Resonances and Twist in Volume-Preserving Mappings

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March 3, 2010

Abstract

The phase space of an integrable volume-preserving map with one action is foliated by a one-parameter family of invariant tori. Perturbations lead to chaotic dynamics with interesting transport properties. We show that near a rank-one resonant torus the mapping can be reduced to a volume-preserving standard map. This map is a twist map only when the frequency map crosses the resonance curve transversely. We show that these maps can be reduced using averaging theory to the usual area-preserving twist or nontwist standard maps. In the volume-preserving setting the twist condition is shown to be distinct from the nondegeneracy condition of used in KAM theory.

1 Introduction

Volume preserving maps are appropriate models for many systems including fluid [CFP96, MGPM99, SVL01, RML+03, SCVH04, ATPM06, MJM08] and magnetic field line flows [TH85, Gre93, BDS98] and even the motion of comets perturbed by a planet on an elliptical orbit [LS94]. One fundamental question relevant for applications is to understand transport in these maps [PF88, RKKCA93, CFP94, LM09].

Transport is prohibited between any two regions that are separated by a codimension-one surface. For volume-preserving maps, these dividing surfaces are often tori. Even when there is no dividing surface, we may imagine, as in the two-dimensional case [MMP84], that remnants of such surfaces—if they exist—may impede transport. Similarly, transport may be reduced because of the “stickiness” of invariant tori, a phenomena well-established numerically

*JDM was supported in part by NSF grant DMS-0707659 and as a visiting research fellow to the University of Sydney.
for area-preserving maps \cite{MO86, CK08, Ven09} and for nearly integrable Hamiltonian systems \cite{P93, PW94} if not yet completely understood. For three-dimensional maps, stickiness has been observed as algebraic decay of exit time distributions for some systems \cite{MJM08}, not for others \cite{SZ08}.

Codimension-one invariant tori are a common feature of the dynamics of three-dimensional volume-preserving mappings. Indeed these maps could be regarded as “integrable” when their phase space is foliated by a family of two-dimensional tori on which the dynamics is conjugate to a rigid rotation. We can think of this case as a map with two angles and one action \cite{PF88}. KAM theory implies that two-tori are robust features of nearly-integrable, one-action maps \cite{CS90, Xia92}. By contrast, in maps with one angle and two actions (which might also be called as integrable), the invariant circles are apparently not robust \cite{Mez01}.

Tori also arise naturally through bifurcations. For example, a fixed point with multipliers \((e^{2\pi \omega}, e^{-2\pi \omega}, 1)\) can lead to a bifurcation that creates an invariant circle surrounded by a family of two-tori \cite{DM09}. Thus to understand transport in volume-preserving systems a first step is to understand the persistence and destruction of invariant tori.

Tori are often destroyed by resonant bifurcations, and this phenomena is the topic of the current paper. Our goal is to obtain several “standard” volume-preserving mappings that describe, as Chirikov’s area-preserving mapping does, the destruction of these invariant tori as the strength of the resonant terms grows. Particular examples that are similar to ours have also been studied in \cite{GR07}.

2 One-Action Maps

While our main interests are invariant two-dimensional tori in three-dimensions, it is easy to generalize to \(d\)-tori in \(d+1\) dimensions. Suppose that \(f : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}\) has a smooth family of homotopic, codimension-one invariant tori, and that the map on the tori is conjugate to a rigid rotation. We also suppose that the tori in some neighborhood are reducible \cite{JRRV97, Pui06}. Then letting \(x \in T^d\) represent the angle variables and \(z \in \mathbb{R}\) the transverse variable, the map in the neighborhood of a reducible torus has the form

\[
f_0(x, z) = (x + \Omega(z) \mod 1, z) .
\]

Such a map is called a one-action map: the variable \(z\) plays the role of the action, and the \(d\)-angles, \(x_i, i = 1, \ldots, d\), are taken to have period one \cite{CFP96, CFP99, Mez01}. The frequency mapping,

\[
\Omega : \mathbb{R} \to \mathbb{R}^d
\]

defines a curve in the frequency space, \(\omega = \Omega(z)\), as the “action” \(z\) varies, as sketched in Fig. 1.\(^1\)

If we fix a lift for the periodic component of the map \(f_0\) then \(\Omega(z^*)\) is the rotation number for orbits on the torus \(z = z^*\).\(^2\)

A rotation vector \(\omega \in \mathbb{R}^d\) is said to be resonant if there exist a nonzero \((m, n) \in \mathbb{Z}^d \times \mathbb{Z}\) such that

\[
m \cdot \omega = n ;
\]

\(^1\)Or more properly, using the terminology from flows, the “frequency ratio” map.

\(^2\)In fact, the usual freedom of choosing a lift such that \(-\frac{1}{2} < \Omega_i \leq \frac{1}{2}\) may be restricted by the global topology of the original map \(f\) in \(\mathbb{R}^{d+1}\). Suppose that the one-parameter family of tori is compact and embedded in \(\mathbb{R}^{d+1}\). Then there is a notion of outermost torus (from which infinity can be reached without intersecting other family members) and an innermost torus. Contracting the innermost torus onto \(T^{d-1}\) there is a unique contractable cycle. This cycle is unique because adding any other cycle makes it non-contractable. The rotation number associated with the unique cycle can be uniquely defined.
Figure 1: Resonance web, thickened by the Diophantine condition \( \text{(4)} \) and a frequency map \( \Omega \) for a one-action map on \( \mathbb{R} \times \mathbb{T}^2 \). The complement is the set positive measure Cantor set of Diophantine rotation vectors.

if there are no such integers, then \( \omega \) is incommensurate. The collection of the \( d-1 \) dimensional planes
\[
\mathcal{R} \equiv \left\{ \omega \in \mathbb{R}^d : m \cdot \omega = n, (m,n) \in \mathbb{Z}^{d+1} \setminus \{0\} \right\}
\]
is the resonance web; it is a dense subset of \( \mathbb{R}^d \).

Our goal is to understand the effects of perturbation and resonance on the tori of \( (1) \). As we will see, it is important to distinguish between resonances of different ranks. For a given \( \omega \), the resonance module is the sublattice of \( \mathbb{Z}^d \) that corresponds to all integer vectors \( m \) that satisfy \( (2) \), namely,
\[
\mathcal{L}(\omega) \equiv \{ m \in \mathbb{Z}^d : m \cdot \omega \in \mathbb{Z} \}
\]
When \( \omega \) is incommensurate \( \mathcal{L}(\omega) \) is trivial: it contains only the origin. When this set is \( r \)-dimensional, i.e., is the integer span of \( r \) independent vectors, then the resonance has rank \( r \). When \( \Omega(z^*) \) is \( r \)-resonant, the invariant \( d \)-torus \( z = z^* \) of \( (1) \) consists of families of orbits that lie on (collections of) \( (d-r) \)-tori, see App. \( A \).

The \( d \)-dimensional invariant tori of \( (1) \) are examples of rotational tori; generally, a rotational
torus is a set that is homotopic to the horizontal torus \( \{(x, 0) : x \in \mathbb{T}^d \} \). Upon perturbation, many of the rotational invariant tori of (1) will be destroyed. In particular, the resonant tori are fragile—we expect that they will be immediately destroyed and replaced by a finite number of lower dimensional tori, and perhaps families of nonrotational, \( d \)-dimensional tori. Conversely, KAM theory teaches us that a robust torus must be not only incommensurate, but sufficiently incommensurate—typically it must have a Diophantine frequency vector. The set of Diophantine frequency vectors is

\[
\mathcal{D}(c, \mu) = \{ \omega \in \mathbb{R}^d : |m \cdot \omega - n| \geq c|m|^{-\mu} \quad \forall (m, n) \in \mathbb{Z}^{d+1} \setminus \{0\} \}.
\]

Even though \( \mathcal{R} \) and \( \mathcal{D} \) are disjoint and \( \mathcal{R} \) is dense in \( \mathbb{R}^d \), \( \mathcal{D} \) is a positive measure Cantor set for each \( \mu > d \) and sufficiently small \( c > 0 \), see Fig. 1.

KAM theory can indeed be applied to volume-preserving perturbations of the one-action map (1); it implies, under certain smoothness and nondegeneracy assumptions, that there exists a set of Diophantine invariant \( d \)-tori (with \( \mu > d^2 + d - 1 \)) whose measure approaches one as the perturbation goes to zero [CS90, Xia92]. In these KAM-type theorems the nondegeneracy condition is that the Wronskian of \( \Omega \) must be bounded away from zero.

The curve \( \Omega(z) \) will typically intersect planes corresponding to \( 1 \)-resonances for a dense set of \( z \) values. In this paper we want to emphasize that the dynamical behavior of the perturbed map near these intersections depends in detail on the rank of the resonance and on whether the intersection is transverse or tangent. When the intersection with the resonance plane is transverse, we will show in §4 that the perturbed map can be approximated by what is called the area-preserving standard map that drives additional fast angles. By contrast, when the curve \( \Omega(z) \) is tangent to the resonance plane the map can be approximated by the area-preserving standard nontwist map that drives additional fast angles.

In standard KAM theory of symplectic maps violation of the twist-condition leads to meandering curves [HH84, Sim98, WAFM05] and twistless bifurcations [DMS00]. We will show that in the nonsymplectic volume preserving case the transversality conditions whose violation leads to the same dynamical phenomena is not the violation of the twist-condition of [CS90, Xia92], but instead the condition that the curve \( \Omega(z) \) be transverse to the resonance surface in frequency space. Our condition is weaker: when we have a tangent resonance then the KAM theorems still hold. This is an essential difference to the symplectic case and shows that these KAM theorems “know” little about the qualitative dynamics.

We start in §3 by studying the dynamics of an example of such a perturbation, a map \((x', z') = f_\varepsilon(x, z)\) given by

\[
\begin{align*}
x' &= x + \Omega(z') \mod 1, \\
z' &= z + \varepsilon g(x; \varepsilon),
\end{align*}
\]

with \( g(x + m; \varepsilon) = g(x; \varepsilon) \) for all \( x \in \mathbb{T}^d \) and \( m \in \mathbb{Z}^d \), and \( g(x; 0) = 0 \). This map is a natural generalization of the Chirikov standard map (a one-action, one-angle map) and its symplectic cousin, the Froeschlé map [Mei92] (a two-action, two-angle map). The map (5) is volume preserving with the standard volume form

\[
dz \wedge dx_1 \wedge \ldots \wedge dx_d; \tag{6}
\]

Rotational tori need not be graphs over the angles. More generally, the unique contractible cycle identified in footnote 2 defines a homotopy class of tori that can be called rotational.

Here \(|m|\) is any norm, for example the maximum norm.
it is not hard to see that the Jacobian, $Df_\varepsilon$, has determinant one. The map (5) has no rotational invariant tori unless the function $g$ has zero average:

$$\int_{\mathbb{T}^d} g(x; \varepsilon) dx_1 \ldots dx_d = 0.$$  

This is precisely the condition that $f_\varepsilon$ is an exact volume-preserving map, or equivalently that it has zero net flux, see App. B. Since we are interested the persistence and destruction of invariant tori, we will assume that the zero flux condition (7) holds.

Subsequently, in §4, we will show that the examples of §3 represent the general situation near a rank-one resonance.

### 3 Example: Transverse and Tangent Resonances

We begin by studying an example of a one-action, three-dimensional map of the form (5). We will show in §4 that this example captures the generic behavior near a rank-one resonance.

The frequency curve $\Omega(z)$ can be locally approximated by a parabola. As we will see more generally in §4, coordinates can be chosen near a rank-one resonance so that the frequency map is locally of the form

$$\Omega(z) = (\gamma + z, -\delta + \beta z^2).$$  

This parabola is sketched in Fig. 2. The twist condition of the KAM-type theories in this case is $\beta \neq 0$.

The forcing function $g(x)$ in (5), is periodic; therefore it can be expanded in a Fourier series. Since $g(x)$ has zero mean (7), its $(0,0)$ Fourier component must vanish. For simplicity we choose $g$ to have just three terms, each $O(\varepsilon)$, resulting in the “standard” map

$$x' = x + z' + \gamma,$$
$$y' = y - \delta + \beta z'^2,$$
$$z' = z + \varepsilon g(x, y),$$

with

$$g(x, y) = -a\sin(2\pi x) - b\sin(2\pi y) - c\sin(2\pi (x - y)).$$

The terms with amplitudes $a, b, c$ represent forcing at the resonances $(1,0,n)$, $(0,1,n)$, and $(1,-1,n)$, respectively. These forced resonances $m \cdot \Omega(z^*) = n$ occur at positions shown in Tbl. 1. This map has reversing symmetries since $g$ is an odd function, see App. C.

Generically, when $\Omega(z)$ is resonant, it crosses the resonance line $m \cdot \omega = n$ transversely. When $\varepsilon \ll 1$ the dynamics near a transverse crossing of a rank-one resonance can be analyzed by expanding the map near $z = z^*$ and performing an appropriate average over the nonresonant, fast angle.

For example, the $(1,0,0)$ resonance occurs when $\Omega_x(z_a) = \gamma + z_a = 0$. Generically the second frequency $\Omega_y(z_a) = \beta \gamma^2 - \delta \equiv \nu$ will be irrational at $z_a$ so that the resonance has rank one with the module

$$\mathcal{L}((0,\nu)) = \{(k,0) : k \in \mathbb{Z}\}, \quad \nu \notin \mathbb{Q}.$$

To expand near this resonance, define a new variable $\zeta$ by $z = z_a + \sqrt{\varepsilon}\zeta$. When $\varepsilon \ll 1$, (9) then reduces to

$$x' = x + \sqrt{\varepsilon}\zeta',$$
$$y' = y + \nu + O(\sqrt{\varepsilon}),$$
$$\zeta' = \zeta + \sqrt{\varepsilon}g(x, y).$$
Figure 2: Resonance lines $m \cdot \omega = n$ for $|m| \leq 3$, and the image of the frequency map (8) for $\beta = 2$, $\gamma = \sqrt{5} - 2$, and two values of $\delta$ that give rise to tangencies of $\Omega$ with the forced resonances of (10).

Since, by assumption $\nu \notin \mathbb{Q}$, the angle $y$ is rapidly rotating relative to $x$ and is, to lowest order, nonresonant. Thus the forcing terms in (10) proportional to $b$ and $c$ rapidly fluctuate relative to the term proportional to $a$, and by the averaging theory given in App. D the dynamics on an intermediate time scale $t \sim O(\varepsilon^{-1})$ is governed by the two-dimensional averaged map

$$x' = x + \sqrt{\varepsilon} \zeta', \quad \zeta' = \zeta - \sqrt{\varepsilon} a \sin(2\pi x).$$

Note that this map is symplectic. Indeed, when $\varepsilon \ll 1$, it is approximately the time $\sqrt{\varepsilon}$ map of the one degree-of-freedom pendulum Hamiltonian

$$H(x, \zeta; M, a) = \frac{\zeta^2}{2M} - \frac{a}{2\pi} \cos(2\pi x),$$

with momentum $\zeta$, coordinate $x$, an effective “mass” $M = M_a = 1$ and amplitude $a$.

Consequently, in a $\sqrt{\varepsilon}$ neighborhood of $z = -\gamma$, the orbits lie on contours of constant $H$ to $O(\varepsilon)$ for times $O(\varepsilon^{-1})$, providing $x$ does not cross a resonance. The pendulum’s librating orbits correspond to tubes of the three-dimensional map (9) that are aligned with the $y$-axis; two such
orbits are shown in Fig. 3 (the red and purple orbits). The rotating orbits of (11) correspond to rotational invariant tori of (9). The full width of this resonance in $z$ is the width in of the separatrix contour of $H$, appropriately scaled in $\varepsilon$,

$$W(M, a) = \sqrt{\frac{8\pi}{\pi} |Ma|}.$$  

(12)

When $\varepsilon a = 0.005$ as in Fig. 3, this gives $w_a = W(1, a) \approx 0.11$.

![Figure 3: Phase space of (9) for $\beta = 2$, $\gamma = \frac{1}{4}(\sqrt{5} - 1) \approx 0.618$, $\delta = 2\delta_R \approx 0.0284$, $a = b = c = 1$, and $\varepsilon = 0.005$. Orbits trapped in four resonances are shown. The three blue-toned orbits correspond to the two $(0, 1, 0)$ resonances at $z_b = \pm 0.119$. The green orbit is trapped in the $(1, -1, 0)$ resonance at $z_c = -0.371$, and the two red-toned orbits librate in the $(1, 0, 0)$ resonance at $z_a = -\gamma = -0.618$. Also shown are three rotational tori and a chaotic orbit (grey).]

A similar analysis can be done for the $(0, 1, 0)$ resonance. If $\delta \beta > 0$, the frequency map (8) crosses this resonance at two positions, $z_b^\pm$, given in Tbl. 1. This resonance has rank one when $\Omega_x(z_b) = \gamma + z_b \notin \mathbb{Q}$, in which case the same expansion procedure as before can be applied; however, the fast angle is now $x$. The averaged Hamiltonian is again (11) with parameters $H(y, \zeta; M_b, b)$. The effective mass $M_b$, given in Tbl. 1, is real precisely when the resonance crossing is transverse: $\delta \beta > 0$. In this case the resonance width is $w(M_b, b)$ with the same function (12). Since the effective masses of the $z_b^\pm$ resonances have opposite signs, the resonance with $M_b > 0$ is an island centered at $y = 0$, while that with $M_b < 0$ is phase shifted, and is centered at $y = \frac{1}{2}$. The three blue-toned orbits shown in Fig. 3 are trapped in these resonances. For the parameters of this figure the resonances are centered at $z_b^\pm = \pm 0.119$ and have widths $w_b \approx 0.16$. Since the separation between the resonances exceeds the sum of their half-widths, the Chirikov overlap criterion leads to the expectation that there are rotational invariant tori.
between the two resonances. Indeed, this is what we observe numerically; one such separating torus (in grey) is shown in Fig. 3.

\[
\begin{array}{cccc}
(m, n) & \text{Amplitude} & z^* & M & \delta_T \\
(1, 0, n) & a & n - \gamma & 1 & \text{none} \\
(0, 1, n) & b & \pm \sqrt{\frac{\delta + n}{\beta}} & \frac{1}{2\beta z^*} & -n \\
(1, -1, n) & c & \frac{1}{2\beta} \left[ 1 \pm \sqrt{1 + 4\beta(\delta + \gamma - n)} \right] & \frac{1}{1 - 2\beta z^*} & n - \frac{1}{4\beta} - \gamma \\
\end{array}
\]

Table 1: Forced resonances. \( m \cdot \Omega(z^*) = n \) for (9), their effective masses \( M \) and values of \( \delta \) for a tangency.

The frequency curve (8) also crosses the \((1, -1, 0)\) resonance transversely at the two points \( z = z_c^\pm \) given in Tbl. 1 provided that \( \beta(\delta + \gamma) > -\frac{1}{4} \). This resonance can be treated as before upon defining new angle variables

\[
(\xi, \eta) = (x - y, y),
\]

aligned with the resonance. Letting \( z = z_c + \sqrt{\varepsilon} \zeta \), the map (9) becomes

\[
\begin{align*}
\xi' &= \xi + \sqrt{\varepsilon} \frac{\zeta'}{M_c}, \\
\eta' &= \eta + \eta + \mathcal{O}(\sqrt{\varepsilon}), \\
\zeta' &= \zeta + \sqrt{\varepsilon} g(\xi + \eta, \eta),
\end{align*}
\]

with \( \eta = \gamma + z_c \) and \( M_c \) given in Tbl. 1. The resonance has rank one when \( \eta \notin \mathbb{Q} \), and averaging over the fast angle \( \eta \) now eliminates the terms proportional to \( a \) and \( b \) in \( g \). The resulting system is again approximated by the time \( \sqrt{\varepsilon} \) map of a pendulum Hamiltonian with parameters \( H(\eta, \zeta; M_c, c) \). In Fig. 3 only the resonance at \( z_c^+ \approx -0.371 \) is shown; it has a width \( w_c^+ \approx 0.072 \).

In the phase portrait Fig. 3 the resonances are all separated by rotational invariant tori. Indeed the Chirikov overlap parameters between neighboring, forced resonances,

\[
\begin{align*}
\sigma_{b+b^-} &= \frac{w^+_b + w^-_b}{2|z^+_b - z^-_b|} \approx 0.66, \\
\sigma_{b^-c^-} &= \frac{w^-_b + w^-_c}{2|z^-_b - z^-_c|} \approx 0.47, \\
\sigma_{c^-a} &= \frac{w^-_c + w^-_a}{2|z^-_c - z^-_a|} \approx 0.37,
\end{align*}
\]

are less than the phenomenological threshold value \( s = \frac{2}{3} \) for “global chaos” [LL92, Mei07], so we expect—and observe—that rotational invariant tori separate these resonances. There are many chaotic trajectories as well for the parameters of Fig. 3, only one of which is shown in the figure. As an example, the \((1, 0, 0)\) resonance is surrounded by a large chaotic layer (grey in the figure) because of overlap with a \((1, -1, -1)\) resonance at \( z^* = -0.691 \) that is not shown in the figure. This resonance and the \((1, 0, 0)\) resonance have an overlap parameter \( s = 1.2 \), and are not separated by any rotational tori.

As \( \delta \) is varied the resonance positions, \( z^* \) and their corresponding widths change as shown in Fig. 4. The averaged Hamiltonian description discussed above is valid near each transversal
crossing provided that the resonance widths are small enough to avoid overlap. Of course, even in this case the region near the separatrix of the averaged Hamiltonian is replaced by a chaotic zone for the full map. As we will see in §4, a similar structure governs the dynamics near any rank-one resonance whenever $\Omega(z)$ crosses the resonance transversely.

Figure 4: Resonance centers $z^*$ (curves) and widths (grey regions) as a function of $\delta$, for the parameters of Fig. 3.

However, when the resonance intersection is nontransversal, the dynamics is radically different. The curves $z^*(\delta)$ for the $(1, 0, 0)$ and $(1, -1, 0)$ resonances in Fig. 4 are parabolas with turning points, $z^*(\delta_T) = z_T$ that signal nontransversal crossings. Tangency with an $(m, n)$ resonance occurs for $z = z_T$ when

$$m \cdot \Omega(z_T) = n \quad \text{and} \quad m \cdot D\Omega(z_T) = 0 . \quad (14)$$

Since a frequency curve $\Omega(z)$ will generically intersect a resonance line for some $z^*$, tangency is a codimension-one phenomena—it will generically occur if the frequency map $\Omega$ depends upon a single parameter. For example, for the frequency map (8) we can think of $\delta$ as the parameter that unfolds these tangencies. Though the $(1, 0, n)$ resonance is never tangent, tangencies do occur for both the $(0, 1, n)$ and $(1, -1, n)$ resonances at the values where their effective masses diverge. The values of $\delta$ for these tangencies are given in Tbl. 1.
The frequency curve is tangent to the \((0, 1, 0)\) resonance when \(\delta = \delta_T = 0\) at \(z = z_T = 0\). If we assume that \(\Omega_x(z_T) = \gamma \notin \mathbb{Q}\), then the angle \(x\) rapidly rotates, and for \(z \approx z_T\) and \(\delta \approx \delta_T\), the map \((9)\) is approximated by the averaged map

\[
\begin{align*}
y' &= y - \delta + \beta z'^2, \\
z' &= z - \epsilon b \sin(2\pi y).
\end{align*}
\]

Again the averaged system is a two-dimensional, symplectic map. This system should approximate the original dynamics when \(z\) and \(\delta\) are near \(z_T = \delta_T = 0\). When \(\beta = O(1)\), a maximal balance of the terms in this map occurs when

\[
\begin{align*}
z &= z_T + \epsilon^{\frac{3}{2}} \zeta, \\
\delta &= \delta_T + \epsilon^{\frac{3}{2}} \Delta.
\end{align*}
\]

In this case, \((15)\) is approximately the time \(\epsilon^2\) map of the Hamiltonian

\[
H(y, \zeta; \beta, \Delta, b) = \frac{\beta}{3} \zeta^3 - \Delta \zeta - \frac{b}{2\pi} \cos(2\pi y).
\]

When \(\Delta \beta > 0\) this Hamiltonian has four equilibria, at \(y = 0, \frac{1}{2}\) and \(\zeta = \pm \frac{1}{2}\sqrt{3} \beta\). When \(\beta b > 0\) the two equilibria \((0, -\sqrt{\frac{3}{2}} \pi)\) and \((\frac{1}{2}, \sqrt{\frac{3}{2}} \pi)\) are saddles, and the remaining two are centers, see Fig. 5. For \(|\Delta| > |\Delta_R|\) where

\[
\Delta_R^3 \equiv \frac{9}{16\pi^2} \beta b^2,
\]

there are two independent island chains. The separatrices of these islands correspond to the energies \(E_{sx} = \pm \frac{2}{3\sqrt{3}}(\Delta_R^2 - \Delta^2)\). These coincide at \(\Delta = \Delta_R\), where there is a “reconnection bifurcation”, see Fig. 5. Note that reconnection occurs when \(\text{sgn}(\Delta_R) = \text{sgn}(\beta)\) at an unscaled value \(\delta_R = \delta_T + \epsilon^{\frac{3}{2}} \Delta_R\). The elliptic and saddle equilibria on the lines \(y = 0\) and \(y = \frac{1}{2}\) continue to move together as \(\Delta\) decreases until they are destroyed in saddle-center bifurcations at \(\Delta = 0\).

This structure can be most easily seen in the three-dimensional map by using slices. A slice is an approximate, two-dimensional phase portrait obtained by plotting orbits only when they land within a thin slab. In Fig. 6 we show two such slices, \(|y| < 0.001\) on the left and \(|x| < 0.001\) on the right—thus on average to obtain one point in either slice the map must be iterated \(O(500)\) times. These slices show the same orbits. The fast dynamics of the \((0, 1, n)\) and \((1, -1, n)\) resonances are transverse to the left slice, and so the islands that the slow dynamics generates can be seen. In particular the two \((0, 1, 0)\) resonances at \(z_b^\pm\). Since \(\delta_T = 0\), and \(\delta = 2\delta_R\) in this figure, the phase portraits of these resonances are well described by the upper portrait of Fig. 5. There are also three \((1, -1, n)\) resonances indicated in the figure with their widths computed from \((12)\). The \((1, 0, n)\) resonances do not appear in this slice since their librating orbits move approximately parallel to the slice; however, the chaotic orbit above the \((1, -1, 1)\) resonance is due to its overlap with \((1, 0, 0)\). Two \((1, 0, n)\) island do appear in the right slice; the diagonal \((1, -1, n)\) resonances also appear again.

Similar pairs of slices of the three-dimensional map are shown in Fig. 7 and Fig. 8 for \(\delta = \delta_R\) and \(\frac{1}{2} \delta_R\), respectively. Again, the averaged Hamiltonian portraits from Fig. 5 are closely mimicked near the \((0, 1, 0)\) resonance in these figures except that the separatrices become chaotic. In particular, note that the invariant tori trapped between the \(z_b^\pm\) resonances in Fig. 8 are “meandering”—they are not graphs over the unperturbed angles.
Figure 5: Contours of the Hamiltonian (17) for three values of $\Delta$ when $\beta b > 0$.

Tangency similarly occurs for the $(1, -1, 0)$ resonance when $\delta = \delta_T$ as given in Tbl. 1 when $z = z_T = \frac{1}{\Delta_R}$, where the effective mass $M_c$ is infinite. As before, we must use the aligned angles $(\xi, \eta)$ of (13) to construct the averaged map, and scale $z$ and $\delta$ as in (16). The result is exactly the Hamiltonian (17) with parameters $H(\xi, \zeta, -\beta, -\Delta, c)$. Thus the same reconnection scenario applies to this resonance near $4\beta(\delta + \gamma) = -1$. A three-dimensional phase portrait near the tangency is shown in Fig. 9. For these parameters, $\delta_T \approx -0.743$, and $\delta_R \approx -0.729$. The figure corresponds to $\Delta = 0.21\Delta_R$. The chaotic layer around the $(1, -1, 0)$ resonances overlaps with that around the $(1, 0, 1)$ resonance, as indicated in Fig. 4.
Figure 6: Two dimensional slices $|x| < 0.001$ (left pane) and $|y| < 0.001$ of the map (9) for the parameters of Fig. 3. The arrows indicate the widths computed using (12). The $x$-slice of the $(0, 1, 0)$ resonances resembles the top pane of Fig. 5.

Figure 7: Two dimensional slices of the map (9) with the parameters of Fig. 6, except $\delta = \delta_R \approx 0.0142$. The $x$-slice of the $(0, 1, 0)$ resonances resembles the middle pane of Fig. 5.
Figure 8: Two dimensional slices of the map (9) with the parameters of Fig. 6, except $\delta = \frac{1}{2} \Delta R \approx 0.0071$. The $x$-slice of the $(0, 1, 0)$ resonances resembles the bottom pane of Fig. 5.

Figure 9: Orbits of the map (9) for $\beta = 2$, $\gamma = \frac{1}{2}(\sqrt{5} - 1)$, $\delta = -0.74$, $\varepsilon = 0.001$, $a = 4$ and $b = c = 1$. Shown are orbits in the two, nearly tangent $(1, -1, 0)$ resonances (shades of red and light blue) and in the nearby $(1, 0, 1)$ resonance (green and dark blue). A chaotic orbit (grey) wanders among all three resonances and an invariant torus (yellow) below bounds the trajectories.
4 The General Rank-One Resonance

In this section we consider more generally the case of a rank-one resonance of a volume-preserving map that is an \( \varepsilon \ll 1 \) perturbation \( f_\varepsilon : \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R} \) of the integrable map (1):

\[
\begin{align*}
    x' &= x + \Omega(z') + \varepsilon X(x, z; \varepsilon), \\
    z' &= z + \varepsilon Z(x, z; \varepsilon),
\end{align*}
\]

(19)

where \( X \) and \( Z \) are smooth functions. We suppose that this map is exact, volume-preserving with respect to the form \( zdx_1 \wedge \ldots \wedge dx_d \), see App. B. We will show that (19) reduces essentially to (9) near a rank-one resonance at \( z = z^* \).

Suppose that frequency vector \( \Omega^* = \Omega(z^*) \) obeys a resonance condition

\[
m \cdot \Omega^* = n
\]

for exactly one coprime vector \( (m, n) \in \mathbb{Z}^d \times \mathbb{Z} \) so that the resonance module \( [4] \) is one-dimensional.

As we discussed in §3 the frequency curve will generically intersect a resonant plane transversely. However, under variation of a parameter, the intersection may become tangent. These two cases can be treated simultaneously by expanding about the resonant torus, setting \( z = z^* + \mathcal{O}(\varepsilon^p) \), and choosing the exponent \( p \) to be \( \frac{1}{2} \) or \( \frac{1}{3} \), respectively. However, to unfold the codimension-one tangency, it is convenient to expand about a torus slightly shifted from the resonant one. Thus instead of expanding about \( z^* \), we expand about a shifted point \( \bar{z} \),

\[
    z = \bar{z} + \varepsilon^p \zeta, \quad 0 < p \leq \frac{1}{2},
\]

(20)

and make the assumption that \( \bar{z} = z^* + \mathcal{O}(\varepsilon^{2p}) \) to define the frequency shift \( \delta \) by

\[
m \cdot \Omega(\bar{z}) = n - \varepsilon^{2p} \delta.
\]

(21)

Substituting this expansion into (19) gives

\[
\begin{align*}
    x' &= x + \Omega(\bar{z}) + \varepsilon^p D\Omega(z^*) \zeta' + \frac{1}{2} \varepsilon^{2p} D^2 \Omega(z^*) \zeta'^2 + \mathcal{O}(\varepsilon, \varepsilon^3), \\
    \zeta' &= \zeta + \varepsilon^{1-p} Z(x, z^*, 0) + \mathcal{O}(\varepsilon).
\end{align*}
\]

(22)

Though the components of the vector \( (m, n) \) are coprime by assumption, the components of \( m \) need not be coprime, see App. A. Let \( k = \gcd(m) \in \mathbb{N} \) be their greatest common divisor so that \( \hat{m} = m/k \) is a coprime, integer vector. As discussed in App. A, there exists a basis for the lattice \( \mathbb{Z}^d \) with \( \hat{m} \) as one basis vector; equivalently, there exists a matrix \( M \in SL(d, \mathbb{Z}) \) whose first row is \( \hat{m}^T \). Thus we can define new coordinates \( (\xi, \eta) \in \mathbb{T} \times \mathbb{T}^{d-1} \) as

\[
\begin{pmatrix}
    \xi \\
    \eta
\end{pmatrix}
\equiv
\begin{pmatrix}
    Mx \\
    \hat{m}^T x
\end{pmatrix},
\]

so that the first angle \( \xi \) is aligned with the resonance. Since \( M \) is a one-to-one transformation of \( \mathbb{Z}^d \), the perturbation \( Z \) can be written in terms of the new angles as

\[
g(\xi, \eta) \equiv Z(M^{-1}(\xi, \eta)^T, z^*, 0) = \sum_{j \in \mathbb{Z}^d} \tilde{Z}_j e^{2\pi i (\xi, \eta)^T M^{-T} j} = \sum_{l \in \mathbb{Z}^d} \tilde{g}_l e^{2\pi i l^T (\xi, \eta)}
\]

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where \( l \in \mathbb{Z}^d \), and \( g \) has the Fourier coefficients \( \tilde{g}_l = \hat{Z}_{M^T l} \). Since the map (19) is exact, then by (7) the 0 Fourier component of \( Z \) vanishes, implying that \( \tilde{g}_0 = \hat{Z}_0 = 0 \) as well.

Since \( \bar{z} = z^* + O(\epsilon^2 p) \),

\[
M\hat{\Omega}(z) = \left( \frac{\eta}{\hat{\Omega}^*} \right) - \frac{1}{k} \epsilon^2 p \delta \left( O(1) \right),
\]

where \( \hat{\Omega}^* = \hat{M} \hat{\Omega}^* \). In the new coordinates the map (22) becomes

\[
\begin{align*}
\xi' &= \xi + \frac{1}{k} \left( n + \epsilon^p \alpha \zeta' + \epsilon^{2p} (\beta \zeta'^2 - \delta) \right) + O(\epsilon, \epsilon^3 p) \\
\eta' &= \eta + \hat{\Omega}^* + O(\epsilon^p) \\
\zeta' &= \zeta + \epsilon^{1-p} g(\xi, \eta) + O(\epsilon)
\end{align*}
\]

where

\[
\alpha \equiv m \cdot D\hat{\Omega}(z^*) , \quad \beta \equiv \frac{1}{2} m \cdot D^2 \Omega(z^*).
\]

Since \( \hat{\Omega}^* \) satisfies only a single resonance condition and the columns of \( \hat{M}^T \) are independent of \( \hat{m} \), whenever \( l \in \mathbb{Z}^{d-1} \) is nonzero \( l \cdot \hat{\Omega}^* = l \cdot \hat{M} \hat{\Omega}^* = (\hat{M}^T l) \cdot \hat{\Omega}^* \notin \mathbb{Z} \). In particular none of the components of this transformed frequency can be rational: the angles \( \eta \) are “rapidly varying.” First order averaging theory implies that to lowest order—over a finite number of iterates \( O(\epsilon^{p-1}) \gg 1 \)—these terms can be averaged away (see App. D) reducing the system to the two-dimensional map

\[
\begin{align*}
\xi_k &= \xi_0 + \epsilon^p \alpha \zeta_k + \epsilon^{2p} (\beta \zeta_k^2 - \delta) + O(\epsilon, \epsilon^3 p) \\
\zeta_k &= \zeta_0 + \epsilon^{1-p} \bar{g}(\xi_0) + O(\epsilon)
\end{align*}
\]

where

\[
\bar{g}(\xi) = \sum_{l=0}^{k-1} \int \int_{\mathbb{T}^{d-1}} g(\xi + j \frac{n}{k}, \eta) d\eta = k \sum_{l \in \mathbb{Z}} \tilde{g}_{kl,0} e^{2\pi i kl \xi}
\]

contains only the Fourier coefficients of \( g \) that are multiples of \((k,0)\). Thus

\[
\bar{g} \left( \xi + \frac{1}{k} \right) = \bar{g}(\xi). \]

There are two distinguished limits corresponding to transversality or tangency of the frequency curve with the resonant plane; these can be treated by selecting the exponent \( p \). If \( \alpha \neq 0 \), the frequency curve \( \hat{\Omega} \) crosses the resonant plane \( m \cdot \hat{\Omega} = n \) transversely (since the vector \( m \) is perpendicular to the resonant plane, and \( D\hat{\Omega} \) is tangent to the frequency curve). In this case we set \( p = \frac{1}{2} \) and the averaged map (25) is approximately the time \( \sqrt{\epsilon} \) flow map for the Hamiltonian

\[
H(\xi, \zeta) = \frac{\alpha}{2} \epsilon^2 + V(\xi).
\]
with \( DV(\xi) = -\hat{g}(\xi) \). This is the standard pendulum system and gives a resonance of size \( \sqrt{\varepsilon} \) about \( z^* \) in the original coordinates, as discussed in \[3\].

On the other hand, if \( \alpha = 0 \), the frequency curve is tangent to the resonance plane. If \( \beta \neq 0 \), then its curvature at the tangency is nonzero. If we set \( p = \frac{1}{3} \), then the map (25) limits on the time \( \varepsilon^{\frac{2}{3}} \) flow map for the Hamiltonian

\[
H(\xi, \zeta) = \frac{\beta}{3} \zeta^3 - \delta \zeta + V(\xi) .
\]

This is the standard Hamiltonian that describes vanishing twist for an area-preserving map.

We assumed that the perturbed map (19) is exact volume preserving; however, in the construction we did not use this property. Thus the same result holds with the weaker assumptions that the unperturbed map is volume preserving and that the average of \( Z(x, z; \varepsilon) \) over the angles is zero. However, for longer time scales nonvolume preserving effects will become important, and on these time scales all invariant tori may be destroyed. Consequently, the qualitative picture conveyed by the reduced Hamiltonians is only good for the case of exact volume-preserving perturbations.

In conclusion, the general structure of the dynamics near a rank-one resonance has been reduced to the normal forms that we studied in \[3\].

5 Double Resonance

For the one-action map (1) in three dimensions, the image of the frequency map is a two-dimensional frequency vector \( \omega \). If its resonance has rank two, then \( \omega \) must be rational, say,

\[
\omega = \left( \frac{l_1}{d_1}, \frac{l_2}{d_2} \right).
\]

Note that orbits of the integrable torus map (29) with this rational frequency are periodic with period \( T = \text{lcm}(d_1, d_2) \). In this case, the near integrable map (19) has two slow angles, and as we will see, the map cannot be reduced in dimension by averaging.

To obtain the normal form, we first address the problem of finding an appropriate basis for the module \( \mathcal{L}(\omega) \). Note that, without loss of generality, we can assume that \( \gcd(l_i, d_i) = 1 \); however, the denominators \( d_1 \) and \( d_2 \) need not be coprime. When \( \gcd(d_1, d_2) = 1 \) the only way to achieve \( m \cdot \omega \in \mathbb{Z} \) is to cancel the denominators \( d_i \), implying that the resonance module has a basis \( m_1 = (d_1, 0) \) and \( m_2 = (0, d_2) \). By contrast, when

\[
g \equiv \gcd(d_1, d_2) \neq 1
\]

the problem of finding a basis for \( \mathcal{L}(\omega) \) is more interesting. One basis vector is easy, either \( (d_1, 0) \) or \( (0, d_2) \) suffices. However, together these two vectors do not form a basis, since there is additional cancellation. If for concreteness we choose \( m_1 = (d_1, 0) \), then upon writing \( d_i = g\hat{d}_i \), the equation to solve for the second vector becomes

\[
gm_2 \cdot \omega = \frac{l_1}{\hat{d}_1} x + \frac{l_2}{\hat{d}_2} y = gn , \quad x, y, n \in \mathbb{Z} .
\]

This implies that \( x = \hat{d}_1 i \) and \( y = \hat{d}_2 j \), so we arrive at

\[
l_1 i + l_2 j = gn , \quad i, j \in \mathbb{Z} ,
\]

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where by construction \( \gcd(l, g) = 1 \). To make \( m_2 \) independent of \( m_1 \), we must choose \( j \neq 0 \).

We can always find a solution with \( j = 1 \), since the Diophantine equation

\[
 gn - l_1i = l_2
\]

is always solvable for integers \( i \) and \( n \) when \( \gcd(g, l_1) = 1 \). Thus a second basis vector is \( m_2 = (i \hat{d}_1, \hat{d}_2) \). Note that the determinant of the matrix of basis vectors is \( d_1 \hat{d}_2 = \gcd(d_1, d_2) = T \), which is the period of orbits with rotation vector \( \omega \).

**Example.** The module for \( \omega = (\frac{1}{3}, \frac{1}{2}) \) has \( g = 1 \), so we can take \( m_1 = (3, 0) \) and \( m_2 = (0, 2) \). The module is the set

\[
 \mathcal{L} = \{(3i, 2j), (i, j) \in \mathbb{Z}^2\}
\]

Of course any unimodular transformation of \( m_1 \) and \( m_2 \) also gives a basis; however, the selected basis is minimal in the sense that no other basis has smaller 1-norm.

**Example.** The module for \( \omega = (\frac{1}{2}, \frac{1}{4}) \) has \( T = 4 \) and \( g = 2 \). In this case the choice \( m_2 = (0, 4) \) with the same \( m_1 \) misses some lattice points, for example \((1, 2)\). The process outlined above gives \( m_1 = (2, 0) \) and \( m_2 = (1, 2) \), which is the minimal basis in the 1-norm.

**Example.** If \( \omega = (\frac{1}{12}, \frac{1}{3}) \) then \( T = 36 \) and \( g = 3 \). If \( m_1 = (12, 0) \), we must solve the equation \( i + j = 3n \). For \( j = 1 \) a solution is \( i = 2 \), so we have a second vector \( m_2 = (8, 3) \). The minimal basis is \((4, -3), (8, 3)\).

Once a basis for the module is known, the analysis of App. A can be used to choose a basis on the torus for which the angle dynamics is diagonal. In particular, we can diagonalize the angle dynamics using the transformation (30), \( M = PKQ \) where \( P \) and \( Q \) are unimodular matrices, and, since we have only two angles in this case, \( K = \text{diag}(k_1, k_2) \). Letting \( \xi = Qx \), then reduces the unperturbed angle dynamics on the resonant torus, \( \Omega(z^*) = \Omega^* \) of (19) to

\[
 \xi' = \xi + Q\Omega^* = \xi + \tilde{\omega}
\]

where \( \tilde{\omega} = k_2 \). Note that this map as period \( T = k_2 \), since, as discussed in App. A, \( k_1 \) divides \( k_2 \).

We now follow the expansions done in §4: expand (19) about the integrable torus using (20) with \( p = \frac{1}{2} \), set \( \delta = 0 \), and iterate the resulting system \( T \) times. This gives

\[
 \begin{align*}
 \xi' &= \xi + \sqrt{\varepsilon}\alpha\zeta' \\
 \zeta' &= \xi + \sqrt{\varepsilon}\tilde{g}(\xi) + O(\varepsilon)
\end{align*}
\]

where \( \tilde{g}(\xi) \) is obtained by iterating \( T \) times along the unperturbed torus map

\[
 \tilde{g}(\xi) = \sum_{j=0}^{T-1} g(\xi + j\tilde{\omega})
\]

with \( g(\xi) = Z(Q^{-1}\xi, z^*, 0) \). Here we assume that the twist vector, \( \alpha = TQD\Omega(z^*) \), is nonzero. This map is a near identity map that is, to lowest order, the time \( \sqrt{\varepsilon} \) map of the flow of the incompressible vector field

\[
 \begin{align*}
 \dot{\zeta} &= \tilde{g}(\xi) \\
 \dot{\xi} &= \alpha\zeta
\end{align*}
\]

This is the “quasi-periodic pendulum” studied in [DIM06].
6 Conclusion

We have shown that analogues of the standard twist and standard nontwist maps naturally arise near resonance for one-action maps upon perturbation. These two cases were treated in essentially the same way by allowing for different scalings in the derivation. For the volume-preserving case, the dynamics of these standard maps is surprisingly similar to that of ordinary area-preserving standard maps. In particular we have shown, using first-order averaging theory, that the dynamics can be approximately reduced to those of area-preserving standard maps, \cite{25}. These maps have twist when the image of frequency ratio map $\Omega(z)$ crosses a resonance plane $m \cdot \omega = n$ transversely, but become nontwist maps when there is a tangency, i.e., when $m \cdot D\Omega(z) = 0$ at resonance. In particular this implies that Chirikov’s resonance overlap criterion can be used to estimate parameters at which invariant tori breakup near a transverse resonance crossing.

Recall that in the symplectic setting, vanishing twist is related to local non-invertibility of the frequency ratio map \cite{DIM06}. Such singularities could also be considered for the volume-preserving case. These singularities could be global—when the frequency ratio map is not one-to-one, or local—when the map is not smooth at some point. An example is the development of a cusp singularity, modeled for example by the unfolded cusp normal form:

$$\Omega(z) = (\nu + z^2, \delta z + z^3)$$

We plan to study such cusp singularities in a future paper.

The results outlined above hold for the standard maps obtained from expansion near a rank-one resonance. By contrast, a very different reduced map \cite{28} is found near a rank-two (or double) resonance. Again, it is a variation of the standard map, but here a standard map with a quasi-periodic potential with two frequencies is found.
A Resonant Integrable Torus Maps

Consider the map
\[ \theta' = \theta + \omega \pmod{1} \quad (29) \]
on \mathbb{T}^d. Suppose that \( \omega \) as a rank-\( r \) resonance, that is the resonance module (3) for \( \omega \in \mathbb{T}^d \) has dimension \( r = \dim \mathcal{L}(\omega) \), where \( 0 \leq r \leq d \). Let \( m_i \in \mathbb{Z}^d, i = 1\ldots r \) be a basis for \( \mathcal{L}(\omega) \), that is, a set of independent vectors so that
\[ \mathcal{L}(\omega) = \text{span}_\mathbb{Z}(m_1, \ldots, m_r) \]
The module \( \mathcal{L} \) is a sublattice of \( \mathbb{Z}^d \). It is said to be primitive if
\[ \mathcal{L} = \text{span}_\mathbb{R}(m_1, \ldots, m_r) \cap \mathbb{Z}^d \]
that is, every integer point in the plane spanned by the vectors \( m_i \) is a point in \( \mathcal{L} \).

Let \( M \) be the \( r \times d \) dimensional matrix
\[
M = \begin{pmatrix}
m_1^T \\
m_2^T \\
\vdots \\
m_r^T 
\end{pmatrix}
\]
By assumption \( \text{rank}(M) = r \).

Following Lochak and Meunier [LM88] and the standard normal form construction for matrices [MB93, Chap. XI.7], there exist unimodular matrices \( P \) and \( Q \) such that
\[ M = PKQ \quad (30) \]
where \( K \) is the \( r \times d \) diagonal matrix
\[ K = \text{diag}(k_1, \ldots, k_r) \]
with \( k_i \in \mathbb{N} \) and \( k_i \) divides \( k_j \) for \( i < j \). The integers \( k_i \) are the invariants of \( \mathcal{L} \). They determine when the lattice is primitive:

**Lemma 1.** A module is primitive only if all its invariants are one, \( k_i = 1 \).

Defining the \( r \)-dimensional vector \( n \) by \( n_i = \omega \cdot m_i, i = 1, \ldots, r \) then
\[ n = M\omega = PKQ\omega \]
If we define \( \hat{\omega} = Q\omega \), then we have
\[ K\hat{\omega} = P^{-1}n = \hat{n} \]
thus \( \hat{\omega}_i = \frac{n_i}{k_i} \) for \( i = 1, \ldots, r \), while \( \hat{\omega}_i \) for \( i > r \) are incommensurate.

We use the \( d \times d \) matrix \( Q \) to construct a new set of angles on \( \mathbb{T}^d \)
\[ \psi = Q\theta \]
so that the map \([29]\) transforms to \(\psi' = \psi + \hat{\omega}\), or in components
\[
\psi'_i = \psi_i + \frac{\hat{n}_i}{k_i}, \quad i = 1, \ldots, r
\]
\[
\psi'_k = \psi_k + \hat{\omega}_k, \quad k = r + 1, \ldots, d
\]
The orbits of the first \(r\) phases are periodic with period
\[
T = \text{lcm}(k_1, \ldots, k_r) = k_r
\]
and since by assumption, there are no additional resonance relations, the orbits of the last \(d - r\) components are dense on \(\mathbb{T}^{d-r}\). Thus we may conclude that

**Lemma 2.** When \(\omega\) is \(r\)-resonant, i.e., \(r = \dim \mathcal{L}(\omega)\), the orbit of each initial condition of the integrable torus map \([29]\) is dense on a family of \(T\), \([31]\), disjoint \(d - r\)-dimensional tori that are linked by iteration

**Example.** For \(d = 3\), the frequency \(\omega_a = (\sqrt{5}, 2\sqrt{5} + \frac{1}{2}, \sqrt{2})\) is 1-resonant. In order that \(m \cdot \omega = n\), then \(m_3 = 0\), \(m_1 + 2m_2 = 0\), and \(m_2 \in 2\mathbb{Z}\). thus the module for this resonance is the set
\[
\mathcal{L}(\omega_a) = \{(-4j, 2j, 0) : j \in \mathbb{Z}\} \subset \mathbb{Z}^3
\]
which has dimension one, with a basis vector \(m = (-4, 2, 0)\). The order of this resonance is \(p = 4\). This resonance is not primitive since the components of the minimal basis vector \(m\) are not coprime. Indeed the line \(mt\) contains the integer vector \((-2, 1, 0)\) that is not in \(\mathcal{L}(\omega_a)\).

For this case, the normalization \([30]\) results in
\[
M = \begin{pmatrix} -4 & 2 & 0 \end{pmatrix} = (1) \begin{pmatrix} 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = PKQ
\]
Thus \(\hat{n} = Pn = 1\), and
\[
\hat{\omega} = Q\omega = \begin{pmatrix} \frac{1}{2} \\ \sqrt{5} \\ \sqrt{2} \end{pmatrix}
\]
so that we have \(K\hat{\omega} = \hat{n}\) as required. Note that the transformation \(Q\) is orientation reversing \(\det Q = -1\), but we could fix this by exchanging the last two columns.

**Example.** The frequency \(\omega_b = (\sqrt{5}, \sqrt{5}, 2)\) has rank 2 with module
\[
\mathcal{L}(\omega_b) = \{j_1(1, -1, 0) + j_2(0, 0, 1) : j_i \in \mathbb{Z}\}.
\]
Here
\[
M = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
is a minimal basis for \(\mathcal{L}\) and thus \(p = 1\). This module is primitive.

**Example.** The frequency \(\omega_c = (2\sqrt{2} + \frac{1}{2}, -\sqrt{2}, \frac{1}{2})\) is 2-resonant. A basis is
\[
M = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix},
\]
with \( n = (1, 2)^T \).

The normalization (30) is

\[
P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

so that

\[
\hat{\omega} = Q\omega = \begin{pmatrix} 1 \\ 1 \\ 1 - \sqrt{2} \end{pmatrix}, \quad \hat{n} = P^{-1}n = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

Example. If \( \omega \in \mathbb{R}^3 \) has rank three, then its components are rational. Indeed, if \( m_i \cdot \omega = n_i \) for three independent \( m_i \), then \( \omega = M^{-1}n \), where the \( 3 \times 3 \) matrix \( M \) is nonsingular by assumption. Note, however, that this does not imply that the module is primitive. Instead, the torus decomposes into period \( T \) orbits.

### B Volume Preserving Maps

Let \( M \) be an \( n \)-dimensional manifold and \( \Lambda^k(M) \) denote the \( k \)-forms on \( M \). A \( C^1 \) map \( f : M \to M \) is volume preserving with respect to the volume form \( \nu \in \Lambda^n \) if

\[
f^*\nu = \nu.
\]

Here \( f^* \) is the pullback, defined for any \( k \)-form by

\[
(f^*\alpha)_x(V_1, V_2, \ldots, V_k) \equiv \alpha_{f(x)}(Df(x)V_1(x), \ldots, Df(x)V_k(x)),
\]

where \( V_1, V_2, \ldots, V_k \) are vector fields.

A map is is exact volume preserving if there exists an \( \alpha \in \Lambda^{n-1}(M) \) such that \( d\alpha = \nu \) and

\[
f^*\alpha - \alpha = d\lambda
\]

is exact. Sevryuk calls this condition globally volume preserving [Sev06]; we call it “exact” by analogy with exact symplectic maps [Mei92]. It is possible that \( \nu \) is exact with respect to a number of different forms; if there are two such forms, say \( \tilde{\alpha} \) and \( \alpha \) such that \( \tilde{\alpha} - \alpha \) is not exact, then the condition of exactness for \( f \) depends on the choice. The collection of exact volume-preserving diffeomorphisms on \( M \) with respect to a given \( \alpha \) is a group under composition that we denote \( \text{Diff}_\alpha(M) \). The form \( \lambda \) can be used as a generator for volume-preserving maps, as well as to compute the volume of lobes between stable and unstable manifolds, for more discussion see [LM09a, LM09b].

Consider the integrable, one-action map (1) on \( M = \mathbb{T}^d \times \mathbb{R} \). Since \( \det(Df_0) = 1 \), this map is volume preserving with respect to the standard volume form

\[
\nu = dz \wedge dx_1 \wedge dx_2 \wedge \ldots dx_d = dz \wedge \tau
\]

where \( \tau = dx_1 \wedge \ldots \wedge dx_d \in \Lambda^d(M) \). The integrable map is also exact with respect to the form

\[
\alpha = z\tau.
\]
In particular,
\[ f^*_0 \alpha - \alpha = d(i_W \tau) = d\lambda_0 \]
where \( W = \sum W_i \partial_{x_i} \) is the vector field with components
\[ W_i = \int z \Omega_i'(z) dz \]
and the inner product of \( \tau \) with \( W \) is defined as the \((d - 1)\)-form
\[ i_W \tau = \tau(W, \cdot, \cdot, \cdot). \] (34)

The easiest way to see that \( f \) is exact is to note that \( f \) is the composition of two shears:
\[ f = s_2 \circ s_1 \] (35)
where \( s_1(z, x) = (z + g(x), x) \) and \( s_2(z, x) = (z, x + \Omega(z)) \). Each of these is obviously volume preserving.

Since \( \text{Diff}_\alpha(M) \) is a group, the map \( f \) is exact with respect to \( \alpha \) whenever the shears \( s_1 \) and \( s_2 \) are exact. The shear \( s_1 \), however, is only exact if
\[ s_1^* \alpha - \alpha = g(x) \tau = d\lambda_1 \]
which is equivalent to the condition of zero net flux
\[ \int_{\mathbb{T}^d} g \tau = 0. \] (36)
By (7) exactness means that the zero Fourier component of \( g \) must vanish, so that
\[ g(x) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \hat{g}_k e^{2\pi i k \cdot x}. \]
In this case
\[ \lambda_1 = i G \tau \]
where \( G \in T\mathbb{T}^d \) is the vector field with components
\[ G_i(x) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{\hat{g}_k}{2\pi i k_i} e^{2\pi i k \cdot x}. \]

When \( s_1 \) and \( s_2 \) are both exact, then \( f = s_2 \circ s_1 \) is exact with \( \lambda = \lambda_1 + s_1^* \lambda_2 \).

Geometrically, the zero net flux condition means that the volume “below” a rotational torus \( \mathcal{T} = \{(z, x) : z = Z(x), x \in \mathbb{T}^d\} \),
\[ V(\mathcal{T}) = \int_{\mathbb{T}^d} Z(x) \tau, \]
is invariant under iteration. Indeed under \( s_1 \), we have \( \mathcal{T} \to \mathcal{T}' = \{(Z(x) + g(x), x) : x \in \mathbb{T}^d\} \),
and the volume under \( \mathcal{T}' \) is clearly that under \( \mathcal{T} \) under the assumption (7):
\[ V(\mathcal{T}') = \int_{\mathbb{T}^d} (Z(x) + g(x)) \tau = V(\mathcal{T}). \]

Similarly under \( s_2 \) a graph \( \mathcal{T} \) undergoes a rigid horizontal translation and thus bounds the same volume. Obviously, the map (5) has no rotational invariant tori unless it has zero net flux.
C Symmetries

The map \( [5] \) is reversible if \( g \) is an odd function of \( x \in \mathbb{Z}^d \). The reversor is

\[
R(x, z) = (-x, z + g(x))
\]

since

\[
R \circ f \circ R = R(f(-x, z + g(x))) = R(-x + \Omega(z + g(x) + g(-x)), z + g(x) + g(-x)) = R(-x + \Omega(z), z + g(-x + \Omega(z)) = (x - \Omega(z), z - g(x') = f^{-1}(x, z)
\]

Note that

\[
\text{Fix } R = \{ x = 0 \}
\]

is a line. Roberts and Lamb [RL95] call such reversors Type I, where the “T” represents the dimension of the fixed set. On its fixed set, a Type I reversor is locally conjugate to the diagonal matrix \( \text{diag}(1, -1, -1) \), and so is orientation preserving.

Since \( g(x) \) is periodic, the translation \( T_m(x, z) = (x + m, z) \) for \( m \in \mathbb{Z}^d \) is a symmetry of \( [5] \). Consequently the operator \( R_m = R \circ T_m \) given by \( R_m(x, z) = (-x - m, z + g(x)) \) is also a reversor for \( [5] \). Note that \( \text{Fix } R_m = \{ x = m/2 \} \).

Since any composition \( f^k \circ R \) is also a reversor, we can consider the second reversor in the family

\[
f \circ R(x, z) = f(-x, z + g(x)) = (-x + \Omega(z), z)
\]

which has the fixed sets

\[
\text{Fix } fR = \{ x = \frac{1}{2} \Omega(z) \}
\]

and

\[
\text{Fix } fRT_m = \{ x = \frac{m}{2} + \frac{1}{2} \Omega(z) \}.
\]

Note that this second involution must also be Type I, since it too must be orientation preserving.

If the frequency is an odd function of \( z \), then \( f \) has an additional symmetry. When \( \Omega(z) = -\Omega(-z) \), then the reflection through the origin is a symmetry, \( S_o(x, z) = (-x, -z) \). Then

\[
S_o \circ f \circ S_o(x, z) = S_o(-x - \Omega(z), -z - g(x)) = f(x, z)
\]

This also implies a second reversor, \( R_o = R \circ S_o(x, z) = (x, -z - g(x)) \):

\[
R_o \circ f \circ R_o(x, z) = R_o(x + \Omega(-z), -z) = f^{-1}(x, z)
\]

whose fixed set is the torus \( \text{Fix } R_o = \{ z = \frac{1}{2}g(x) \} \). This is analogous to the second symmetry of the standard mapping.

In the case that \( \Omega \) is even and \( g \) satisfies the condition \( g(x + k/2) = -g(x) \), for some integer \( k \), the second symmetry is \( S_e(x, z) = (x + k/2, -z) \), since

\[
S_e \circ f \circ S_e(x, z) = S_e(x + k/2 + \Omega(-z + g(x + k/2)), -z + g(x + k/2)) = f(x + 1 + \Omega(z + g(x), z + g(x))) = f(x, z)
\]

\(^5\) Since \( g(x + m) = g(x) \) and \( g(-x) = -g(x) \) we have \( g(x) = -g(-x + m) \), so \( g(m/2) = 0 \) for every \( m \in \mathbb{Z}^d \).
Note that our example has a function $g$ that satisfies such a relation only if $abc = 0$, i.e., if one of the parameters is zero. For example if $b = 0$ then $k = (1,0)$ works, and if $c = 0$ then $k = (1,1)$ works. Otherwise we do not have this symmetry.

The symmetry can be generalized to having $\Omega$ even about a point $z^*$, by setting $S_\epsilon(x,z) = (x + k/2, 2z^* - z)$. Similar cases for the two-dimensional nontwist map have been studied in [Pet01].

### D First Order Averaging Theory

When a dynamical system can be separated into slow and fast components, averaging can often be used to describe the slow dynamics on a long time scale [LMS88 SVM07]. In most cases averaging theory is formulated for ODEs, though there are some results specifically for maps [DEV04]. Here we present a result that we use in §4 and is appropriate for the map (23) that results in the averaged system (25). We mimic the standard techniques as presented, e.g., in [SVM07].

**Lemma 3** (First Order Averaging). Suppose that $\{\zeta_t, \theta_t\}$ is an orbit of the map on $\mathbb{T}^d \times M$ defined by

\[
\begin{align*}
\theta' &= \theta + \omega_0 + \epsilon g(\zeta, \theta) \\
\zeta' &= \zeta + \epsilon f(\zeta, \theta)
\end{align*}
\]  

for some manifold $M$, and $\{z_t\}$ is an orbit of the averaged map defined by

\[
z' = z + \epsilon \bar{f}(z)
\]

on $M$ where $\bar{f}(z) = \int_{\mathbb{T}^d} f(z, \theta) d\theta$. Assume that $f$ is analytic with the Fourier expansion

\[
 f(\zeta, \theta) = \sum_{k \in \mathbb{Z}^d} f_k(\zeta) e^{ik \cdot \theta}
\]

and that $f_k$ and $g$ are Lipschitz functions on $M$, and $\omega_0 \in \mathcal{D}(c, \mu)$ for some $c(\epsilon)$ and $\mu > d$.

Then if $z_0 = \zeta_0$, then whenever $\epsilon$ is small enough, the sequence

\[
|\zeta_t - z_t| = \mathcal{O}(\epsilon), \forall t \leq \mathcal{O}(\epsilon^{-1})
\]

**Proof.** We do this in two steps. First we find a coordinate transformation $\zeta \to y$,

\[
\zeta = y + \epsilon S(y, \theta)
\]

to eliminate the fluctuating terms to $\mathcal{O}(\epsilon)$ in the $\zeta$-equation. That is the dynamics for $y$ should be those of (38) to first order in $\epsilon$.

Substituting (39) into (37) and expanding to first order in $\epsilon$ gives a homological equation of the form

\[
S(y, \theta + \omega_0) - S(y, \theta) = f(y, \theta) - \bar{f}(y).
\]  

(40)

Formally, this equation can be solved by Fourier transformation, setting

\[
S(y, \theta) = \sum_{k \in \mathbb{Z}^d} S_k(y) e^{ik \cdot \theta},
\]

24
and noting that $S_0(y) = f(y)$ to obtain, for $k \neq 0$,

$$S_k(y) = \frac{f_k(y)}{e^{ik \omega_0} - 1}.$$

As is well known, if $f$ is analytic and $\omega_0$ Diophantine, then this formal series converges and the resulting function $S$ is also analytic, for example see [Per68, see Lemma 7]. Consequently, $\zeta$ and $y$ obviously remain within $O(\varepsilon)$ for all time.

For the second step we need to show that $y$ and $z$ remain close for $t = O(\varepsilon^{-1})$. The map for $y$ is

$$y' = \zeta' - \varepsilon S(y', \theta') = y + \varepsilon [f(\zeta, \theta) + S(y, \theta) - S(y', \theta')].$$

This can be written

$$y' = y + \varepsilon \tilde{f}(y) + \varepsilon^2 h(y, \theta, \varepsilon)$$

where $h$ is defined implicitly through

$$h(y, \theta, \varepsilon) = \frac{1}{\varepsilon} [f(y + \varepsilon S(y, \theta), \theta) - \tilde{f}(y) + \varepsilon S(y, \theta)$$

$$- \varepsilon S(y + \varepsilon \tilde{f}(y) + \varepsilon^2 h(y, \theta, \varepsilon), \theta + \omega_0 + \varepsilon g(y + \varepsilon S(y, \theta), \theta))].$$

Note that (40) implies that $h$ is well-defined as $\varepsilon \to 0$; indeed, $h(y, \theta, 0) = D_y f(y, \theta) S(y, \theta) - D_y S(y, \theta) \tilde{f}(y) - D_{\theta} S(y, \theta + \omega_0) g(y, \theta)$.

Since $f_k$ is Lipschitz, so are $\tilde{f} = f_0$ and $S$. Consequently, the implicit function theorem implies that for small enough $\varepsilon$, $h$ is analytic. The Grönwall lemma and Cor. 5 imply that the orbit $\{y_t\}$ is bounded over the finite time $t = O(\varepsilon^{-1})$, thus $h$ is Lipschitz on this bounded set as well.

We now compare the dynamics of $y$ in (41) with those of $z$ in (38). Upon defining $\delta_t = |y_t - z_t|$ we have

$$\delta_{t+1} \leq \delta_t + \varepsilon|\tilde{f}(y_t) - \tilde{f}(z_t)| + \varepsilon^2|h(y_t, \theta_t, \varepsilon) - h(z_t, \theta_t, \varepsilon)|$$

$$\leq k\delta_t + \varepsilon^2|z_t - z_{t+1}|.$$

where

$$k = (1 + \varepsilon L_f + \varepsilon^2 L_h)$$

and $L_f$ and $L_h$ are the Lipschitz constants for $f$ and $h$, respectively. This is almost of the form appropriate to applying a discrete version of Grönwall’s lemma, see App. E except the affine term is time dependent; nevertheless, a similar strategy works. Iteration of the inequality yields

$$\delta_t \leq k^t \left( \delta_0 + \varepsilon^2 \sum_{n=0}^{t-1} \frac{|h(z_n, \theta_n, \varepsilon)|}{k^{n+1}} \right).$$

Thus if $t \leq 1/\varepsilon$, $k^t = (1 + \varepsilon L_f + \varepsilon^2 L_h)^t \leq e^{(\varepsilon L_f + \varepsilon^2 L_h)t} \leq e^{L_f + \varepsilon L_h}$, so that

$$\delta_t \leq e^{L_f + \varepsilon L_h} \left( \delta_0 + \varepsilon^2 \sum_{n=0}^{t-1} |h(z_n, \theta_n, \varepsilon)| \right).$$
Say we can prove the global bound \( |h(z_t, \theta_t)| < H \) for all \( t \leq O(\varepsilon^{-1}) \). Then we can estimate 
\[
\delta_t \leq e^{L_f+\varepsilon L_h} (\delta_0 + \varepsilon^2 t H) .
\]
The initial \( \delta_0 \) is zero, hence again using \( t \leq 1/\varepsilon \) gives 
\[
\delta_n \leq \varepsilon e^{L_f+\varepsilon L_h} H .
\]
The needed bound on \( h(z_t, \theta_t) \) can be found by applying Cor. 5 in App. E to the map (38) using the assumed Lipschitz constant \( \varepsilon L_f \) for \( \varepsilon \bar{f} \) to obtain 
\[
|z_t - z_0| \leq |\bar{f}(z_0)| \frac{(1 + \varepsilon L_f)^t - 1}{L_f} \leq |\bar{f}(z_0)| \frac{e^{L_f} - 1}{L_f} ,
\]
so that \( z_t \) is contained in a ball about \( z_0 \). Hence the bound \( H \) is given by the maximum of \( |h(z, \theta)| \) for arbitrary \( \theta \) and \( z \) in this ball.

Thus \( \delta_t = |z_t - y_t| = O(\varepsilon) \) for all \( t \leq O(\varepsilon^{-1}) \). Since \( y_t \) is within \( \varepsilon \) of \( \zeta_t \), we have proven the lemma. \( \square \)

E Grönwall Lemma for Maps

For the proof of the averaging theorem it is convenient to state a discrete version of the standard Grönwall Lemma.

**Lemma 4** (Grönwall for Maps). Suppose that \( k_t \geq 0 \) for each \( t \in \mathbb{N} \), \( c > 0 \), and
\[
\delta_n \leq c + \sum_{t=1}^{n-1} k_t \delta_t , \quad n > 0 ,
\]
then
\[
\delta_n \leq c \prod_{t=1}^{n-1} (1 + k_t) .
\]

**Proof.** Define \( \Delta_n = c + \sum_{t=1}^{n-1} k_t \delta_t \). Since \( \delta_n \leq \Delta_n \) by assumption
\[
\Delta_{n+1} - \Delta_n = k_n \delta_n \leq k_n \Delta_n .
\]
Setting \( \Delta_n = \eta_n \prod_{t=1}^{n-1} (1 + k_t) \) and \( \Delta_1 = \eta_1 = c \), we then have
\[
(\eta_{n+1} - \eta_n) \prod_{t=1}^{n} (1 + k_t) \leq 0 ,
\]
which implies that the sequence \( \eta_n \) is nonincreasing. Consequently
\[
\Delta_n = \eta_n \prod_{t=1}^{n-1} (1 + k_t) \leq c \prod_{t=1}^{n-1} (1 + k_t) ,
\]
and since \( \delta_n \leq \Delta_n \), we have the result. \( \square \)

One simple consequence of this Grönwall result is the boundedness of orbits of a first order difference equation.
Corollary 5. Suppose \( \{z_t, \theta_t\} \) is an orbit of the map
\[
\begin{align*}
\theta' &= g(z, \theta) \\
z' &= z + f(z, \theta)
\end{align*}
\]
\( \mathbb{T}^d \times M \), and that \( f|_B \) is Lipschitz with constant \( L \) on \( \mathbb{T}^d \times B \), where \( B \subset M \). Then for each \( n \in \mathbb{N} \) if \( z_0 \in B \),
\[
|z_n - z_0| \leq R = c \left( 1 + \frac{L}{L} \right)^{n-1} - 1
\]
where \( c = \max_{\theta \in \mathbb{T}^d} |f(z_0, \theta)| \), so long as this ball of radius \( R \) is contained in \( B \).

Proof. For each \( t \in \mathbb{N} \),
\[
z_t - z_{t-1} = f(z_0, \theta_0) + \sum_{j=1}^{t-1} [f(z_j, \theta_j) - f(z_{j-1}, \theta_{j-1})].
\]
Defining \( \delta_t \equiv |z_t - z_{t-1}| \) and using the Lipschitz property of \( f \) gives
\[
\delta_t \leq c + L \sum_{j=1}^{t-1} \delta_j.
\]
Then Lem. 4 with \( k_t = L \) gives \( \delta_t \leq c(1 + L)^{t-1} \), so that finally
\[
|z_n - z_0| \leq \sum_{t=1}^{n} \delta_t = c \sum_{t=1}^{n} (1 + L)^{t-1},
\]
implying the promised the result. \( \square \)

References


