Optimal Robust Controller Design for the Ball and Plate System

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Abstract—The problem of controlling an open-loop unstable system presents many unique and interesting challenges and ball and plate system is specific example for these kinds of systems. Ball and plate system is inherently nonlinear and under-actuated system. This paper proposes an optimal robust controller design via H-infinity approach. Simulation results show that the proposed controller has the strong robustness and satisfactory and eliminate the effect of linearization problems, unknown external disturbances and time-varying uncertain friction while the ball rolling on the plate surface, thus the trajectory tracking precision is improved.

Keywords—Ball and plate system; nonlinear system; robust control; optimal control; trajectory tracking; H-infinity; uncertainty; under-actuate;

I. INTRODUCTION

The ball and plate system is extension of the well-known ball and beam system. The latter has two degree of freedom where a ball can roll on a rigid beam while the former is the four degree of freedom system consisting of a ball that can roll freely on a plate. However, it is more complicated than ball and beam due coupling of multi-variables. This under-actuated system has only two actuators and should be stabilized by just two control inputs. The system is a benchmark to test various nonlinear control schemes and it is influenced by nonlinearities such as motion resistance between the ball and plate, backlash in transmission set and so on. The experimental platform includes a flat plate, a ball, motors and their driving system, a CCD camera and so on. The plate rotates around its x and y axis in two perpendicular directions. Inclinations of the plate are changed by a servo control system mainly including photo encoders, two step motors and two step motor drivers. Position of the ball is measured by a machine vision system.

Difficulty of creating a ball and plate mechanism with respect to a ball and beam mechanism is the main drawback of such systems but the enormous potential of performing manifold control strategies on ball and plate systems make them desirable.

Various control methods have been introduced for ball and plate system [1]-[9]. For instance, a controller design for two dimensional Electro-mechanical ball and plate system based on the classical and modern control theory [1]. A supervisory fuzzy controller was proposed for studying motion control of the system included the set-point problem and the tracking problem along desired trajectory, which was composed of two layers [2]. A nonlinear velocity observer for output regulation of ball and plate system proposed in [3] where ball velocities are estimated by state observer. In [4] position of the ball regulated with double feedback loops. The sufficient conditions for the controllability of affine nonlinear control systems on Poisson manifolds are discussed in [5]. The active disturbance rejection control is applied to the trajectory tracking design of the ball and plate system in [6] and the PID neural network controller based on genetic algorithm [7] is another work in this topic. Also, recently some works proposed based on sliding mode control [8]-[9].

This paper proposes an optimal robust controller design for linear and nonlinear model of ball and plate system with uncertainties. H-infinity approach is considered. The problems of linearization, unknown external disturbances and time-varying uncertain friction make large estimation and output error in system response, so it is necessary to design a controller which eliminate all external and internal perturbations.

The rest of this paper is organized as follows. Section 2 present ball and plate system model. Section 3 introduces the H-infinity optimal controller design and finally the effectiveness of the proposed method is demonstrated through simulation results in section 4.

II. SYSTEM DESCRIPTION

The mathematical model for ball and plate system is shown in Fig. 1. The plate rotates around its x and y axis in two perpendicular directions. Kinematics differential equations of the ball and plate system are obtained using Lagrange method.

![Fig. 1 The scheme of the model simplification for ball and plate system](image)

System variables are selected as following: x(m) is the displacement of the ball along the x-axis, y(m) is the displacement of the ball along the y-axis, α (rad) is the angle between x-axis of the plate and horizontal plane, β (rad) is the angle between y-axis of the plate and horizontal plane, τx (N·m) is the torque exerted on the plate in x-axis, τy (N·m) is the torque exerted on the plate in the y-axis. Ball and plate...
system can be simplified into a particle system made by two rigid bodies. The plate has three geometry limits in the translation along the x-axis, y-axis and z-axis. It also has a geometry limit in rotation about z-axis. The plate has two degree of freedom (DOF) in the rotation about x-axis and y-axis. The two DOF are depicted in α and β. They are limited in a certain range. The ball has a geometry limit in the translation along the z-axis, and it has two DOF in translations along x-axis and y-axis. Generalized coordinates are chosen as, \( q_1 = x \), \( q_2 = y \), \( q_3 = \alpha \), \( q_4 = \beta \). Generalized forces or torque \( Q \) corresponding to \( \alpha \) and \( \beta \) respectively. Motion resistance includes friction between the ball and plate [10], collision power between the ball and plate and etc.

Suppose the ball remains in contact with the plate and the rolling occurs without slipping at any time. Dynamical equations of the n-DOF system are obtained using Euler-Lagrange’s equation [11].

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} &= Q_k \\
\text{Where} \ L &= \text{the deference of the kinetic and potential energy,} \\
Q_k &= \text{generalized force associated with} \ q_k \ \text{and} \ k = 1, ..., n. \\
\text{Kinetic energy of the ball} \ T_{\text{ball}} &= \frac{1}{2} \left( \left( l_1 \right)^2 \dot{x}_2 + \left( l_2 \right)^2 \dot{y}_2 + \left( m + l_b \right) \left( \dot{a}^2 + \dot{b}^2 \right) + m \left( \dot{x}_a + \dot{y}_b \right)^2 \right) (2) \\
\text{Kinetic energy of the ball and plate system is} \ T_{\text{plate}} &= \frac{1}{2} \left( \left( l_1 \dot{x}_2 + \left( l_2 \right)^2 \dot{y}_2 + \left( m + l_b \right) \left( \dot{a}^2 + \dot{b}^2 \right) + m \left( \dot{x}_a + \dot{y}_b \right)^2 \right) (3) \\
\text{Potential energies of system along the x-axis and y-axis are} \ V_x = mg \sin \alpha, \ V_y = mg \sin \beta \\
\text{Applying the Euler-Lagrange’s equation the mathematical model for ball and plate system is shown as:} \\
\begin{align*}
(m + \frac{b}{r^2}) \ddot{x} - mx \dot{a}^2 - my \dot{b}^2 + mg \sin \alpha &= 0 \\
(m + \frac{b}{r^2}) \ddot{y} - mx \dot{a} \dot{b} + mg \sin \beta &= 0 \\
(l_{px} + l_b + mx^2) \ddot{a} + 2mx \dot{a} \dot{x} + mxy \dot{b} + mxy \dot{b} + m \dot{x} \dot{a} \dot{b} + mg \cos \alpha &= \tau_x \\
(l_{py} + l_b + my^2) \ddot{b} + 2my \dot{a} \dot{x} + mxy \dot{a} + mxy \dot{a} + m \dot{x} \dot{a} \dot{b} + mg \cos \beta &= \tau_y \\
\text{Where} \ m &= \text{the mass of the ball,} \ g = \text{gravity acceleration,} \ l_b = \text{ball inertia,} \ l_{px} = \text{plate inertia to x-axis and} \ l_{py} = \text{plate inertia to y-axis state variables are selected as} \\
x_1 = x, x_2 = \dot{x}, x_3 = \alpha, x_4 = \dot{\alpha} \\
x_5 = y, x_6 = \dot{y}, x_7 = \beta, x_8 = \dot{\beta} \\
\text{State equations of the ball and plate system are:} \\
x_1 &= x_2 \\
x_2 &= C(x_1 x_2 + x_1 x_4 x_8 - g \sin x_3) \\
x_3 &= x_4 \\
x_4 &= u_x \\
x_5 &= x_6 \\
x_6 &= C(x_3 x_5 + x_4 x_5 x_8 - g \sin x_7) \\
x_7 &= x_8 \\
x_8 &= u_y \\
\text{Where} \ C = \frac{mr^2}{2} + l_b = \frac{2}{5} mr^2 \ \text{for a spherical ball so it easily can be shown} \ C = \frac{5}{7}. \ \text{The value of} \ u_x \ \text{and} \ u_y \ \text{considered to be zero.} \ \text{In the ball and plate system, it is supposed that the ball remains in contact with the plate and the rolling occurs without slipping, which imposes a constraint on the rotation acceleration of the plate. Because of the low velocity and acceleration of the plate rotation, the mutual interactions of both coordinates can be negligible. So the model of the ball and plate system can be approximately decomposed as follows:} \\
\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\dot{x}_7 \\
\dot{x}_8 \\
\end{bmatrix} &= \begin{bmatrix}
x_2 \\
C(x_1 x_2 + g \sin x_3) \\
x_4 \\
0 \\
x_6 \\
C(x_3 x_5 + g \sin x_7) \\
x_8 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
t_x \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \\
\begin{bmatrix}
x_1 \\
0 \\
0 \\
0 \\
x_3 \\
x_4 \\
0 \\
0 \\
\end{bmatrix} &= A \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
\end{bmatrix} + B \begin{bmatrix}
t_x \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix} u_x \\
u_y \end{bmatrix} \\
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
\end{bmatrix} &= \begin{bmatrix}
x_2 \\
C(x_1 x_2 + g \sin x_3) \\
x_4 \\
0 \\
x_6 \\
C(x_3 x_5 + g \sin x_7) \\
x_8 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
t_x \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \\
\text{Namely, the ball-plate system can be regarded as two individual sub-systems and both coordinates can be controlled independently. Linear state space model along x-axis is:} \\
\begin{align*}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= 0 \\
\dot{x}_3 &= 0 \\
\dot{x}_4 &= 0 \\
\dot{x}_5 &= 0 \\
\dot{x}_6 &= 0 \\
\dot{x}_7 &= 0 \\
\dot{x}_8 &= 0 \\
\end{align*} \text{and} \begin{array} {c}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
\end{bmatrix} = \begin{bmatrix}
A \\
B \\
C \\
D \end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \
\end{bmatrix} \\
\dot{x} = Ax + Bu \\
\dot{y} = Cy + Du 
\end{array}
\text{The state space model is similar along y-axis. According to continue system control theory, system is state completely controllable if and only if vector B, AB, ..., A^{n-1}B are linear independent, that is the rank of matrix [B AB ... A^{n-1}B] equal to n, the dimension of state system. Moreover, system (10) is completely observable if and only if the rank of matrix [C CA ... C A^{n-1}] equal to n, too. Therefore the Rank of controllability matrix and observability matrix of system (9) are both 4 which are computed in MATLAB, which equal to the dimension of system state. Thus some kind of controller can be designed to make the system (9) stable.}
A. Robust Control and Uncertainty

Robust control is a branch of control theory that explicitly deals with uncertainty in its approach to controller design. The key idea of robust control is to separate the known part and the uncertain part from the knowledge about the uncertain system under investigation. This is illustrated in Fig. 2, where $M(s)$ denotes the known part of the uncertain system and $\Delta(s)$ denotes the uncertain part [12]. Usually, we have some limited knowledge about $\Delta(s)$ such as the upper bound information.

![Fig. 2 general structure of uncertain system](image)

A general description of robust control system structure is shown in Fig. 3, where $P(s)$ is the augmented plant model and $F(s)$ is the controller model. The transfer function from the input $u_1(t)$ to the output $y_1(t)$ is denoted by $T_{y1u1}(s)$. It should be emphasized at this point that the block diagram shown in Fig. 3, is fairly general. The signal vector $u_1(t)$ can include both reference and disturbance signals. $P(s)$ can include both the plant model and the disturbance generation model. Moreover, uncertainties can also be included in $P(s)$.

![Fig. 3 standard feedback control](image)

The unstructured uncertainties can be classified into the additive and multiplicative uncertainties. The feedback system structure with uncertainties is shown in Fig. 4. In general, the uncertain model can be represented by

$$G_{p}(s) = \Delta(s) + G(s) \left[ I + \Delta M(s) \right]$$

(11)

If $\Delta_0(s) \equiv 0$, one has $G_p(s) = G(s) \left[ I + \Delta_0(s) \right]$, and the uncertainty is referred to as the multiplicative uncertainty. When $\Delta_M(s) \equiv 0$, the uncertainty is referred to as the additive uncertainty with the model $G_p(s) = G(s) + \Delta_0(s)$. Based on small gain theorem [12] with no loss of generality, if we assume that the uncertainty norm bound shown in Fig. 2, is unity, then we can concentrate on Fig. 3, if there were no uncertainty in $P(s)$. But now our control design task amounts to designing $F(s)$ such that $\|T_{y1u1}(s)\|_\infty < 1$. We should understand that it is always possible to scale $\Delta(s)$ such that the scaled uncertainty bound is less than 1.

![Fig. 4 Feedback control with uncertainties](image)

B. $H_\infty$ Controller Design

The system structure for $H_\infty$ control described in Fig. 3 considered. Based on the above arguments, considering on the configuration shown in Fig. 3, where an augmented plant model can be constructed as the controller can be represented by

$$p(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$

(12)

With the augmented state space description as follows:

$$\dot{x} = Ax + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(13-a)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x \\ D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(13-b)

Straightforward manipulations give the following closed-loop transfer function:

$$T_{y1u1}(s) = P_{21}(s) + P_{22}(s) \left[ I - F(s) P_{22}(s) \right]^{-1} F(s) P_{21}(s)$$

(14)

The above expression is also known as the linear fractional transformation (LFT) of the interconnected system. The objective of robust control is to find a stabilizing controller $u_2(s) = F(s) y_2(s)$ such that $\|T_{y1u1}(s)\| < 1$.

Based on (14), the standard $H_\infty$ robust control obtained by

$$\|Ty1u1(s)\| < 1.$$  

The design objective is to find a robust controller $F_c(s)$ guaranteeing the closed-loop system with an $H_\infty$-norm bounded by a given positive number $\gamma$ [13].

$$F_c(s) = \begin{bmatrix} A_f & -ZL \\ F & 0 \end{bmatrix}$$

(15)

Where

$$A_f = A + \gamma^{-2} B_1 B_1^T X + B_2 F + ZLC_2$$

$$F = -B_2^T X, L = -YC_2^T, Z = (I - \gamma^{-2} YX)^{-1}$$

And $X$ and $Y$ are, respectively, the solutions of the following two AREs:

$$AX + X^T A^T + \gamma^{-2} B_1 B_1^T X - B_2 B_2^T X + C_1 C_1^T = 0$$

$$AY + Y^T A^T + \gamma^{-2} C_1 C_1 - C_2 C_2 Y + B_2^T B_1 = 0$$

The conditions for the existence of an $H_\infty$ controller are as follows:

- $D_{11}$ is small enough such that $D_{11} < \gamma$.
- The solution $X$ of the controller ARE is positive-definite.
- The solution $Y$ of the observer ARE is positive-definite.
- $\lambda_{max}(XY) < \gamma^2$, which indicates that the eigenvalues of the product of the two Riccati equation solution matrices are all less than $\gamma^2$.

The returned tree variable $F_c(s)$ is the designed $H_\infty$ controller in state space form.

In optimal $H_\infty$ controller design, the optimal criterion is defined as:

$$\max_{\gamma} \|Ty1u1\| < \frac{1}{\gamma}$$

(16)
IV. SIMULATION RESULTS

With the discussion above, the proposed optimal H\(_\infty\) controller can be realized. The simulation object for tracking problem is to make ball follow a specific path. The desired trajectory for tracking is a circle. Simulation results show reasonable output for both linear and nonlinear models. The radius of desired circle is 400 mm. Simulation results for linear model by applying robust controller are shown in Fig. 5 and Fig. 6. For linear model system error shown in Fig. 6 is less than 0.5 mm and ball settle down in circle path after 4 second.

![Fig. 5 Circular trajectory tracking for linear system](image)

![Fig. 6 Circular trajectory tracking error for linear system](image)

Simulation results for nonlinear system are shown in Fig. 7 and Fig. 8. Results demonstrate the effectiveness of controller on nonlinear system.

![Fig. 7 Circular trajectory tracking for nonlinear system](image)

![Fig. 8 Circular trajectory tracking error for nonlinear system](image)

For nonlinear model tracking error shown in Fig. 8 is less than 1mm at the steady state.

V. CONCLUSION

This paper proposes a trajectory tracking optimal robust controller for the ball and plate system with uncertainty. Simulation is conducted in MATLAB, and the results demonstrate that the proposed controller can make the ball precisely track the given trajectory despite of the existence of uncertainty. The controller can be implemented on the ball and plate system. Despite being based on a linear model, the controllers performed extremely well with the nonlinear system. Also, the problems of linearization, unknown external disturbances and time-varying uncertain friction that make large estimation and output error in system response eliminated with designed controller.

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