</td>
<td>[0.6985, 1.5846]</td>
</tr>
<tr>
<td>7</td>
<td>[0.1574, 0.7118]</td>
<td>[0.2223, 0.3548]</td>
<td>[1.0000, 2.2180]</td>
<td>[0.8516, 1.4893]</td>
</tr>
<tr>
<td>8</td>
<td>[0.1000, 0.4523]</td>
<td>[0.7276, 1.0000]</td>
<td>[0.8516, 1.4893]</td>
<td>[1.4843, 2.3460]</td>
</tr>
</tbody>
</table>
sector, risk evaluation is considered a very important issue. Therefore, financial institutes or individual investors must certainly evaluate the performance of banks in the banking industry before attempting to invest in this sector. The 20 remaining commercial banks are considered to be pessimistic non-efficient.

The evaluations above are from different viewpoints and may as well have different results (see the ranking of the commercial banks in the second column of Table 2). Any conclusion of evaluation that considers only one of these two perspectives will undoubtedly be one-sided, unrealistic, and unconvincing (Wang et al., 2007). In order to obtain an overall evaluation of the performance of each commercial bank, we consider the results of both optimistic and pessimistic perspectives simultaneously. The measures obtained from Eq. (13) are reported in the fourth column of Table 1. Furthermore, the second column of Table 2 shows the ranking of the commercial banks based on the overall efficiency interval. It is noteworthy that the commercial bank #19 has the first position based on the optimistic and pessimistic ranking. It is therefore considered the best commercial bank. Ranking results are based on the degree of preference approach. From the optimistic point of view, DMU$_{19}$ $\succ$ DMU$_{20}$, i.e., the performance of DMU$_{19}$ is 51.10% better than DMU$_{20}$. From the pessimistic point of view, DMU$_{19}$ $\succ$ DMU$_{13}$, i.e., the performance of DMU$_{19}$ is 100% better than DMU$_{13}$. According to the overall efficiency interval, DMU$_{19}$ $\succ$ DMU$_{13}$, i.e., the performance of DMU$_{19}$ is 100% better than DMU$_{13}$.

**Example 2.** Performance evaluation of 8 telecommunication company (TELECOM) branches (Cooper, Park, & Yu, 2001b).

Performances of 8 telecommunication company branches (DMUs) were evaluated for three inputs and three outputs, defined below:

- $x_1$: Manpower
- $x_2$: Operations cost (million dollars)
- $x_3$: Management level
- $y_1$: Revenue (million dollars)
- $y_2$: Facilities success rate (%) $y_3$: Call completion rate (% ratio)

The dataset for this analysis is taken from Cooper et al.’s (2001b) work. Table 3 shows the input and output data of 8 TELECOM company branches. Specifically, this example shows how exact and bounded data, as well as ordinal preference information, are integrated as a unified approach in the proposed SBM models. Here, management level is included as a qualitative input and call completion rate is considered as an output with bounded data.

Since models (7)–(10) have been developed for dealing with interval data and Table 3 contains ordinal preference information, the ordinal preference information has to be converted to interval numbers. In the case of exact data, they can be considered a special case of interval data in which the lower and upper bounds are equal. For this example, the preference intensity parameter and the ratio parameter about strong ordinal preference information are given (estimated) as $\eta_3 = 1.12$ and $\sigma_3 = 0.1$, respectively (Azizi, 2013a, 2014b). To illustrate the conversion technique described in Wang et al. (2005), the interval estimation for the third input in case of DMU$_3$ is calculated as follows:

$$
\hat{x}_{3i} \in [\sigma_3 \eta_{3i}^{-1}, \eta_{3i}^{-1}] = [0.1 \eta_3^{-1}, 1.2 \eta_3^{-1}] = [0.1405, 0.6355]
$$

Interval estimation of the third input for all DMUs is shown in the second column of Table 4. Therefore, all input and output data are now converted to interval numbers and can be evaluated using the proposed SBM models. Table 4 shows the efficiencies of the eight DMUs evaluated by models (7)–(10) and the measures in (13). According to Table B.3, these eight DMUs are ranked based on the overall efficiency interval as follows:

$$
\text{DMU}_8 \succ \text{DMU}_2 \succ \text{DMU}_1 \succ \text{DMU}_5 \succ \text{DMU}_6 \succ \text{DMU}_4
$$

where DMU$_9$ $\succ$ DMU$_2$ means that the performance of DMU$_9$ is 72.20% better than that of DMU$_2$. It is evident that DMU$_9$ has the best performance, followed by DMU$_2$, DMU$_1$, and DMU$_5$. The detailed ranking results are presented in Appendix B.

Using the pessimistic efficiency interval, we were able to identify the pessimistic inefficient units. On the other hand, using the

<table>
<thead>
<tr>
<th>Table B.1</th>
<th>Degree of preference matrix and rankings based on the optimistic efficiency interval from models (7) and (8).</th>
</tr>
</thead>
<tbody>
<tr>
<td>DMU</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>0.5008</td>
</tr>
<tr>
<td>3</td>
<td>0.0000</td>
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<tr>
<td>5</td>
<td>0.0000</td>
</tr>
<tr>
<td>6</td>
<td>0.0000</td>
</tr>
<tr>
<td>7</td>
<td>0.0000</td>
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<tr>
<td>8</td>
<td>0.5021</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table B.2</th>
<th>Degree of preference matrix and rankings based on the pessimistic efficiency interval from models (9) and (10).</th>
</tr>
</thead>
<tbody>
<tr>
<td>DMU</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>0.7993</td>
</tr>
<tr>
<td>3</td>
<td>0.5006</td>
</tr>
<tr>
<td>4</td>
<td>0.5451</td>
</tr>
<tr>
<td>5</td>
<td>0.5024</td>
</tr>
<tr>
<td>6</td>
<td>0.8098</td>
</tr>
<tr>
<td>7</td>
<td>0.5000</td>
</tr>
<tr>
<td>8</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
optimistic efficiency interval, we managed to identify the optimistic efficient units, but we were unable to produce a complete ranking of the units. By combining these two interval efficiencies, we succeeded to rank both the optimistic efficient units and the pessimistic inefficient units in a logical manner (see Table 5). For example, the pessimistic efficiency interval shows that DMU1 succeeded to rank both the optimistic efficient units and the pessimistic inefficient units. DMU1 is on both the efficient production frontier and the inefficient production frontier—it was both optimistic efficient and pessimistic inefficient. In other words, the two frontiers pass this DMU concurrently. This condition is likely to happen when the number of evaluated DMUs is small, and it can be interpreted as follows: Even though optimistic efficient units have a good performance, some pessimistic efficient units have a worse performance than others. Similarly, even if we expect pessimistic inefficient units to have a poor performance, some pessimistic inefficient units perform better than others. Therefore, if a DMU is both optimistic efficient and pessimistic inefficient, this means that its performance is neither the best nor the worst, like DMU1 in this example (see Tables B.1, B.2, and B.3).

7. Conclusion

Using only the optimistic perspective for evaluating DMU performances in decision-making problems is undoubtedly a one-sided, unrealistic, and unconvincing approach. Because of the complexity of real-world decision-making problems, integration of the measures obtained from both the optimistic and pessimistic perspectives can offer a more realistic framework for taking into account this uncertainty as compared with the measures obtained from the optimistic perspective. However, integration of the measures obtained from the optimistic and pessimistic perspectives is still a research topic that warrants more investigation.

In this paper, we obtained the SBM models based on the concept of DEA with double frontiers. The models, which we presented for dealing with imprecise data in DEA, have a more complete capability for using the conventional DEA models with imprecise data. The integrated measures obtained from the proposed SBM models determine an overall efficiency interval for each DMU from both the optimistic and pessimistic perspectives. A simple and effective approach based on degree of preference ranking was used for comparing the interval efficiencies of the DMUs. Two numerical examples illustrated the application of DEA with double frontiers. Since fuzzy data can be converted to interval data using α-cuts, DEA with double frontiers is also applicable to fuzzy data. Consequently, it can be used extensively for decision-making problems modeled as interval and fuzzy data.

Acknowledgment

The authors would like to thank two anonymous reviewers whose comments were valuable in improving this article.

Appendix A

A.1. Examining unit invariance and constant returns to scale properties in the proposed SBM models

Models (7) and (8) are obtained from the following pair of fractional programming problems:

\[
\begin{align*}
\text{min} & \quad \rho^u_i = \frac{1 - \frac{1}{m \sum_{j=1}^{m} s_j^+}}{1 + \frac{1}{m \sum_{j=1}^{m} s_j^-}} \quad \rho^u_i = \frac{1}{y^u_{it}} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij}^u + s_j^+ = x_{ij}^u, \quad i = 1, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_{ij}^u - s_j^- = y_{ij}^u, \quad r = 1, \ldots, s, \\
& \quad s_j^+ \geq 0, \quad s_j^- \geq 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, j = 1, \ldots, n.
\end{align*}
\]

(7)

\[
\begin{align*}
\text{min} & \quad \rho^l_i = \frac{1 - \frac{1}{m \sum_{j=1}^{m} s_j^+}}{1 + \frac{1}{m \sum_{j=1}^{m} s_j^-}} \quad \rho^l_i = \frac{1}{y^l_{it}} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij}^l + s_j^- = x_{ij}^l, \quad i = 1, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_{ij}^l - s_j^+ = y_{ij}^l, \quad r = 1, \ldots, s, \\
& \quad s_j^- \geq 0, \quad s_j^+ \geq 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, j = 1, \ldots, n.
\end{align*}
\]

(8)

Now, we will show that the objective functions of (A.1) and (A.5) preserve unit invariance. In other words, if \( x_{ij} = [x_{ij}^l, x_{ij}^u] \) and \( y_{ij} = [y_{ij}^l, y_{ij}^u] \) (\( r = 1, \ldots, s; j = 1, \ldots, n \)), where \( \alpha_i > 0 \) (\( i = 1, \ldots, m \)) and \( \beta_r > 0 \) (\( r = 1, \ldots, s \)), there will be no change in the efficiency values of the models.

Let

\[
\begin{align*}
& x_{ij}^l \rightarrow \alpha_i x_{ij}^l, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n, \\
& x_{ij}^u \rightarrow \alpha_i x_{ij}^u, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n, \\
& y_{ij}^l \rightarrow \beta_r y_{ij}^l, \quad r = 1, \ldots, s; \quad j = 1, \ldots, n, \\
& y_{ij}^u \rightarrow \beta_r y_{ij}^u, \quad r = 1, \ldots, s; \quad j = 1, \ldots, n.
\end{align*}
\]

(9)

(10)

(11)

(12)

First we prove that \( \rho^u_i \) does not change. Substituting (A.9) and (A.12) in (A.2) and (A.3), we have:

\[
\alpha_i x_{ij}^l - \sum_{j=1}^{n} \lambda_j (\alpha_i x_{ij}^l) = \alpha_i \left( x_{ij}^l - \sum_{j=1}^{n} \lambda_j x_{ij}^l \right) = \alpha_i s_j^-, \quad i = 1, \ldots, m.
\]

(13)
\[ \sum_{j=1}^{n} \beta_i (y_i^j - y_i^0) = \beta_i \left( \sum_{j=1}^{n} y_i^j - y_i^0 \right) = \beta_i s_i^+, \quad r = 1, \ldots, s. \quad (A.14) \]

Also, substituting in the objective function of (A.1), we have:
\[ x_j^U - \sum_{j=1}^{n} \beta_i x_j^0 = x_j^U - \sum_{j=1}^{n} \beta_i x_j^0 = x_j^U, \quad i = 1, \ldots, m. \quad (A.15) \]

Therefore, \( p_i^U \) is stable regarding unit invariance in inputs and outputs.

Now, we prove that the value of \( p_i^U \) does not change. Substituting (A.9)–(A.12) in (A.6) and (A.7), we have:
\[ x_j^U - \sum_{j=1}^{n} \beta_i x_j^0 = x_j^U - \sum_{j=1}^{n} \beta_i x_j^0 = x_j^U, \quad i = 1, \ldots, m. \quad (A.17) \]

Also, substituting in the objective function of (A.5), we have:
\[ x_j^U - \sum_{j=1}^{n} \beta_i x_j^0 = x_j^U - \sum_{j=1}^{n} \beta_i x_j^0 = x_j^U, \quad i = 1, \ldots, m. \quad (A.18) \]

Therefore, \( p_i^U \) is stable regarding unit invariance in inputs and outputs. Similarly, it can be proved that models (9) and (10) have unit invariance.

In addition, it is obvious that if any of the slack variables \( s_i \) \( (i = 1, \ldots, m) \) and \( s_r \) \( (r = 1, \ldots, s) \) increases, \( p_i^U \) and \( p_i^L \) will decrease monotonically.

Regarding the constant returns to scale property of the proposed approach, it should be pointed out that the convexity constraint that is related to the nature of returns to scale is relaxed and it is not imported in the model. This relaxation guarantees that the nature of returns to scale is constant.

**Appendix B**

Detailed ranking results for the 8 TELECOM company branches.

**References**


