Characteristics analysis of joint space inverse mass matrix for the optimal design of a 6-DOF parallel manipulator

Jiang Hong-Zhou *, He Jing-Feng, Tong Zhi-Zhong
School of Mechatronic Engineering, Harbin Institute of Technology, Harbin 150001, China

ARTICLE INFO

Article history:
Received 27 August 2009
Received in revised form 30 November 2009
Accepted 8 December 2009
Available online 13 January 2010

Keywords:
Optimal design
6 DOF Parallel manipulator
Gough–Stewart platform
Joint space mass matrix
Centro symmetric matrix, Dynamic isotropy

ABSTRACT

An optimal design method for the Gough–Stewart platform manipulators based on dynamic isotropy is proposed. First, a dynamic isotropy measure is derived from the analysis of the natural frequencies of a Stewart platform at a neutral pose using the inverse of the joint space mass matrix. Next, considering a specific Gough–Stewart platform (SGSP), it is found that, when the payload inertia matrix is diagonal, the neutral pose joint space inverse generalized mass matrix is a symmetric Centro symmetric matrix. Using this property, we derive symbolic expressions for the Eigen values and Eigenvectors and discuss and present various predominant definitions in terms of the Eigen values. We show that one can obtain spatial isotropy based on dynamics whereas it is impossible to get spatial isotropy from kinematics and statics. Finally, we present an optimal design method based on the dynamic isotropy for SGSP. We also give a numerical example to illustrate the design procedure.

© 2009 Elsevier Ltd. All rights reserved.

1. Introduction

It is still difficult to find an optimal design for 6 DOF parallel manipulators because there are so many performance measures such as workspace volume, manipulability, dexterity, singularity, accuracy, actuators interference, etc., to consider. Depending on the wide variety of application areas, e.g. machining tool, motion simulation, vibration isolation, and precise positioning, etc., we can select one or several measures simultaneously for an optimal design.

Isotropy, which fits into three categories including kinematic, static and dynamic isotropy, is one of the common measures of performance of a manipulator [1–21]. Designing a parallel robot that is isotropic in one pose or over its full workspace is often considered a design objective [2]. Through investigating the characteristics of Jacobian matrix related isotropy performance indices such as the condition number of the Jacobian, separate force and moment transformation matrices, a structural stiffness matrix, a joint space generalized mass matrix, and generalized natural frequencies, etc., several researchers [1–21] have proposed different approaches to deal with the dimensional synthesis of the Gough–Stewart platform (GSP) and found a family of 6 DOF parallel manipulators with isotropic performances.

Stocco et al. [4–6] defined a global isotropy index that measures the worst-case performance of a GSP over a workspace. Gosselin and Angeles [7] also defined a global isotropy index. Pittens and Podhorodeski [8] found GSPs numerically through minimizing the condition number of the inverse Jacobian at a single point. McInroy [9,10] combined static input–output transformations with hexapod geometric design based on two decoupling algorithms for vibration isolation applications. Jafari and McInroy [11,12] used analytic methods to characterize all orthogonal Gough–Stewart platforms (OGSPs) and to study their properties over a small workspace. This characterization is used to design optimal OGSPs for precision applications that achieve a desired hyperellipsoid of velocities. Kim [13] presented a new formulation of a dimensionally
homogeneous Jacobian matrix for parallel manipulators to avoid the unit inconsistency problem in the conventional Jacobian matrix. Yi et al. [14] explored the class of OGSPs defined by Jafari and McInroy [9], and developed methods for generating classes of orthogonal Gough–Stewart platforms (OGSPs). They extended these designs to redundant platforms, where more struts are introduced to achieve redundancy without sacrificing orthogonality or isotropy. Bandyopadhyay and Ghosal [16] proposed an algebraic method to study and design spatial parallel manipulators that demonstrate isotropy in the force and moment distributions. They used the force and moment transformation matrices separately, and derive conditions for their isotropy individually as well as in combination.

Many researchers [17–21] have pointed out that it is essential to take dynamics into account for higher precision and faster motion controls. Several measures of dynamic performance of manipulators have been introduced in previous work. Ma and Angeles [18,19] introduced the concept of a dynamic conditioning index or DCI. A low value of DCI signifies that the inertia matrix is close to an ideal isotropic generalized inertia matrix. Mukherjee et al. [20] put forward a dynamic stability index based on natural frequencies of a flexible manipulator. Wang et al. [21] proposed an optimal design method based on a generalized natural frequency to expand the bandwidth for the control of large hydraulic Stewart platforms.

So far, to the best of our knowledge, no one has used dynamic isotropy analysis in an analytical formulation to discover the relationship between the mass-geometry characteristics and dynamic isotropy for the specific Gough–Stewart platform (SGSP) considered in this paper, which will lead to an optimal design based on dynamic isotropy. In this paper, we aim to determine to what extent, considering some dynamic isotropy measures, the payload inertial parameters including mass, inertia, and the center of mass can affect the optimal design.

This paper is organized as follows. In Section 2, on the basis of the analysis of the generalized natural frequencies of the Generalized Gough–Stewart platforms (GGSP), we propose a dynamic isotropy measure and dynamic cross-coupling indices based on the inverse mass matrix in joint space, for an optimal design of the GGSP. In Section 3, an analytical formulation of the inverse mass matrix in joint space for the GGSP is derived. Considering the SGSP at a neutral pose, the Eigen values and Eigenvectors of the inverse of the joint space mass matrix are expressed as functions of configuration parameters in analytical form. In Section 4, the dynamic isotropy conditions, including rotation, translation, complete and combined isotropy, are also derived in terms of the Eigen values. Therefore, an optimal design method is proposed for determining the geometry configuration parameters using an iterative procedure. The relationship between the mass-geometry characteristics (mass, inertia, the center of mass and geometry configuration parameters of the SGSP) and the dynamic isotropy is also expressed explicitly. Finally, we present our conclusions.

2. Problem formulation

In this section, we propose a dynamic isotropy measure and dynamic cross-coupling indices derived from a generalized natural frequencies analysis for an optimal design of the GGSP.

2.1. Generalized natural frequency of GGSP

**Definition 1 (Generalized Gough-Stewart Platform (GGSP) [22]).** Any system connected to the environment through six parallel joints with five (passive) rotational and one (active) translation DOF is considered a GGSP.

This is a 6 DOF closed kinematic chain mechanism consisting of a fixed base and a moveable platform with six linear actuators supporting it, see Fig. 1. At neutral position the body axes \( \{M\} \) attached to the movable platform are parallel to, and coincide with, the inertial frame \( \{B\} \) fixed to the base. \( a_i \) denotes the 3 \( \times \) 1 vector of the upper joint center point in the body axes, \( b_i \) represents the 3 \( \times \) 1 vector of the lower joint center point in the fixed base frame, and the subscript \( i \) is the actuator number.

Generally the equations of motion in the task space of the GGSP [22–24] are written as:

\[
J_i^T \mathbf{f}_i = M_i(sx)x + C_i(sx) \hat{x} + G_i(sx)
\]

where \( sx = [x \ y \ z \ \varphi \ \theta \ \psi]^T \) is the 6 \( \times \) 1 vector of the platform position with respect to the fixed base frame, and contains the translation and Euler angles. \( \dot{x} \) and \( \ddot{x} \) are the 6 \( \times \) 1 platform velocity and acceleration vectors, and both contain translation and angular components. \( f_i \) is the 6 \( \times \) 1 vector of the actuator output forces. \( M_i(sx) \) is the 6 \( \times \) 6 mass matrix found in the base frame. \( C_i(sx, \dot{x}) \) is the 6 \( \times \) 6 Coriolis/centripetal coefficients matrix. \( G_i(sx) \) are the gravity terms. \( J_i^T(sx) \) is the 6 \( \times \) 6 Jacobian matrix relating the platform velocities to the actuators length rates in the joint space, i.e. \( J_i \hat{x} = I \)

\[
J_i^T = \begin{bmatrix}
L_{n1}^T (T^{a_{n1}}) \times L_{n1})^T \\
L_{n2}^T (T^{a_{n2}}) \times L_{n2})^T \\
L_{n3}^T (T^{a_{n3}}) \times L_{n3})^T \\
L_{n4}^T (T^{a_{n4}}) \times L_{n4})^T \\
L_{n5}^T (T^{a_{n5}}) \times L_{n5})^T \\
L_{n6}^T (T^{a_{n6}}) \times L_{n6})^T
\end{bmatrix} = [L_i^T (T^{a_{n1}}) \times L_{n1})^T],
\]
here \( \mathbf{I}_{nl} \), \( i = 1, 2, \ldots, 6 \) is the \( 3 \times 1 \) spatial unit vector of the \( i \)th actuator, \( \mathbf{T} \) is the \( 3 \times 3 \) rotation matrix described with Euler angles.

We have assumed that the fixed base and top platforms are rigid, and only the stiffness of the actuators is considered. In this paper, we are interested only in the platform dynamic response to external disturbances, and the Coriolis/centripetal, gravity and other non-linear terms are assumed to be external disturbances or negligible in this system. In fact, it is unfair to drop these terms to define dynamic isotropy measures; however, in many practical cases, as will be analyzed in the next sections, it is still possible to obtain some valuable results to guide the design under these assumptions.

To analyze the natural frequencies of the GGSP, we can reformulate Eq. (1) in the joint space. For conciseness, we introduce directly the linearized version of the dynamics of the GGSP in joint space, proposed in [20]. For the analysis of the free vibration of the platform, the characterization equation [20] can be described as:

\[
(M_{\text{act}}^2 + \mathbf{K})\mathbf{l} = 0
\]  

where \( \mathbf{l} = (l_j) \) with \( j = 1, 2, \ldots, 6 \), \( s \) is the Laplace derivative operator, \( \mathbf{K} \) is a \( 6 \times 6 \) stiffness matrix in joint space. The mass matrix, \( M_{\text{act}} = \mathbf{J}^\top \mathbf{M} \mathbf{J} \), known as the joint space generalized mass matrix as seen from the actuators, contains the platform pose \( \mathbf{s} \) dependent Jacobian matrix and an almost constant platform mass matrix \( \mathbf{M} \) [22].

The undamped Eigen frequencies of the mechanical system can be predicted by calculating the square roots of the Eigen values of \( M_{\text{act}}^2 \mathbf{K} \). Solving Eq. (2), we can get six natural frequencies and six mode shapes for a particular position and orientation of a Gough–Stewart platform:

\[
\omega_h = \sqrt[2]{\lambda(M_{\text{act}}^2 \mathbf{K})}
\]  

where \( \lambda(\cdot) \) depicts the Eigen value of a matrix. Note that, the natural frequencies and mode shapes of the Stewart platform also depend on position and orientation. Assuming that the six actuators have identical stiffness, i.e. \( \mathbf{K} = k_i \mathbf{I} \), here \( k_i \) is a scalar and \( \mathbf{I} \) is an identity matrix, Eq. (3) can be rewritten as:

\[
\omega_h = \sqrt{k_i \lambda(M_{\text{act}}^2)}
\]  

since the mass matrix is positive definite symmetric, and the eigenvector matrix can be made unitary. This results in the singular value decomposition [22, 24]:

\[
M_{\text{act}}^{-1} = \mathbf{USU}^\top
\]  

where \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) \) are singular values, and \( \mathbf{U} \) is a pose dependent unitary orthogonal matrix. These singular values can be interpreted as the inverse of the generalized masses, which the actuators will have to accelerate moving along the six orthogonal directions described by the columns of the unitary decoupling matrix \( \mathbf{U} \). Each column of the matrix, \( \mathbf{U} \), is defined as the rigid body modal direction of the Gough–Stewart platform [22]. Consequently, Eq. (4) can be formulated as:

\[
\omega_{hi} = \sqrt{k_i \sigma_i}, \quad i = 1, 2, \ldots, 6
\]
It is very important to consider the dynamic stability at an earlier design stage. As pointed out by Parthajit [1], a very low value of the lowest natural frequency signifies dynamic instability of the manipulator. The lowest natural frequency plays the same role in dynamic stability as does the least singular value of the force transformation matrix in statics. The aim of this design is to maximize the lowest natural frequency and to ensure that the natural frequencies are as close as possible in the neutral position.

2.2. Local dynamic isotropy measure

Thus, an optimal design problem in one pose can be formulated by:

\[
\text{Minimizing } k = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} = \sqrt{\frac{\lambda_{\text{max}}(M^{-1}_{\text{act}})}{\lambda_{\text{min}}(M^{-1}_{\text{act}})}} = \sqrt{\frac{\sigma_{\text{max}}}{\sigma_{\text{min}}}}
\]

(7)

We call a parallel manipulator completely isotropic if \(k = 1\). We can use \(k\) as a local dynamic isotropy measure for an optimal design. Therefore, we can study the dynamic isotropy by exploiting the properties of \(M^{-1}_{\text{act}}\), only if the six actuators have identical stiffness.

2.3. Dynamic cross-coupling indices

Defining the new variables \(\dot{x} = U^{T}i\), with the modal directions, the system can be decoupled into six SISO subsystems in parallel [22,24]. The physical meaning of \(U\) can be explained as a new generalized Jacobian matrix which denotes the linear mapping from the generalized velocity in modal space to the length variation rate of the actuators in physical joint space [22]:

\[
\dot{i} = UX
\]

Substituting \(J_{6}\dot{x} = \dot{i}\) into Eq. (8), we can obtain:

\[
\dot{x} = (U^{T}J_{6})\dot{x} = C_{T}\dot{x}
\]

(9)

Eq. (9) shows the mode shapes in physical task space, and \(C_{T}\) denotes the dynamic coupling between the DOFs. To measure the degree of coupling effects between the DOFs quantitatively, we introduce a local cross-coupling index proposed by Gogu [25].

**Definition 2** (Local cross-coupling index). The local cross-coupling index \(\zeta_{ij}\) is a measure of decoupling between two column vectors \(A_{i}\) and \(A_{j}\) of the matrix \(A\) and is given by the sine of the intersection angle \(\theta_{ij}\) between \(A_{i}\) and \(A_{j}\):

\[
\zeta_{ij} = \sin \theta_{ij} = \sqrt{1 - \left(\frac{A_{i} \cdot A_{j}}{|A_{i}| |A_{j}|}\right)^{2}}
\]

(10)

Consequently the dynamic cross-coupling index for the GGSP can be expressed by:

\[
\zeta_{ij} = \sqrt{1 - \left(\frac{C_{i} \cdot C_{ij}}{|C_{i}| |C_{ij}|}\right)^{2}}
\]

(11)

Since the angle of interaction between column vectors is in the range of \((0, \pi)\) we can see that \(0 \leq \zeta_{ij} \leq 1\). If the two column vectors corresponding to the DOF of \(i\) and \(j\) are fully coupled, then \(\zeta_{ij} = 0\). When \(\zeta_{ij} = 1\) the two column vectors \(C_{i}\) and \(C_{ij}\) are mutually orthogonal and the DOF motions are completely decoupled. The name of this index is derived from the fact that its maximum value corresponds to a decoupled motion [25].

So far, we have proposed two dynamic measures, which are in accord with one another. We want to find an optimal configuration as close to the ideal isotropy as possible and simultaneously with fewer dynamics coupling effects.

3. Characteristics analysis of the joint space inverse mass matrix of GSP

In this section, an analytical formulation of the inverse mass matrix in joint space for Generalized Gough–Stewart platforms (GGSP) is derived. Then, considering the SGSP, the Eigen values and Eigen vectors of inverse mass matrix in joint space are expressed as functions of configuration parameters in analytical form by exploiting its symmetry properties. Finally, the dynamic coupling effects between the DOFs at neutral pose are analyzed using the coupling matrix \(C_{T}\).

3.1. Inverse mass matrix in joint space for GGSP

Generally, taking the geometric center of the platform as the origin of the coordinate system, which does not coincide with the payload’s center of mass, then the payload mass matrix can be described as:
\[ M_i = \begin{bmatrix} mI - m\tilde{\rho}_c \\ m\tilde{\rho}_c \\ \mathbb{I}_0 \end{bmatrix} \]  

where the center of mass \( \rho_c = \mathbf{T}\rho_c^0 \) is a 3 \times 1 vector, the superscript \( ^m \) denotes body axes, \( m \) is the payload mass, \( \mathbb{I}_0 \) is the inertia matrix with respect to the origin of the coordinate system, \( \tilde{\rho}_c \) denotes a skew matrix of a spatial vector \( \rho_c \).

In addition, without considering the inertial effects of the actuators, the inverse matrix of \( M_\tau \) can be calculated by:

\[ M_\tau^{-1} = \begin{bmatrix} \frac{1}{m} (I + m\tilde{\rho}_c I_\tau^{-1} \tilde{\rho}_c^{-1} \tilde{\rho}_c I_\tau^{-1}) \\ -I_\tau^{-1} \tilde{\rho}_c \end{bmatrix} \]  

Here, \( I \) is 3 \times 3 identity matrix. \( I_c \) is the inertia with respect to the center of mass.

Thus we can have the joint space inverse mass matrix term described as:

\[ M_{act}^{-1} = J_n M_i^{-1} J_n^T = \begin{bmatrix} L_n^T (\mathbf{T}A^m \times L_a)^T \end{bmatrix} \begin{bmatrix} \frac{1}{m} (I + m\tilde{\rho}_c I_\tau^{-1} \tilde{\rho}_c^{-1} \tilde{\rho}_c I_\tau^{-1}) \\ -I_\tau^{-1} \tilde{\rho}_c \end{bmatrix} \begin{bmatrix} L_n \\ \mathbf{T}A^m \times L_a \end{bmatrix} = \frac{1}{m} L_n^T L_a + (\mathbf{T}A^m \times L_a)^T I_c^{-1} \mathbf{T}A^m \times L_a \]  

where \( A^m = [a_{i1}^m - \rho_c^m, a_{i2}^m - \rho_c^m, a_{i3}^m - \rho_c^m, a_{i4}^m - \rho_c^m, a_{i5}^m - \rho_c^m, a_{i6}^m - \rho_c^m] \).

The element of the inverse matrix in row \( i \) and column \( j \) can be written as:

\[ M_{act}^{-1}(i,j) = \frac{1}{m} L_n^T L_{nj} + (\mathbf{T}(a_{i}^m - \rho_c^m) \times L_a)^T I_c^{-1} \mathbf{T}(a_{j}^m - \rho_c^m) \times L_a \]  

where \( i, j = 1, 2, \ldots, 6 \).

Note in Eqs. (14) and (15), the joint space inverse mass matrix could be decomposed into the sum of the mass related matrix and the inertia related matrix which explicitly reveals the coupling effects between all the actuators. With the formulation given, the influence of the payload inertial parameters (mass, inertia, the center of mass and geometry parameters of the GGSP) on the joint space inverse mass matrix can be clearly be seen. It is easy to see that \( M_{act}^{-1} \) is independent of the location of the origin of the frame \{M\}.

3.2. Eigen values and Eigen vectors of the inverse mass matrix in joint space for SGSP

It is very difficult to solve the Eigen problems analytically for the GGSP due to large number of parameters that must be considered. As many researchers \[2,9,10,13,16\] have done before, we choose the SGSP at neutral pose as the object of our study. The motivations for our choice were: (a) its widespread technical applications, (b) it is the most studied spatial parallel manipulator, and (c) its symmetry at neutral pose allows us to obtain its Eigen values and vectors in a concise symbolic form. We define neutral pose as the pose that has zero orientation and center position with \( \mathbf{s}_x = [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T \). Thus, the optimum design problem is to find the architecture variables such that the dynamic isotropy measure of the corresponding inverse mass matrix in joint space is minimal.

Fig. 2. Specific Gough–Stewart platforms.
Definition 3 (Specific Gough–Stewart platforms (SGSP) [22]). See Fig. 2, both the upper and lower gimbal points virtually form a platform, which looks like two blunt equilateral triangles rotated 180 degrees with respect to each other. All the upper and the lower gimbal points lie on two circles with radius \( r_a \) and \( r_b \), respectively. The circles lie in parallel planes if the actuator lengths are equal. The actuators and gimbal points, which are numbered 1–6, are distributed pair wise counterclockwise over the circle. All the actuators must have the same properties.

The architecture of such a manipulator can then be fully defined by parameters \((r_a, r_b, H, h, \alpha, \beta)\), as shown in Fig. 2, where \( r_a \) is the distribution radius of upper gimbal points, \( r_b \) is the distribution radius of lower gimbal points, \( H \) is the height of the moveable platform with respect to the base, \( h \) is the height of the payload’s center of mass with respect to the origin of Frame (M), \( \alpha \) is the half central angle of the short edge of the moveable platform, \( \beta \) is the half central angle of the short edge of the base.

To solve the Eigen problems analytically, without loss of generality, supposing that the platform is at neutral pose, the inertial effects of all the actuators are not considered, the payload mass–inertia matrix \( \mathbf{M} \), found with respect to the base frame, must be diagonal, and the frame must be selected to coincide with the orientation of the principal axes of the payload:

\[
\mathbf{M} = \begin{bmatrix}
    ml & 0_{l 	imes 3} \\
    0_{3 	imes 3} & \text{diag}(l_{xx}, l_{yy}, l_{zz})^T
\end{bmatrix}
\]

where \( l_{xx}, l_{yy}, l_{zz} \) are the moments of inertia.

3.2.1. Centrosymmetric matrix

Fortunately, some symmetry properties of the SGSP at a nominal configuration can be used to reduce the order of Eigen problems. We introduce the notion of a centro symmetric matrix below.

Definition 4. Let \( \mathbf{P} \in \mathbb{R}^{n \times n} \) be the matrix with ones along the secondary diagonal and zeros elsewhere, i.e. \( \mathbf{P} = (\mathbf{e}_n \ e_{n-1} \ldots \ e_1) \) and \( \mathbf{e}_i \) is the \( i \)th column of identity matrix \( \mathbf{I}_n \), the matrix \( \mathbf{D} \in \mathbb{R}^{n \times n} \) is called centro symmetric if \( \mathbf{PDP} = \mathbf{D} \).

It is not difficult to verify that, if \( \mathbf{D} = (d_{ij}) \) is a centro symmetric matrix, then one can have \( d_{ij} = d_{n+1-i,n+1-j} \).

Property 1. For SGSP, at neutral pose, when the payload mass–inertia matrix is diagonal, \( \mathbf{M}_{\text{act}}^{-1} \) is a symmetric Centro symmetric matrix:

\[
\begin{cases}
    \mathbf{M}_{\text{act}}^{-1}(i, j) = \mathbf{M}_{\text{act}}^{-1}(i, i) \\
    \mathbf{M}_{\text{act}}^{-1}(i, j) = \mathbf{M}_{\text{act}}^{-1}(6 + 1 - i, 6 + 1 - j)
\end{cases}
\]

where \( i, j = 1, 2, \ldots, 6 \).

Proof. Considering its geometrical symmetry at neutral pose, the unit vector of any actuator can be calculated through manipulating the unit vector of Actuator 1 with rotation along the Z axis or/and reflecting its transformation relative to the plane XOZ. The unit vector of Actuator 6 can be written as a reflecting transformation of Actuator 1, \( \mathbf{l}_{61} = \text{diag}(1, -1, 1)^T \mathbf{I}_{n1} \), the others can be calculated by rotating \( \mathbf{I}_{n1} \) or \( \mathbf{I}_{n6} \) with \( \pm 2/3 \pi \) along Z axis, \( \mathbf{l}_{12} = \mathbf{R}(z, -2/3 \pi) \mathbf{l}_{n6}, \mathbf{l}_{14} = \mathbf{R}(z, 2/3 \pi) \mathbf{l}_{n6}, \mathbf{l}_{13} = \mathbf{R}(z, -2/3 \pi) \mathbf{l}_{n1}, \) and \( \mathbf{l}_{15} = \mathbf{R}(z, 2/3 \pi) \mathbf{l}_{n1} \); here \( \mathbf{R}(z, \gamma) \) is a rotation matrix with angle \( \gamma \) along Z axis, \( \mathbf{l}_{n1} = \begin{bmatrix} l_{n1x} & l_{n1y} & l_{n1z} \end{bmatrix}^T \). Similarly, \( \mathbf{a}_{n1} \), the unit vector of the upper gimbal point can also be described by functions of \( \mathbf{a}_{n1} \); here \( \mathbf{a}_{n1} = \begin{bmatrix} a_{n1x} & a_{n1y} & a_{n1z} \end{bmatrix}^T \). According to the symmetry properties of the SGSP, we have:

\[
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & -1 & 0 \\
    0 & 0 & 1
\end{bmatrix}
\]

Defining \( \mathbf{v}_i = r_i \mathbf{a}_{ni} \times \mathbf{l}_{ni} \), here \( r_i \) is a positive scalar representing the norm of the upper gimbal point vector of \( \mathbf{a}_i \), we can obtain:

\[
\mathbf{v}_i = \begin{bmatrix}
    -1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & -1
\end{bmatrix}
\]

Substituting these terms into Eq. (17):

\[
\mathbf{M}_{\text{act}}^{-1}(7 - i, 7 - j) = \frac{1}{m} \mathbf{l}_{n(7-i), n(7-j)}^T \mathbf{I}_{n(7-i)} \mathbf{I}_{n(7-j)}^{-1} \mathbf{v}_j \]

(18)
For the first term of Eq. (18):

\[
\frac{1}{m} \mathbf{I}^T_m (\mathbf{I} - \mathbf{I}) = \frac{1}{m} \mathbf{I}^T_m \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right] \mathbf{I} = \frac{1}{m} \mathbf{I}^T_m \mathbf{I} \\
\text{If } \mathbf{I}^{-1} \text{ is diagonal, then the second term of Eq. (18) can be rewritten as:}
\]

\[
\mathbf{v}_j^T \mathbf{I}^{-1} \mathbf{v}_j = \mathbf{v}_j^T \left[ \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right] \mathbf{I}^{-1} = \mathbf{v}_j^T \mathbf{I}^{-1} \mathbf{v}_j
\]

Obviously, we can obtain \( \mathbf{M}^{-1}_{\text{act}} (7 - i, 7 - j) = \mathbf{M}^{-1}_{\text{act}} (i, j) \). This shows that at neutral pose \( \mathbf{M}^{-1}_{\text{act}} \) is a Centro symmetric matrix. Consequently, we can have:

\[
P \cdot \mathbf{M}^{-1}_{\text{act}} \cdot P = \mathbf{M}^{-1}_{\text{act}}
\]

Centro symmetric matrices have practical applications in information theory, linear system theory, and in physics. The inverse Eigen problems, and the solvability and reducibility of Centro symmetric matrices are the subject of much attention and have potential to resolve engineering problems, for example, signal processing, structural design, vibration design, etc. We refer the reader to [26–31]. In this paper, we make use of some results from previous work [26,27] to deal with the Eigen problems for the SGSP. Further, \( \mathbf{M}^{-1}_{\text{act}} \) can be written as a block matrix:

\[
\mathbf{M}^{-1}_{\text{act}} = \begin{bmatrix} \mathbf{A} & \mathbf{B}P \\ \mathbf{P}\mathbf{B} & \mathbf{P}\mathbf{AP} \end{bmatrix}
\]

where \( \mathbf{A}, \mathbf{B} \) are \( 3 \times 3 \) matrices. Given an unitary orthogonal matrix \( \mathbf{Q} \):

\[
\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{I}_{3 \times 3} \\ -\mathbf{P}_{3 \times 3} & \mathbf{P}_{3 \times 3} \end{bmatrix}
\]

Then:

\[
\mathbf{Q}^T \mathbf{M}^{-1}_{\text{act}} \mathbf{Q} = \begin{bmatrix} \mathbf{A} - \mathbf{B} & 0 \\ 0 & \mathbf{A} + \mathbf{B} \end{bmatrix}
\]

Eq. (22) shows us that the 6th order Eigen problem is reduced to a 3rd order, which is easy to solve analytically using the method proposed by Bandyopadhyay et al. [16].

3.2.2. Eigen values

The Eigen problem can be simplified further. From our experience using this manipulator and examining each element of \( \mathbf{M}^{-1}_{\text{act}} \) carefully, we find the following properties:

**Property 2.** There exists an all ones Eigen vector \( \mathbf{q}_1 = [1 \ 1 \ 1 \ 1 \ 1]^T \) corresponding to an Eigen value \( \lambda_1 \).

\[
\lambda_1 = \frac{6}{m} r^2 n_{1,z}
\]

\( \lambda_1, \delta \mathbf{q}_1 = [1 \ 1 \ 1 \ 1]^T \) is also one pair of the Eigen values and vectors of \( \mathbf{A} + \mathbf{B} \).

**Proof.** This implies that all the sums of each row of \( \mathbf{M}^{-1}_{\text{act}} \) are equal. It is easy to verify this by substituting \( \mathbf{q}_1 \) into the characteristics equation of \( \mathbf{M}^{-1}_{\text{act}} \). This mode direction agrees with the (z) direction in physical task space.

**Property 3.** There exists an Eigen vector with alternate components 1 and \(-1\), \( \mathbf{q}_2 = [1 \ -1 \ 1 \ -1 \ 1]^T \) corresponding to an Eigen value \( \lambda_2 \):

\[
\lambda_2 = \frac{6}{l_{1z}} r^2 \left( a_{\text{n}1_y} \cdot l_{\text{n1}_x} - a_{\text{n}1_x} \cdot l_{\text{n1}_y} \right)^2 = \frac{6}{l_{1z}} r^2 \left| l_{\text{n1}_x} \ a_{\text{n1}_x} \ - \ l_{\text{n1}_y} \ a_{\text{n1}_y} \right|^2
\]

where \( r \) represents the distance from the payload’s center of mass to each upper gimbal point. \( \lambda_2, \delta \mathbf{q}_2 = [1 \ -1 \ 1]^T \) is also one pair of the Eigen values and vectors of \( \mathbf{A} - \mathbf{B} \).

**Proof.** We can prove it by substituting \( \mathbf{q}_2 \) into the characteristics equation of \( \mathbf{M}^{-1}_{\text{act}} \). This also means that, if we take the sum of odd-numbered column vectors minus the sum of even-numbered column vectors, then the absolute values of all the rows are the same and equal to \( \lambda_2 \), and the sign for the odd numbered row is positive, and the sign for even-numbered row is negative. This mode direction agrees with yaw (rz) direction in physical task space.
These two properties can be validated using symbolic tools; for conciseness the detail proof is not given here. Note that the translation and rotation along Z axis are independent of other DOFs.

So far, the Eigen problem has been further simplified, and the third-order problem is transformed into a second-order problem. Considering the relation of the roots and coefficients for the cubic equation, and with the aid of the Matlab symbolic tool, the other four Eigen values can be obtained as:

\[
\lambda_{3,4} = \frac{3}{2m} \left( \left( \Delta_{xx} + \Delta_{yy} \right) I_{xy} + \Delta_{x,y} n_{n,x} n_{n,y} \right) + \left( \left( \Delta_{xx} + \Delta_{yy} \right) I_{xx} + \Delta_{x,y} n_{n,x} n_{n,y} \right)^2 - 4\mu_{xy} n_{n,x} n_{n,y} \right)^{\frac{1}{2}}
\]

\[
\lambda_{5,6} = \frac{3}{2m} \left( \left( \Delta_{xx} + \Delta_{yy} \right) I_{xx} + \Delta_{x,y} n_{n,x} n_{n,y} \right) + \left( \left( \Delta_{xx} + \Delta_{yy} \right) I_{xx} + \Delta_{x,y} n_{n,x} n_{n,y} \right)^2 - 4\mu_{xx} n_{n,x} n_{n,y} \right)^{\frac{1}{2}}
\]

where

\[
\Delta_{xx} = \begin{vmatrix} a_{n,x} & l_{n,x} \\ a_{n,y} & l_{n,y} \end{vmatrix}, \Delta_{yy} = \begin{vmatrix} a_{n,x} & l_{n,x} \\ a_{n,y} & l_{n,y} \end{vmatrix}, \Delta_{x,y} = \begin{vmatrix} a_{n,x} & l_{n,x} \\ a_{n,y} & l_{n,y} \end{vmatrix}, \mu_{xx} = \frac{m r^2}{I_{xx}}, \mu_{yy} = \frac{m r^2}{I_{yy}}
\]

Note in Eq. (25) we can see that these Eigen values are not only relevant to the payload’s mass \( m \) but also dependant on the moments of inertia \( I_{xx}, I_{yy} \). Obviously the translation and rotation motion are coupled heavily along the x and y axes respectively.

For ease of use, the Eigen values can be expressed as functions of the configuration parameters \( (r_a, r_b, H, \alpha, \beta) \) and the payload inertial parameters \( (m, I_{xx}, I_{yy}, I_{zz}, h) \).

Since

\[
L = \begin{bmatrix} r_a \cos \alpha - r_b \cos \left( \frac{\pi}{3} - \beta \right) \\ r_a \sin \alpha - r_b \sin \left( \frac{\pi}{3} - \beta \right) \\ -H \end{bmatrix}^T L
\]

\[
a_{n,1} = \left[ \begin{array}{c} r_a \cos \alpha - r_b \sin \left( \frac{\pi}{3} - \beta \right) \\ r_a \sin \alpha - r_b \sin \left( \frac{\pi}{3} - \beta \right) \\ H \end{array} \right]
\]

where \( \left. \begin{array}{c} L = \sqrt{r_a^2 + r_b^2 - 2r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) + H^2} \\ r = \sqrt{h^2 + r_a^2} \end{array} \right| \\text{subject to} \\ r_a \geq 0, \ r_b \geq 0, \ H \geq 0, \ 0 \leq \alpha, \ \beta \leq \frac{\pi}{3}.
\]

Substituting Eq. (26) into Eqs. (23) and (25), we obtain:

\[
\lambda_1 = \frac{6}{m} \frac{H^2}{r_a^2 + r_b^2 - 2r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) + H^2}
\]

\[
\lambda_2 = \frac{6}{I_{zz}} \frac{r_a^2 r_b^2 \sin^2 \left( \frac{\pi}{3} - \alpha - \beta \right) + H^2}{r_a^2 + r_b^2 - 2r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) + H^2}
\]

\[
\lambda_{3,4} = \frac{3 \left( K_{yy} \pm \sqrt{K_{yy}^2 - 4 \frac{m}{L_{yy}} H r_a^2 r_b^2 \sin^2 \left( \frac{\pi}{3} - \alpha - \beta \right) + H^2} \right)}{2m(r_a^2 + r_b^2 - 2r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) + H^2)}
\]

where

\[
K_{yy} = \frac{m}{L_{yy}} \left( r_a^2 + r_b^2 - 2r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) \right) h^2 + 2r_a H \left( r_a - r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) \right) h + r_a^2 H^2
\]

\[
\lambda_{5,6} = \frac{3 \left( K_{xx} \pm \sqrt{K_{xx}^2 - 4 \frac{m}{L_{xx}} H r_a^2 r_b^2 \sin^2 \left( \frac{\pi}{3} - \alpha - \beta \right) + H^2} \right)}{2m(r_a^2 + r_b^2 - 2r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) + H^2)}
\]

where

\[
K_{xx} = \frac{m}{L_{xx}} \left( r_a^2 + r_b^2 - 2r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) \right) h^2 + 2r_a H \left( r_a - r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) \right) h + r_a^2 H^2
\]

To avoid singularities, we must satisfy the constraints \( H \neq 0 \), and \( \alpha + \beta \neq \frac{\pi}{3} \), see Eqs. (27a) and (27b).
3.2.3. Eigen vectors

First, we introduce the definition of symmetric (skew symmetric) vectors.

**Definition 5.** Vector \( z \in \mathbb{R}^n \) is called symmetric (skew symmetric) if \( Pz = z \) or \( Pz = -z \).

**Property 4.** The Eigen vectors of \( M_{\text{act}}^{-1} \) are symmetric, for those derived from \( A + B \), or skew symmetric, for those derived from \( A - B \).

**Proof.** We need to analyze the characteristics of the Eigen values of \( M_{\text{act}}^{-1} \), \( A - B \), \( A + B \) and their relationships. Defining \( x_1, x_2 \) as the Eigen vectors of \( M_{\text{act}}^{-1} \), \( A - B \), \( A + B \) respectively, for \( A - B \), we can obtain:

\[
Q^\top M_{\text{act}}^{-1} Q y = \begin{bmatrix} A - B & 0 \\ 0 & A + B \end{bmatrix} y = \begin{bmatrix} A & B \\ 0 & A - B \end{bmatrix} y_1 = \lambda y_1
\]

where \( \lambda \) is one of the Eigen values corresponding to \( A - B \). Thus:

\[
x = Q y = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{3 \times 3} & I_{3 \times 3} \\ -P_{3 \times 3} & P_{3 \times 3} \end{bmatrix} y_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} y_1 \\ -P y_1 \end{bmatrix}
\]

Similarly, for \( A + B \):

\[
Q^\top M_{\text{act}}^{-1} Q y = \begin{bmatrix} A - B & 0 \\ 0 & A + B \end{bmatrix} y = \begin{bmatrix} A & B \\ 0 & A + B \end{bmatrix} y_2 = \lambda^+ y_2
\]

where \( \lambda^+ \) is one of the Eigen values corresponding to \( A + B \). Thus:

\[
x = Q y = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{3 \times 3} & I_{3 \times 3} \\ -P_{3 \times 3} & P_{3 \times 3} \end{bmatrix} y_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} y_2 \\ -P y_2 \end{bmatrix}
\]

Note in Eqs. (29) and (31), we can see that, each Eigen vector of \( M_{\text{act}}^{-1} \) is symmetric or skew symmetric. This can guide us in identifying the linear components relative to the Eigen vectors for any vector through analyzing its symmetric characteristics.

In addition, given one of the Eigen vectors, we can obtain the other unknowns according to their orthogonality. For \( A + B \), defining its Eigen vector as \( \delta q = [\delta q_1, \delta q_2, \delta q_3]^T \), since it is orthogonal to \( [1 \ 1 \ 1]^T \), we can obtain the equations:

\[
\begin{cases}
(\langle A + B \rangle_{11} - \lambda_{3,4}) \delta q_1 + (\langle A + B \rangle_{12} \delta q_2 + (\langle A + B \rangle_{13} \delta q_3 = 0 \\
\delta q_1 + \delta q_2 + \delta q_3 = 0
\end{cases}
\]

where \( \langle A + B \rangle_{ij} \) indicates the element of \( A + B \) in row \( i \) and column \( j \).

Let \( \delta q_3 = 1 \), solving Eq. (32), the Eigen vectors corresponding to \( \lambda_{3,4} \) can be obtained as:

\[
\begin{cases}
\delta q_1 = \frac{\langle A + B \rangle_{13} - \langle A - B \rangle_{11}}{\langle A + B \rangle_{12} + \langle A - B \rangle_{11}} \\
\delta q_2 = \frac{\langle A + B \rangle_{11} \delta q_3 - \langle A + B \rangle_{13}}{\langle A + B \rangle_{12} + \langle A - B \rangle_{11}} \\
\delta q_3 = 1
\end{cases}
\]

where

\[
\langle A + B \rangle_{12} + \lambda_{3,4} - (\langle A + B \rangle_{11}) = 2m \left( \mu_{yy} \left( \Delta_{xz} + \sqrt{3} \Delta_{yz} \right) \left( 3 \Delta_{xz} + \sqrt{3} \Delta_{yz} \right) 
+ 3 \left( \frac{2 \sqrt{3}}{3} l_{n1, x} n_{1, y} - l_{n1, x} \right) \right) + \left( \Delta_{xz} + \Delta_{yz} \right) \left( \mu_{yy} + l_{n1, x}^2 + l_{n1, y}^2 \right) 
- 4 \mu_{yy} l_{n1, x}^2 \Delta_{xy}^2 \right)^{1/2}
\]

\[
\langle A + B \rangle_{13} - (\langle A + B \rangle_{11}) = \frac{2 \sqrt{3}}{m} \left( \Delta_{xz} \Delta_{yz} \mu_{yy} + l_{n1, x} l_{n1, y} \right) \left( \langle A + B \rangle_{11} - \lambda_{3,4} - (\langle A + B \rangle_{11}) 
+ \frac{2 \sqrt{3}}{3} l_{n1, x} l_{n1, y} \right) \left( \Delta_{xz} + \Delta_{yz} \right) 
- 3 \left( \frac{2 \sqrt{3}}{3} l_{n1, x} l_{n1, y} \right) \mu_{yy} l_{n1, x}^2 + l_{n1, y}^2 
\pm \left( \left( \Delta_{xz}^2 + \Delta_{yz}^2 \right) \mu_{yy} + l_{n1, x}^2 + l_{n1, y}^2 \right)^{1/2} \right)^{1/2}
\]

For \( A - B \), defining its Eigen vector as \( \delta q = [\delta q_1, \delta q_2, \delta q_3]^T \), since it is orthogonal to \( [1 \ -1 \ 1]^T \), we can obtain the equations as:

\[
\begin{cases}
(\langle A - B \rangle_{11} - \lambda_{5,6}) \delta q_1 + (\langle A - B \rangle_{12} \delta q_2 + (\langle A - B \rangle_{13} \delta q_3 = 0 \\
\delta q_1 - \delta q_2 + \delta q_3 = 0
\end{cases}
\]
where \((A - B)_i\) indicates the element of \(A - B\) in row \(i\) and column \(j\). Let \(\delta q_3 = 1\), solving Eq. (34), the Eigen vectors corresponding to \(\lambda_{5, 6}\) can be obtained as:

$$
\begin{align*}
\delta q_1 &= \frac{(A - B)_{13} - (A - B)_{12}}{(A - B)_{12} - \lambda_{5, 6} = (A - B)_{11}} \\
\delta q_2 &= \frac{(A - B)_{11} - (A - B)_{13}}{(A - B)_{12} - \lambda_{5, 6} = (A - B)_{11}} \\
\delta q_3 &= 1
\end{align*}
$$

where

$$
(A - B)_{12} - \lambda_{5, 6} + (A - B)_{11} = -\frac{1}{2m} \left( \mu_{xx} (\Delta_{xx} + \sqrt{3} \Delta_{yy}) (3 \Delta_{xx} - \sqrt{3} \Delta_{yy}) + 3 \left( \frac{2\sqrt{3}}{3} l_{n1x} l_{n1y} + l_{n1x}^2 - l_{n1y}^2 \right) \right) - (A - B)_{13} - (A - B)_{12}
$$

$$
= \frac{2\sqrt{3}}{m} \left( \Delta_{xx} \Delta_{yy} \mu_{xx} + l_{n1x} l_{n1y} \right)
$$

$$
(A - B)_{11} - \lambda_{5, 6} - (A - B)_{13} = -\frac{1}{2m} \left( \mu_{xx} (\Delta_{xx} - \sqrt{3} \Delta_{yy}) (3 \Delta_{xx} + \sqrt{3} \Delta_{yy}) + 3 \left( -\frac{2\sqrt{3}}{3} l_{n1x} l_{n1y} + l_{n1x}^2 - l_{n1y}^2 \right) \right)
$$

Normalizing Eqs. (33) and (35) and substituting them into Eqs. (29) and (31), we can obtain the Eigen vectors \(q_3, q_4, q_5, q_6\) corresponding to the Eigen values \(\lambda_3, \lambda_4, \lambda_5, \lambda_6\) of \(M^{-1}_{\text{ww}}\), respectively. The unitary orthogonal transform matrix can be obtained as:

$$
U = [q_1, q_2, q_3, q_4, q_5, q_6]
$$

### 3.2.4. Dynamic coupling effects between DOFs

To analyze dynamic coupling effects between DOFs, we first give the Jacobian matrix as a function of \(a_{11}\) and \(l_{11}\):

$$
J_0 = \begin{bmatrix}
\frac{1}{2} l_{11x} & \frac{1}{2} l_{11y} & l_{11z} & r(a_{11x} l_{11x} - a_{11x} l_{11y}) & r(a_{11x} l_{11x} - a_{11y} l_{11y}) & r(a_{11x} l_{11x} - a_{11z} l_{11z}) \\
-\frac{1}{2} l_{11x} & -\frac{1}{2} l_{11y} & l_{11z} & -r(a_{11x} l_{11x} - a_{11x} l_{11y}) & -r(a_{11x} l_{11x} - a_{11y} l_{11y}) & -r(a_{11x} l_{11x} - a_{11z} l_{11z}) \\
\frac{1}{2} l_{11x} - \frac{1}{2} l_{11y} & \frac{1}{2} l_{11x} + \frac{1}{2} l_{11y} & l_{11z} & -r(a_{11x} l_{11x} - a_{11x} l_{11y}) & -r(a_{11x} l_{11x} + a_{11y} l_{11y}) & -r(a_{11x} l_{11x} - a_{11z} l_{11z}) \\
-\frac{1}{2} l_{11x} - \frac{1}{2} l_{11y} & -\frac{1}{2} l_{11x} + \frac{1}{2} l_{11y} & l_{11z} & r(a_{11x} l_{11x} - a_{11x} l_{11y}) & -r(a_{11x} l_{11x} + a_{11y} l_{11y}) & -r(a_{11x} l_{11x} - a_{11z} l_{11z}) \\
\frac{1}{2} l_{11x} + \frac{1}{2} l_{11y} & \frac{1}{2} l_{11x} - \frac{1}{2} l_{11y} & l_{11z} & -r(a_{11x} l_{11x} - a_{11x} l_{11y}) & r(a_{11x} l_{11x} + a_{11y} l_{11y}) & r(a_{11x} l_{11x} - a_{11z} l_{11z}) \\
\frac{1}{2} l_{11x} - \frac{1}{2} l_{11y} & -\frac{1}{2} l_{11x} - \frac{1}{2} l_{11y} & l_{11z} & -r(a_{11x} l_{11x} - a_{11x} l_{11y}) & r(a_{11x} l_{11x} + a_{11y} l_{11y}) & r(a_{11x} l_{11x} - a_{11z} l_{11z})
\end{bmatrix}
$$

Some properties of the Jacobian can be found as follows:

**Property 5.** \(J_0(\cdot, 1), J_0(\cdot, 5), J_0(\cdot, 2), J_0(\cdot, 4)\) are orthogonal to \(q_1 = [1, 1, 1, 1, 1, 1]^T\) and \(q_2 = [1, -1, 1, 1, -1, -1]^T\).

**Proof.** Examine each column of the Jacobian, it is not difficult to prove it by calculating the inner products of the column vectors with \(q_1, q_2\). □

**Property 6.** \(J_0(\cdot, 1) + J_0(\cdot, 5)\) can only be expressed as a linear function of the Eigen vectors \(q_1, q_3, q_5, q_6\) deduced from \(A + B\). \(J_0(\cdot, 2)\) and \(J_0(\cdot, 4)\) can only be expressed as a linear function of the Eigen vectors \(q_3, q_5, q_6\) deduced from \(A - B\).

**Proof.** First inspect each column of Jacobian Matrix, and note that column 1 and 5, the elements of \(J_0(\cdot, 1), J_0(\cdot, 5)\) are symmetric, and let a vector \(y\) be composed of their first three elements, then \(J_0(\cdot, 1)\) or \(J_0(\cdot, 5)\) can be described as \(y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T\).

Similarly, column 2 and 4, \(J_0(\cdot, 2), J_0(\cdot, 4)\) are skew symmetric, and can be described in the form as \(y = \begin{bmatrix} y_1 & -Py_1 \end{bmatrix}^T\).

Then start with \(J_0(\cdot, 2)\). According to Property 5, \(J_0(\cdot, 2)\) cannot be a linear function of \(q_1, q_2\), thus it is can only be a linear function of \(q_3, q_4, q_5, q_6\), and we can have:

$$
J_0(\cdot, 2) = k_1 q_3 + k_2 q_5 + k_3 q_3 + k_4 q_4
$$

where \(k_{ij}, i = 1, 2, 3, 4\) is a nonzero real scalar. □
Further, Eq. (38) can be expanded as:

$$J_{i}(\ast, 2) = \begin{bmatrix} k_{y_{1}} \delta q_{5} + k_{y_{2}} \delta q_{6} + k_{y_{3}} \delta q_{3} + k_{y_{4}} \delta q_{4} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \ddots \\ 0 \end{bmatrix} \begin{bmatrix} \cdots \\ \ddots \\ \cdots \end{bmatrix} + \begin{bmatrix} 0 \\ \ddots \\ 0 \end{bmatrix} \begin{bmatrix} \cdots \\ \ddots \\ \cdots \end{bmatrix} = \begin{bmatrix} y_{1} \\ -py_{1} \end{bmatrix} \quad (39)$$

$$\begin{cases} k_{y_{1}} \delta q_{5} + k_{y_{2}} \delta q_{6} + k_{y_{3}} \delta q_{3} + k_{y_{4}} \delta q_{4} = y_{1} \\ -k_{y_{1}} \delta q_{5} - k_{y_{2}} \delta q_{6} + k_{y_{3}} \delta q_{3} + k_{y_{4}} \delta q_{4} = -y_{1} \end{cases} \quad (40)$$

Since $PP = I$, the second equation of Eq. (40) is left-multiplied by $P$, then plus the first equation, we can obtain:

$$k_{y_{2}} \delta q_{3} + k_{y_{4}} \delta q_{4} = 0 \quad (41)$$

Since $\delta q_{3}$ and $\delta q_{4}$ are linear independent. Thus, only if $k_{3}$ and $k_{4}$ are zeroes, can Eq. (41) be satisfied. Therefore, Eq. (38) can be rewritten as:

$$J_{i}(\ast, 2) = k_{y_{1}} q_{3} + k_{y_{2}} q_{6} \quad (42)$$

The proof is conclusive. The other properties can be proved in a similar fashion. Consequently the Jacobian can be expressed as linear combinations of a set of Eigen vectors of $M_{act}^{-1}$:

$$J_{x} = \begin{bmatrix} k_{x_{1}} q_{1} + k_{x_{2}} q_{4} + k_{x_{3}} q_{5} + k_{x_{4}} q_{6} + k_{x_{1}} q_{5} + k_{x_{2}} q_{6} + k_{x_{3}} q_{3} + k_{x_{4}} q_{4} \\ r(a_{x_{1}} l_{x_{1,y}} - a_{x_{1}} l_{x_{1,x}}) \end{bmatrix} \quad (43)$$

where the coefficients $k_{x_{1}}, k_{x_{2}}, k_{x_{3}}, k_{x_{4}}, k_{x_{5}}, k_{x_{6}}$ are real scalars.

Then substituting Eqs. (36) and (43) into Eq. (9), we can obtain:

$$C_{y} \ddot{x} = U^{T} J_{y} x = \begin{bmatrix} 0 & 0 & 6l_{x_{1},z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} k_{y_{1}} q_{1} \delta q_{5} \delta q_{3} \delta q_{4} \delta q_{6} + k_{y_{2}} q_{4} \delta q_{5} \delta q_{3} \delta q_{4} \delta q_{6} + k_{y_{3}} q_{3} \delta q_{3} \delta q_{4} \delta q_{6} + k_{y_{4}} q_{4} \delta q_{3} \delta q_{4} \delta q_{6} \end{bmatrix} \begin{bmatrix} \cdots \\ \ddots \\ \cdots \end{bmatrix} + \begin{bmatrix} \cdots \\ \ddots \\ \cdots \end{bmatrix} \begin{bmatrix} \cdots \\ \ddots \\ \cdots \end{bmatrix} \begin{bmatrix} 0 \\ \ddots \\ 0 \end{bmatrix} \begin{bmatrix} \cdots \\ \ddots \\ \cdots \end{bmatrix} \begin{bmatrix} \cdots \\ \ddots \\ \cdots \end{bmatrix} \begin{bmatrix} \cdots \\ \ddots \\ \cdots \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{bmatrix} \quad (44)$$

Eq. (44) represents the DOFs coupling caused by the payload inertial properties, and can also be explained as mode directions in the physical coordinate system. Generally the mode directions do not completely agree with the generalized degrees of freedom described by translation and Euler angles. We can also see that the mode shapes of the Eigen values $\lambda_{i,2}$ correspond to translation and rotation along the $Z$ axis respectively, and $\lambda_{i,3,4}$ correspond to translation along the $x$ axis and rotation along the $y$ axis, and $\lambda_{i,5,6}$ correspond to translation along the $y$ axis and rotation along the $x$ axis.

4. Dynamic isotropy based optimal design

The Eigen values of $M_{act}^{-1}$ in the concise analytical form allow us to scrutinize the relationship between geometry and inertial parameters under some design requirements, e.g. dynamic isotropy. We shall see that, as long as we select one of the dynamic isotropy conditions proposed here as the design aim, the dimensional synthesis of the SGSP is determined completely by the payload’s inertial parameters.

In this section, we present a novel optimal design method based on some dynamic isotropy conditions for the SGSP. First we analyze the influence of the payload’s center of mass on the dynamic isotropy and coupling effects. Second we obtain four types of dynamic isotropy conditions. Finally we list the optimal design procedures.

4.1. The influence of the height of payload’s center of mass on dynamic isotropy and coupling

See Eqs. (27c) and (27d), the Eigen values $\lambda_{i,2}, \lambda_{i,3,4}, \lambda_{i,5,6}$ are quadratic functions of variable $h$, and when:

$$h = h^{*} = \frac{-r_{a}(r_{a} - r_{b} \cos (\frac{\pi}{2} - \alpha - \beta))}{r_{a}^{2} + r_{b}^{2} - 2r_{a}r_{b} \cos (\frac{\pi}{2} - \alpha - \beta)} \quad (45)$$

there exist extreme values. With regard to the expression of $h^{*}$, if $r_{a} \leq r_{b} \cos (\frac{\pi}{2} - \alpha - \beta)$ is always true, $h^{*}$ can be a nonnegative, and the payload’s center of mass can be placed on or above the upper circle plane of the platform, otherwise it is impossible to approach the extreme values. Several researchers have found some special cases such as $r_{a} = r_{b}$ and $r_{a} = 2r_{a}$ in the literature [8,9,12,13]. In this paper, Eq. (45) leads to a general method of finding the geometry to satisfy a given center of mass criteria.
Substituting $h^*$ into Eq. (27c), we can obtain:

$$K_{yy} = \frac{m r_a^2 r_b^2 \sin^2 \left( \frac{\pi}{3} - \alpha - \beta \right)}{I_{yy} \left( r_a^2 + r_b^2 - 2 r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) \right)} H^2 + \left( \frac{r_a^2 + r_b^2 - 2 r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right)}{I_{yy}} \right)$$

Thus, the extreme values of $\lambda_3^*, \lambda_5^*$ are simplified as:

$$\lambda_3^* = \frac{3 r_a^2 r_b^2 \sin^2 \left( \frac{\pi}{3} - \alpha - \beta \right)}{I_{yy} \left( r_a^2 + r_b^2 - 2 r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) \right)} \left( \frac{r_a^2 + r_b^2 - 2 r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right)}{I_{yy}} \right) H^2$$

$$\lambda_4^* = \frac{3 \left( r_a^2 + r_b^2 - 2 r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) \right)}{m \left( r_a^2 + r_b^2 - 2 r_a r_b \cos \left( \frac{\pi}{3} - \alpha - \beta \right) \right) + H^2}$$

Similarly we can give $\lambda_3^*, \lambda_5^*$ in almost the same form as Eqs. (47) and (48), the only difference is the term $I_{yy}$ is replaced with $I_{xx}$. We can see that, when $h = h^*$, besides the geometry parameters, $\lambda_3^*$ is only determined by $I_{yy}$, $\lambda_5^*$ only depends on $I_{xx}$, while $\lambda_4^*, \lambda_6^*$ are only relevant to the mass $m$. Obviously, this implies the decoupling between the DOFs under the condition of Eq. (45). In fact, we can conclude this through calculating each element of $C_T$. However, it is difficult to give $C_T$ in a concise symbolic form at this stage. Next we shall give a numerical example to demonstrate the influence of the height of the payload’s center of mass on dynamic isotropy and coupling.

Considering a flight simulator motion system as an example, for which the configuration and inertial parameters are listed in Table 1, and then calculating Eqs. (27) and (6), the undamped natural frequencies are obtained, and the curves of $h$ are depicted in Fig. 3.

The variations in height of the payload’s center of mass affect the undamped Eigen frequencies dramatically, and mainly impact on those along the $x$, $y$ axes. The lower the height of $h$, the fewer the natural frequencies corresponding to $\lambda_4, \lambda_6$. In this case, rising from 2 Hz, corresponding to the point where $h^* = 0.7884$ m, the natural frequencies reach their maximum peak value at 4 Hz, and beyond this they fall off gradually. Meanwhile the natural frequencies relative to $\lambda_3, \lambda_5$ decrease and then rise, reaching the minimum bottom value at 6 Hz when $h^* = 0.7884$ m. The other two Eigen frequencies corresponding to the yaw and heave motions are almost constant. For $\lambda_3^*, \lambda_5^*$, they have a minimum gap at the point when $h = h^*$. When $h$ is far

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_a$</td>
<td>Distribution radius of upper joint points</td>
<td>2.1148</td>
<td>m</td>
</tr>
<tr>
<td>$r_b$</td>
<td>Distribution radius of lower joint points</td>
<td>2.5170</td>
<td>m</td>
</tr>
<tr>
<td>$H$</td>
<td>Platform height in neutral position</td>
<td>2.6519</td>
<td>m</td>
</tr>
<tr>
<td>$m$</td>
<td>Payload mass</td>
<td>13642.000</td>
<td>kg</td>
</tr>
<tr>
<td>$I_{xx}$</td>
<td>Moment of inertia</td>
<td>46477.100</td>
<td>kg m²</td>
</tr>
<tr>
<td>$I_{yy}$</td>
<td>Moment of inertia</td>
<td>49396.100</td>
<td>kg m²</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>Moment of inertia</td>
<td>53865.000</td>
<td>kg m²</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Half central angle of Short section of upper platform</td>
<td>0.0541</td>
<td>rad</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Half central angle of Short section of fixed base</td>
<td>0.0454</td>
<td>rad</td>
</tr>
<tr>
<td>$k_c$</td>
<td>Stiffness of the actuators</td>
<td>9.1168e+006</td>
<td>N/m</td>
</tr>
</tbody>
</table>

![Fig. 3. The undamped natural frequencies influenced by the height of the payload's center of mass.](image-url)
away from $h^*$, they tend to separate in opposite directions. In this case, since $\lambda_3 > \lambda_4$, with the rise of $|h - h^*|$, $\lambda_3$ increases correspondingly, but $\lambda_4$ decreases gradually. The closer $h$ is towards $h^*$, the better the dynamic isotropy.

The changes of the height of the payload’s center of mass result in marked effects on the mode directions in physical task space. As shown in Fig. 4, when the undamped natural frequency approaches the extreme values, the DOFs agree with mode directions, this means no coupling between the DOFs at this location.

The curves of the dynamic cross-coupling indices versus $h$ are depicted in Fig. 5, and we can clearly distinguish the degree of the dynamic coupling effects. Note that at point with $h = h^*$ the magnitude of the dynamic cross-coupling indices is equal to 1. This means that the DOFs of $x$ and $ry$, as well as $y$ and $rx$, are fully decoupled.

The literature [9] proposed three types of orthogonal SGSPs. However, from the viewpoint of this paper, generally for any SGSP, there exists a compliance center, and we can adjust the payload’s center of mass to coincide with it, thus achieve the best dynamic isotropy at neutral pose and fully decouple the DOFs. The ratio of the upper platform radius to the lower base determines the position of the compliance center.

### 4.2. Isotropy conditions

According to the analysis of 4.1, only when the payload’s center of mass coincides with the compliance center $h^*$, are all the Eigen values closest to each other, and we can achieve better dynamic isotropy. So we take Eq. (45) as the pre condition for isotropy design.
For ease of analysis, suppose that \( x = \beta = 0 \) and \( I_{yy} \approx I_{xx} \), which is easy to satisfy in practice. The dynamic isotropy can be categorized into four types as follows:

(1) Rotation isotropy

Let \( \lambda_2^* = \lambda_3^* = \lambda_5^* \), we have:

\[
H_m = \sqrt{2} \sqrt{\frac{I_{yy}}{I_{zz}} (r_a^2 + r_b^2 - r_a r_b)}
\]

\[
\lambda_1^* = 3 \sqrt{\frac{I_{zz}}{I_{xx} + 2I_{yy}}}
\]

\[
\lambda_{ISO,m} = \lambda_2^* = \lambda_3^* = \lambda_5^* = \frac{9}{2} \frac{1}{I_{zz} + 2I_{yy}} \frac{n^2}{n^2 - n + 1} r_b^2
\]

\[
\lambda_4^* = \lambda_b^* = \frac{12}{m} \sqrt{\frac{I_{yy}}{I_{zz} + 2I_{yy}}} (50)
\]

(2) Translation isotropy

Let \( \lambda_1^* = \lambda_4^* = \lambda_6^* \), we have:

\[
H_v = \frac{\sqrt{2}}{2} \sqrt{r_a^2 + r_b^2 - r_a r_b} \lambda_2^* = \frac{3}{4} \frac{r_b^2}{I_{yy}} \frac{n^2}{n^2 - n + 1}
\]

\[
\lambda_3^* = \lambda_5^* = \lambda_{ISO,v} = \lambda_1^* = \lambda_4^* = \lambda_6^* = \frac{2}{m}
\]  

(3) Complete isotropy

Depending on the results of the rotation and translation isotropy conditions, let \( \lambda_1^* = \lambda_2^* = \lambda_3^* = \lambda_4^* = \lambda_5^* = \lambda_6^* = \lambda_{ISO} \), to achieve the complete dynamic isotropy, we have to satisfy the requirements specified by:

\[
\begin{align*}
I_{zz} &= 4I_{yy} \\
I_{zz} &= \frac{3}{2} \pi \frac{m}{n^2 - n + 1} mr_b^2
\end{align*}
\]  

(51)

In practice, it is difficult to satisfy the first equation of Eq. (51) which may require laborious work and tedious procedures as reported in literature [24]. Therefore, we conclude that it is impossible to achieve complete dynamic isotropy using only geometry design. This is similar to the result in the literature [16] on static isotropy.

(4) Combined isotropy

In some applications, we may expect that the frequency bandwidths are uniform just in translation and angular motion along the Z axis, or just in the \( x-ry \) and \( y-rx \) directions, or by combining both of them. For flight simulation, a SGSP is often used to replicate the flight motions and provide the pilots motion cueing signals that only contain the signals along the surge (\( x \)), sway (\( y \)), roll (\( rx \)), and pitch (\( ry \)) axes, and the components of the other DOFs are negligible. In addition, simulation fidelity is always emphasized due to the high requirements of flight safety. \( x-ry \) and \( y-rx \) isotropy means that it is possible to reduce the effects caused by bandwidth inconsistency and dynamic couplings on simulation fidelity to a limited extent. The combined isotropy can be categorized into three types listed as follows:

(4a) \( z-rz \) isotropy;
(4b) \( x-ry \) and \( y-rx \) isotropy;
(4c) combining both 4a and 4b

First let \( \lambda_1^* = \lambda_2^* \), we have:

\[
H_1 = \sqrt{m} \frac{f_a r_b}{I_{zz}} \sin \left( \frac{\pi}{3} - \alpha - \beta \right) = \frac{\sqrt{3}}{2} \sqrt{m} \frac{f_a r_b}{I_{zz}}
\]  

(52)
Then let \( \lambda_3 = \lambda_4 \), we can get:

\[
H_2 = \sqrt{\frac{I_{yy}}{m}} \left( \frac{r_a^2 + r_b^2 - 2r_a r_b \cos \left( \frac{\pi}{2} - \alpha - \beta \right)}{r_a r_b \sin \left( \frac{\pi}{2} - \alpha - \beta \right)} \right)^{\frac{1}{3}} = \frac{2 \sqrt{3}}{3} \sqrt{\frac{I_{yy}}{m}} \left( \frac{r_a^2 + r_b^2 - r_a r_b}{r_a r_b} \right)
\]  

(53)

Let \( \lambda_3 = \lambda_0 \), taking \( I_{yy} \approx I_{xx} \), we can get the same formula as Eq. (53). Further let \( H_1 = H_2 \), we have:

\[
\left( 1 - \frac{3}{4} \delta \right) n^2 - n + 1 = 0
\]  

(54)

where \( \delta = \frac{m_a^2}{\sqrt{I_{xx} I_{yy}}} \), \( n = \frac{4}{r_a} \). The roots of Eq. (54) are:

\[
n = \frac{1 \pm \sqrt{3(\delta - 1)}}{2(1 - \frac{3}{4} \delta)}
\]  

(55)

We call Eq. (55) a \( n - \delta \) equation representing the combined isotropy necessary conditions. If we want to guarantee that \( n = \frac{4}{r_a} \) is real, then \( \delta \geq 1 \). On the other hand, for the physical meaning, it is necessary to guarantee the condition of \( n = \frac{4}{r_a} \geq 0 \).

For the case of \( \frac{4}{r_a} = \frac{1 - \sqrt{3(\delta - 1)}}{2(1 - \frac{3}{4} \delta)} \), we can arrive at \( 1 \leq \delta < \frac{4}{3} \) and \( n = \frac{4}{r_a} \in [2, +\infty) \), as shown in Fig. 6a. Obviously, under this condition, \( r_a \geq 2r_b \), there are no examples in industry which could adopt this kind of geometry. For this reason, it is not considered in this paper.

For the case of \( n = \frac{4}{r_a} = \frac{1 - \sqrt{3(\delta - 1)}}{2(1 - \frac{3}{4} \delta)} \), we have \( 1 \leq \delta < +\infty \) and \( n = \frac{4}{r_a} \in [2, 0) \), as shown in Fig. 6b. To satisfy the conditions of Eq. (50), the Eigen values are described as:

\[
\lambda_{ISO,z} = \lambda_1 = \lambda_2 = \frac{6}{m} \cdot \frac{1}{\sqrt{\frac{I_{yy}}{m} + 1}}
\]  

(56a)

\[
\lambda_{ISO,xy} = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \frac{3}{m} \cdot \frac{1}{\sqrt{\frac{I_{yy}}{m} + 1}}
\]  

(56b)

It is interesting to note that the Eigen values are independent of the geometry, and are only functions of the inertial parameters. Under the same situation we can also have:

\[
H_{ISO} = \frac{2 \sqrt{3}}{3} \sqrt{\frac{I_{yy}}{m}} \left( \frac{n^2 - n + 1}{n} \right)
\]  

(56c)

The dynamic isotropy index can be expressed as:

\[
k = \frac{\max(\lambda_{ISO,xy}, \lambda_{ISO,z})}{\min(\lambda_{ISO,xy}, \lambda_{ISO,z})}
\]  

(57)

When \( \frac{I_{yy}}{m} = 1 \), then \( k = \sqrt{2} \). This result is very similar to the local optima found by Pittens [8] and Kim [13].

It is very important for all the isotropy conditions representing interesting mass-geometry characteristics for parallel manipulators, which is a challenging area for further research. We list several typical geometries in Table 2 under the condition of Eq. (55).

In Table 2, although the cases of Nos. 2 and 3 have been found by some researchers, in this paper the relationship between inertial parameters and geometric design based on dynamic isotropy has been treated in depth and in a more general sense. Let \( \lambda_{ISO,xy} = \lambda_{ISO,z} \), we can also deduce the complete isotropy condition:
Table 2

<table>
<thead>
<tr>
<th>No.</th>
<th>n</th>
<th>δ</th>
<th>r_b</th>
<th>H</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>$\sqrt{\frac{45b}{m}}$</td>
<td>$\sqrt{3}\sqrt{\frac{m}{w}}$</td>
<td>$h^* = -H = -\sqrt{3}\sqrt{\frac{m}{w}}$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4/3</td>
<td>$2\frac{3}{4} \sqrt{\frac{45b}{m}}$</td>
<td>$2\frac{3}{4} \sqrt{\frac{m}{w}}$</td>
<td>$h^* = \frac{H}{2} = -\frac{3}{4} \sqrt{\frac{m}{w}}$</td>
</tr>
<tr>
<td>3</td>
<td>1/2</td>
<td>4</td>
<td>$2\sqrt{\frac{45b}{m}}$</td>
<td>$\sqrt{3}\sqrt{\frac{m}{w}}$</td>
<td>$h^* = 0$</td>
</tr>
<tr>
<td>4</td>
<td>1/3</td>
<td>28/3</td>
<td>$2\frac{2}{3} \sqrt{\frac{45b}{m}}$</td>
<td>$\frac{144}{3} \sqrt{\frac{m}{w}}$</td>
<td>$h^* = \frac{3}{2} \sqrt{\frac{m}{w}}$</td>
</tr>
</tbody>
</table>

\[ l_{zz}^2 - 5l_{zz}l_{yy} + l_{yy}^2 = 0 \] (58)

There are two roots of \( l_{zz} = 4l_{yy} \) and \( l_{zz} = l_{yy} \) able to satisfy Eq. (58). After verification, \( l_{zz} = 4l_{yy} \) is the necessary condition of complete isotropy.

4.3. **Optimal design based on dynamic isotropy**

The above four types of dynamic isotropy conditions provide a solid foundation for a novel optimal design method based on dynamic isotropy measures. The iterative design procedures are listed as follows:

(a) For the payload’s center of mass criteria and other constraints to determine \( n = r_b/r_a \), see Eq. (45).

(b) According to the payload inertial parameters, follow the options listed below to determine \( r_b \).

(Option a): \( \min \{\lambda_1, \lambda_4\} \leq \lambda_{ISO,m} \leq \max \{\lambda_1, \lambda_4\} \) for rotation isotropy.

(Or Option b): \( \min \{\lambda_2, \lambda_3\} \leq \lambda_{ISO,v} \leq \max \{\lambda_2, \lambda_3\} \) for translation isotropy.

(Or Option c): Eq. (54) for combined isotropy.

Note that, for Option (a), since \( \lambda_1 \) and \( \lambda_4 \) are invariant and independent of geometry parameters, in order to achieve the best dynamic isotropy, the magnitude of \( \lambda_{ISO,m} \) must be limited in the range specified by \( \lambda_1 \) and \( \lambda_4 \). For Option (b), we can have similar results.

(c) According to the requirements of isotropy conditions, calculate \( H \).

(d) Taking account of other technical specifications such as workspace volume, manipulability, dexterity, singularity, accuracy, actuator interference, etc., evaluate this design. If some modifications are needed, return to step (a), then start another design iteration.

Next, we use an example to illustrate how to discover the architecture parameters based on the above design procedures.

We give the payload inertial parameters of the Delft SIMONA motion system (configuration C, data adopted from literature [22]) as follows:

\[ \rho_c = [0 \ 0 \ -0.45]^T \text{ (m)} \] (59a)
\[ m = 4300 \text{ (kg)} \] (59b)
\[ \mathbf{I}_a = \text{diag}(4100, 4000, 6700) \text{ (kg m}^2) \] (59c)

Now the design problem can be described as: Given the above inertial parameters of the payload, find a SGSP such that it is as close to dynamic isotropy as possible. For simplicity, it is assumed that \( x, \beta \) are zeros. From Eq. (59a), the center of mass should be above the top platform. To satisfy the given center of mass criteria, Eq. (45) tells us \( n \) should be less than 1/2, here we let \( n = 1/3 \). Then we choose the Option (c) i.e. the combined isotropy considering its application in flight simulation areas. According to the combined dynamic isotropy conditions, Eq. (54) determines the size of the platforms \( r_b = 3.3729 \text{ m as well as } r_a = 1.1243 \text{ m. Then we can calculate the initial height of the platform } H_{ISO} = 2.6309 \text{ m and the new compliance center with the value of } h^* = 0.1879 \text{ m.} \)

The dynamic isotropy index can be worked out:

\[ k = \sqrt{\frac{\max \{\lambda_{ISO-z}, \lambda_{ISO-xy}\}}{\min \{\lambda_{ISO-z}, \lambda_{ISO-xy}\}}} = 1.2508 \]

Note that \( h^* \) is not equal to \( \rho_c^2 (3) \), this implies that Eq. (59a) is not satisfied completely. Clearly, if the constraint of Eq. (59a) can be removed, we can make the center of mass coincide with the new compliance center. Otherwise we can only adjust \( x, \beta \) to meet the given center of mass criteria by sacrificing the dynamic isotropy index. Eq. (45) can be rewritten in the form:
Calculating Eq. (60), we obtain $\alpha + \beta = 12.3138^\circ$, and Eq. (59a) is satisfied. However, the dynamic isotropy index decreases slightly:

$$k = \sqrt{\frac{\max\{\lambda\}}{\min\{\lambda\}}} = 1.3554$$

From this numerical example, we can realize that the results obtained from this design method are very close to dynamic isotropy.

5. Conclusions

We propose a dynamic isotropy measure based on the inverse mass matrix in joint space for an optimal design of Generalized Gough–Stewart Platforms. Compared to the other dynamic isotropy measures in the previous work, such as generalized mass matrix in joint space, or natural frequency, this measure can be expressed easily in analytical form, as there is no need to calculate the inverse of the Jacobian.

An analytical formulation of the inverse mass matrix in joint space for Generalized Gough–Stewart platforms is derived. The joint space inverse mass matrix can be decomposed into the sum of the mass related matrix and the inertia related matrix, which explicitly reveals the coupling effects between all the actuators. With the formulation given, the influence of payload inertial parameters (mass, inertia, the center of mass and geometry parameters of the platform) to the joint space inverse mass matrix could be clearly identified.

Considering the SSGP, it is found that when the payload mass-inertia matrix is diagonal, the neutral pose joint space inverse generalized mass matrix is a Centro symmetric matrix, whose inverse matrix, Eigen values, Eigen vectors, and singular value decomposition are expressed as functions of the configuration parameters and the payload inertial parameters in analytical form.

The dynamic coupling effects between the DOFs at neutral pose are expressed by coupling matrix $C_T$ and its dynamic cross-coupling indices. Each row of $C_T$ describes the decoupled rigid body modal direction in physical task space.

From the above results, the conditions of rotation isotropy, translation isotropy, complete, and combined isotropy are also obtained. The relationship between mass-geometry characteristics (mass, inertia, the center of mass and geometrical configuration parameters of the SSGP) and dynamic isotropy is explicit. The compliance center is found and expressed in symbolic form. It is found that, once given one of the dynamic isotropy conditions, the dimension of the SSGP are completely dependent on the inertial parameters of the payload. The complete dynamic isotropy condition is given, thus we can see that it is possible to achieve complete dynamic isotropy by adjusting the payload inertia and the platform dimension, even though it was shown that it was impossible to obtain a spatially isotropic configuration for the SGSP using a static isotropy measure [16].

Finally, we present an optimal design method based on the dynamic isotropy conditions of this paper for the SGSP.

In future work we plan to make the most of reflecting and circular symmetrical features, and extend this method to more general symmetrical parallel manipulators to study their Eigen problems analytically.

Acknowledgments

This work is financially supported by the National Natural Science Foundation of China (Grant No. 50975055). The authors wish to thank the anonymous reviewers for their comments, which have helped us improve the paper.

References


