Simulating the Effects of Quantum Error-correction Schemes

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Abstract

It is important to protect quantum information against decoherence and operational errors, and quantum error-correcting (QEC) codes are the keys to solving this problem. Of course, just the existence of codes is not efficient. It is necessary to perform operations fault-tolerantly on encoded states because error-correction process (i.e., encoding, decoding, syndrome measurement and recovery) itself induces an error. By using simulation, this paper investigates the effects of some important QEC codes (the five qubit code, the seven qubit code and the nine qubit code) and their fault-tolerant operations when the error-correction process itself induces an error. The corresponding results, statistics and analyses are presented in this paper.

1 Introduction

A quantum computer will necessarily interact with its surroundings, which results in decoherence and consequently in the decay of the quantum information stored in the device. Quantum gates (in contrast to classical gates) are unitary transformations consisting of a continuum of possible values and hence cannot be implemented perfectly. The effects of small imperfections in the gates will accumulate, which will cause serious failure.

Therefore, when we build and operate a quantum computer, it is important to protect quantum information from errors. Over the past of few years, several theoretical studies have been made on quantum error-correcting (QEC) codes [14, 1, 17, 3, 8, 5] to protect quantum information against errors. Of course, just the existence of codes is not efficient. It is necessary to perform operations fault-tolerantly on encoded states [6, 15, 12].

The most powerful application of QEC scheme is not only the protection of stored or transmitted quantum information but also the protection of quantum information as it dynamically undergoes computation. However, it seems difficult to analyze the effects of QEC scheme theoretically in the latter case.

For example, the main operations (i.e., computation) are interleaved with the error-correction operations. It is necessary to prevent more errors within a code block than the QEC code can handle. Hence, as more errors happen in the main operations, more often the error-correction operations need to be applied. The error-correction operation itself, however, induces errors. Therefore, too many error-correction operations may increase the total amount of errors.

Hence, in terms of fidelity through simulations, we investigate the effects of QEC schemes and their fault-tolerant circuits that have been developed by many investigators to correct errors in quantum computations, even errors that occur during the error-correction process. Of course, for sufficiently long computations, the concatenated correcting codes are probably the best because of the threshold theorem [4, 2, 5, 13]. However, the concatenated correcting codes require much more qubits than simple (i.e., non-concatenated) correcting codes because concatenation requires multi-level encodings. Taking account of the physical implementation issue, it will be necessary to reduce the number of required qubits as much as possible. Therefore, we investigate how effective simple QEC codes are in the real computation. This time we deal with the seven qubit code, the nine qubit code and the five qubit code as the simple QEC codes because they correct one arbitrary error but they require only less than 10 qubits to encode 1 qubit.

Section 2 describes our simulator for quantum computing. Section 3 briefly explains the error-correction operations for the five qubit code, the seven qubit code and the nine qubit code, respectively. Section 4 shows the effectiveness of these QECC by the simulations in the realistic case. Section 5 discusses related work. Section 6 concludes by summarizing this paper.

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2   Quantum Computer Simulation System (QCSS)

We have proposed a general-purpose fast simulator for quantum computing [9] and implemented it on various multi-computers. The features of our quantum computer simulation system (QCSS) are described briefly as follows.

- QCSS deals with quantum circuits.

  We have implemented fast libraries for fundamental operations such as single qubit operation, controlled operation and measurements. Quantum circuits can be described as a C program as shown below and the fundamental operations are described as procedure calls in the program. The C program is compiled and linked with our libraries to generate executable code.


```
ApplyGate(H, 0); /* Apply Hadamard gate to 0th qubit*/
ApplyControlledGate(X, 0, 1);/* Apply sigma X gate to 1st qubit
                                 and 0th qubit is controlled bit*/
ApplyGate(H, 0); /* Apply Hadamard gate to 0th qubit*/
Measurement(0); /* Measure 0th qubit*/
```
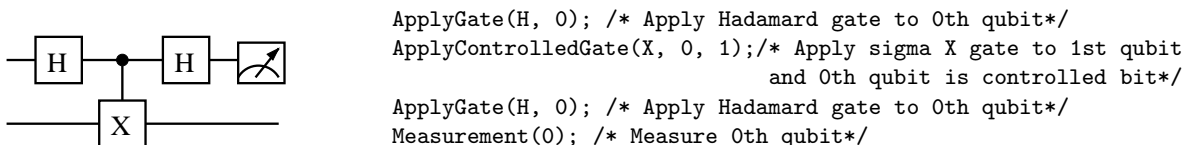
Figure 1: Sample Circuit.

- QCSS is scalable by using the parallel processing.

  QCSS is implemented on multi-computers to reduce simulation time. QCSS now works on not only shared-memory multi-computers (Sun, Enterprise4500) but also distributed-memory multi-computers (16-node Dell PowerEdge cluster and 128-node Sun Blade cluster). On the current implementation, QCSS can deal with about 30 qubits. Of course, this limit is caused by hardware (i.e., memory, processor-speed). It should be noted that QCSS can deal with more qubits as more powerful multi-computers are available.

- QCSS implements time evolution efficiently.

  QCSS requires only $\mathcal{O}(2 \times 2)$ space and $\mathcal{O}(3 \times 2^n)$ arithmetic operations for $n-$qubit time evolution [7].

- QCSS implements the decoherence error model and operational error model.

  QCSS assumes that the quantum depolarizing channel as the decoherence error model. In this channel, with probability $1 - p$, each qubit is left alone. In addition, there are equal probabilities $p/3$ that X (a bit flip error), Z (a phase flip error) or Y (both errors, $Y = XZ$) affects the qubit.

  In general, all of single qubit gates are generated from rotations and phase shifts. QCSS represents inaccuracies by adding small deviations to the angles of rotations and phase shifts. Each error angle is drawn from Gaussian distribution with the standard deviation (σ).

  QCSS does not deal with mixed states. Therefore, in order to compute the fidelity, the experiments were repeated many times and the average values are used.

3   Error-correction Circuits

We deal with three important QECC that correct one arbitrary error. In this section, we introduce the error-correction operations respectively.

3.1   The Nine Qubit Code [14]

The nine qubit code $[[9, 1, 3]]$ is known as Shor code. This code is a combination of the three qubit phase flip codes and bit flip codes. It can be seen as a two level concatenated code. The codewords are given by:

$$|0\rangle \mapsto |0_L\rangle \equiv \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle \mapsto |1_L\rangle \equiv \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle).$$
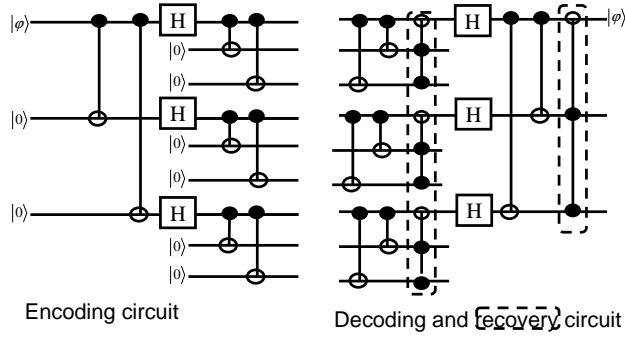
Figure 2: Encoding/decoding/recovery circuit for the nine qubit code

Its encoder, decoder and recovery circuit are shown in Figure 2. Of course, there exists the syndrome measurement and recovery circuit based on the stabilizer formalism [5]. However, we use this circuit, since it is simpler and hence induces less errors. We should notice that this recovery circuit cannot be used without decoding circuit.
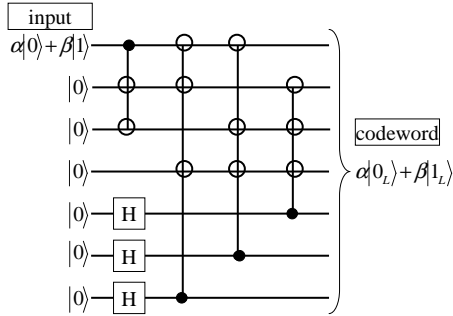
3.2 The Seven Qubit Code [1, 17]

Figure 3: Encoder for the seven qubit code.

The seven qubit code $[[7, 1, 3]]$ is known as Steane code. This code belongs to CSS code and constructed using the $[7, 4, 3]$ (classical) Hamming code. The codewords are given by:

$$|0_L\rangle = \frac{1}{\sqrt{8}}[|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$$

$$+|0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle$$

$$|1_L\rangle = \frac{1}{\sqrt{8}}[|11111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle$$

$$+|1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle.$$

Steane code has nice properties. One of them is that the codewords do no require to be decoded in order to perform main operations. Its encoding circuit is shown in the Figure 3. Its decoding circuit is just the encoding circuit run in reverse. Its syndrome measurement and recovery circuit are shown in the Figure 4.
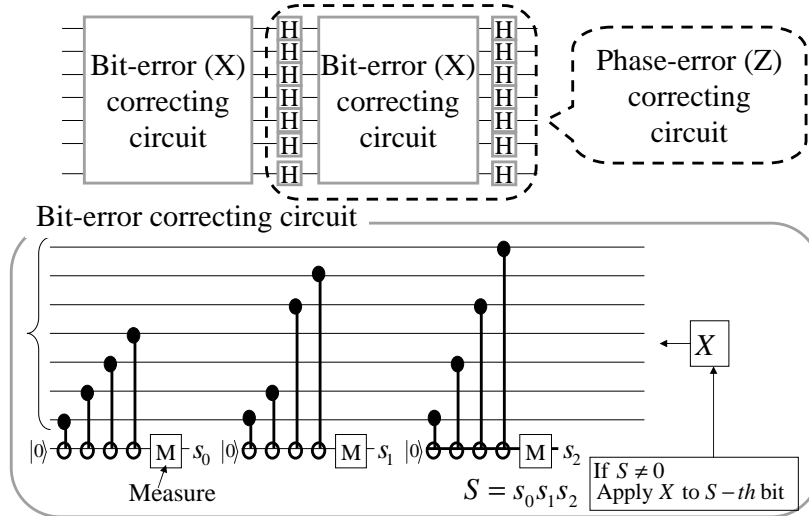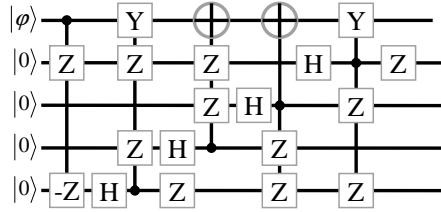

Figure 4: Syndrome measurement and recovery circuit for the seven qubit code.

3.3 The Five Qubit Code [3, 8]

Figure 5: Encoder for the five qubit code.

The five qubit code $[[5, 1, 3]]$ is the perfect non-degenerate code and belongs to the non-CSS stabilizer code. This code is the smallest capable of protecting against a single error. Its encoding circuit is shown in the Figure 5. Its decoding circuit is just the encoding circuit run in reverse. The error syndrome measurement and recovery circuit is shown in the Figure 6. The codewords are given by:

$$
\begin{aligned}
|0_L\rangle &= \frac{1}{4}[|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle + |00101\rangle \\
&\quad -|11110\rangle - |01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle \\
&\quad -|01100\rangle - |00110\rangle - |00011\rangle - |10001\rangle - ||11000\rangle \\
|1_L\rangle &= \frac{1}{4}[|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle + |11010\rangle \\
&\quad -|00001\rangle - |10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle \\
&\quad -|10011\rangle - |11001\rangle - |11100\rangle - |01110\rangle - |00111\rangle]
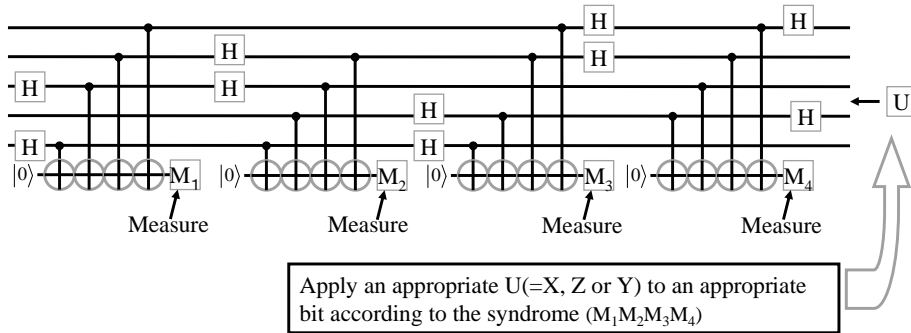\end{aligned}
$$

Figure 6: Syndrome measurement and recovery circuit for the five qubit code

It will be helpful to compare the features of these three QEC codes. Table 1 summarizes the comparison among three QEC codes. Table "Depth of error-correcting circuits" shows the depth of error-correcting circuits

Table 1: Comparison among the nine qubit code, the seven qubit code and the five qubit code
The number of (minimum) required qubits.

	$[[9, 1, 3]]$	$[[7, 1, 3]]$	$[[5, 1, 3]]$
qubits	9	8	6

Depth of error-correcting circuits.

	$[[9, 1, 3]]$	$[[7, 1, 3]]$	$[[5, 1, 3]]$
Encoder (Decoder)	5	4	10
Syn. measurement and recovery	2	32	22

Transversal gate implementation [6].

$[[9, 1, 3]]$	$[[7, 1, 3]]$	$[[5, 1, 3]]$
Hard	Easy	Hard

using the minimum number of qubits. As for error-correction circuit complexity, the nine qubit code is simplest. Furthermore, it does not require an ancilla qubit. About implementation of transversal gates, that is, encoded gates in a bit-wise fashion, the seven qubit code is the easiest.

4 Experimental Verification through Simulations

The most powerful application of quantum error-correction scheme is not only the protection of stored or transmitted quantum information but also the protection of quantum information as it dynamically undergoes computation.

Quantum computation using encoded qubits, is a sequence of computation directly (or indirectly) on the encoded qubits along with periodic error-correction processes. If these processes are not performed sufficiently, then multiple errors may occur between these processes, which results in an uncorrectable state. However, if these processes are performed too much, then the circuit size becomes too large, which increases the total amount of errors. Our simulation aims to investigate how often to perform these operations.
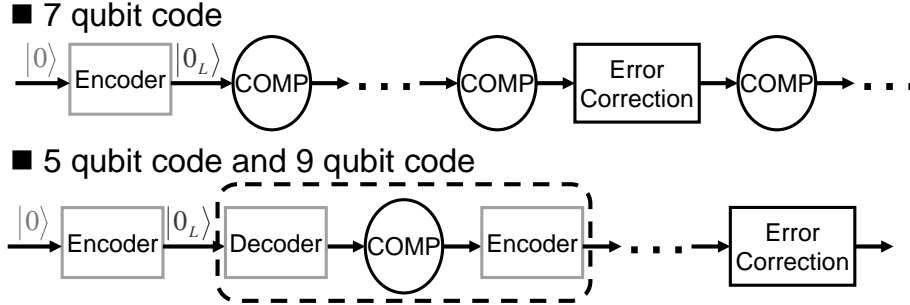
4.1 Simulation Methodology

Figure 7: Simulation Methodology

The simulation is performed using QCSS described in Section 2.

As for the seven qubit code, it is easy to perform computation directly on the encoded qubits, especially about normalizer operations (the Hadamard, phase and controlled-NOT gates).

On the other hand, as for the five qubit code and the nine qubit code, it is hard to perform computation directly on the encoded qubits without using more additional qubits. Therefore, it is necessary to decode codewords in order to perform main computation if we reduce the number of qubits as much as possible. That is, encoding/decoding operations between computation are necessary. Furthermore, as for the nine qubit code, the recovery circuit should be used along with decoding circuit. This means that the error-correction operation is performed at every 1 main gate. As for seven qubit code and five qubit code, error-correction is performed at every 1 or 50 or 100 or 200 or 2000 main gate.

Let p be the decoherence probability. At each time step, that is, as the depth of circuit increases by 1, each qubit has no change with probability $1-p$ and it undergoes rotations by the operation X, Y, Z with equal probabilities $\frac{p}{3}$.

QCSS represents inaccuracies by adding small deviations to the angles of rotations and phase shifts. Each error angle is drawn from Gaussian distribution with the standard deviation (σ).

When we compute fidelity, we trace out ancilla qubits. In order to trace out ancilla qubits, QCSS copies the state and ideally decodes the copied state and measures only ancilla qubits. Of course, the original state has no change in fidelity calculation. Fidelity is calculated as follows

$$fidelity = \sum_{i=1}^{N} \frac{||\langle \phi_{correct}|\phi_{output_i}\rangle||^2}{N}.$$

N means the number of experiments ($10^5 \sim 10^6$). $|\phi_{correct}\rangle$ represents the correct state and $|\phi_{output_i}\rangle$ represents the output state computed by QCSS at i-th experiment.

4.2 Effects of Ancilla Operations

To compute syndrome, ancilla qubits are usually used. The obvious way is to use only one qubit in order to compute each bit of the syndrome. In this case, ancilla preparation is very simple. However, it may propagate the errors backward.

The fault-tolerant way is to use enough qubits to compute each bit of the syndrome transversally. Of course, we prepare the ancilla in a Shor state [15] that reveals the errors without revealing the data, The latter is much more fault-tolerant than the former. However, this requires much more qubits in total and ancilla preparation is much harder.

Thus, we need to check which system is better in terms of fidelity through simulations. We use the seven qubit code and adopt 4000 times Hadamard transformation as main computation. The start state of the quantum register is $|0_L\rangle$ (logical 0 bit). If there are no errors, the final correct state must be $|0_L\rangle$. The latter error-correcting circuit requires 11 qubits in total and its depth is at least 40 *.

Table 2: Final fidelity with decoherence errors for the one qubit ancilla system and the four qubit ancilla system

rate	Ancilla bit	Frequency of recovery process				
		1G	50G	100G	200G	2000G
10^{-5}	1bit	0.5840	0.9836	0.9920	0.9942	0.9925
	4bit	0.5750	0.9808	0.9886	0.9955	0.9922
10^{-4}	1bit	0.4970	0.8290	0.8910	0.8940	0.7110
	4bit	0.5020	0.8010	0.8558	0.8705	0.7164
10^{-3}	1bit	0.4890	0.5070	0.4780	0.5220	0.5070
	4bit	0.4900	0.5082	0.5075	0.4928	0.5034

Table 2 shows the final fidelity with decoherence errors ($rate = 10^{-5} \sim 10^{-3}$) for the one qubit ancilla system and the four qubit ancilla system. For example, The column "50G"represents the results performing the QEC circuit at every 50 main (i.e., Hadamard) gate. From Table 2, we cannot conclude that the four qubit ancilla system is better than the one qubit ancilla system.

Consider the circuit area defined by "number of the qubits" × "depth of the circuit". It is reasonable to suppose that the error probability of the circuit is proportional to its area. The area of the four ancilla qubits system is at least 440 and it is much larger than that of the 1 ancilla qubit system, that is, 256. None the less, the fidelity values are almost the same. This result indicates that the transversal operations in the four ancilla qubits system are fault-tolerant.

4.3   Effects of Frequency of the Error-correction Operations

We adopt 4000 times Hadamard transformation as main computation. The start state of the quantum register is $|0_L\rangle$ (logical 0 bit). Fidelity is calculated at every two main gate, which means that $|\phi_{correct}\rangle$ is $|0_L\rangle$ if there are no errors. From now on, the x-axis in the Figure shows the even number of main computations (i.e., Hadamard Transform). The y-axis in the Figure shows the fidelity.

4.3.1   Decoherence Errors

The Seven Qubit Code
Figure 8 shows how the frequency of the error-correction operation affects the fidelity for decoherence (probability $= 10^{-5}$, 10^{-4} and 10^{-3}) in the seven qubit code case. For example, "QECC/50" in the Figure means the error-correction circuit is performed at every 50 main (i.e., Hadamard) gates and "Physical 1 bit" means the real (physical) 1bit, hence QEC scheme cannot be applied to "Physical 1 bit" case.

Figure 8 shows that fidelity is improved just after the QEC operation with adequate frequency is performed. However, it also shows that the fidelity is not high for the circuit that perform error-correction operations at every 1 main gates ("QECC/1"). When the decoherence probability is 10^{-5}, we can see that "QECC/x" (x=50, 100, 200, 2000) is better than "Physical 1 qubit". When the decoherence probability is 10^{-4}, the result is slightly different. "QECC/2000" is worse than "Physical 1 qubit". In this situation, we consider that "QECC/2000" is not sufficient and multiple errors occur between QEC circuits, which results in an uncorrectable state. When the decoherence probability is 10^{-3}, we can see that "Physical 1 qubit" is better than "QECC/x" (x=1, 50, 100, 200, 2000).

It is reasonable to suppose that computation with QEC scheme is worse than computation without it for any frequency of QEC operations when decoherence probability is more than 10^{-3}. These results show that the appropriate frequency of QEC operations (i.e., every $50 \sim 200$ main gate) makes the 7 qubit scheme really effective when the decoherence probability is not more than 10^{-4}.
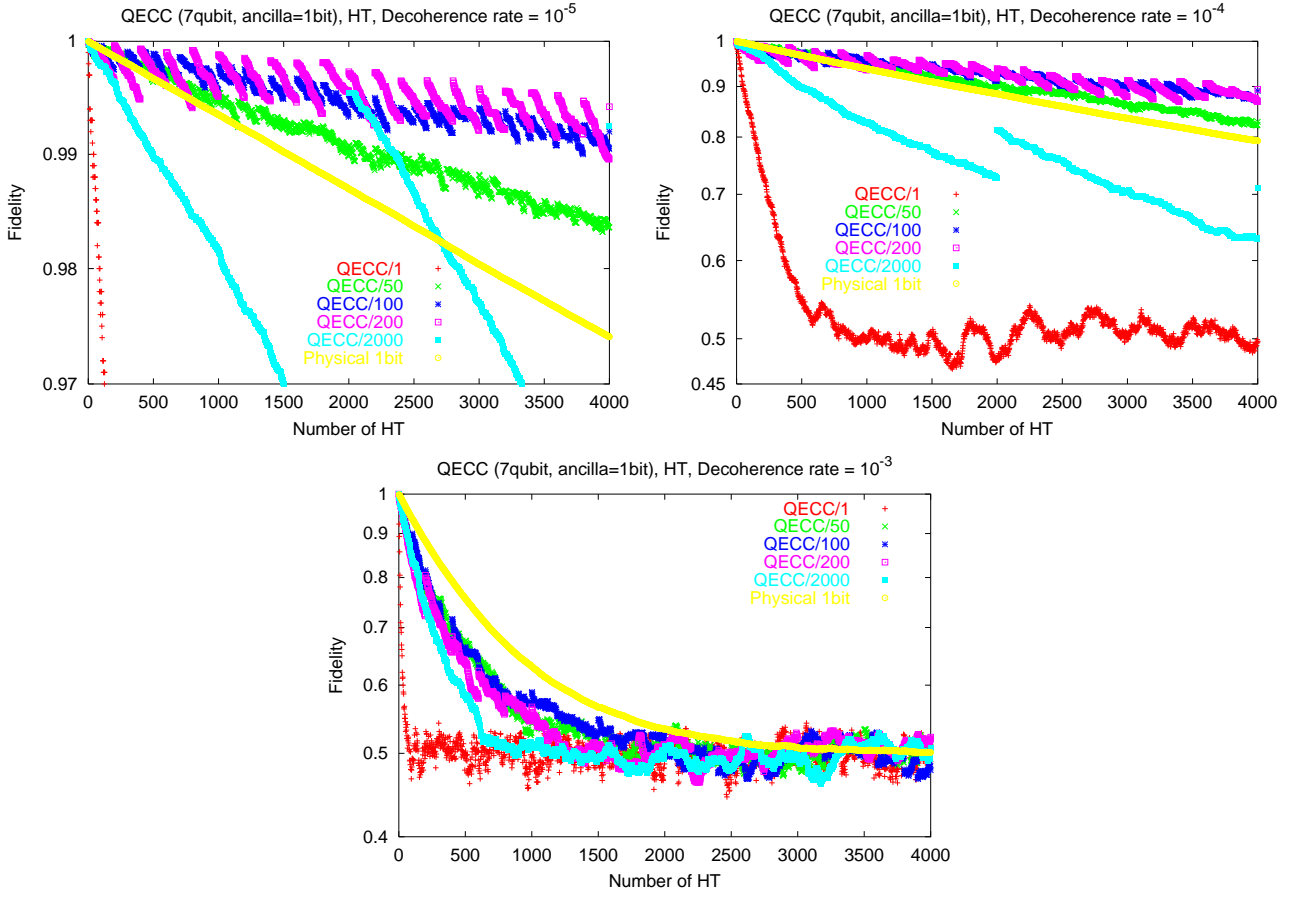
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*The ancilla preparation is repeated until it succeeds.

Figure 8: Fidelity with decoherence errors (10^{-5}, 10^{-4} and 10^{-3}) for the seven qubit code.

Analysis The "Physical 1 qubit" case is theoretically calculated as $\frac{1+(1-\frac{4}{3}p)^n}{2}$ (n means the even number of Hadamard transform and p means the decoherence rate). It is of course the first-order approximation. We consider two consecutive Hadamard gates as the unit. Let $1-P$ be the probability that the fidelity is decreased at the end of the unit. We assume that there are at most 1 error in the unit, that is, before the first Hadamard gate or just before the second Hadamard gate. X and Y before the first Hadamard gate degrade the fidelity at the end of the unit, and Z and Y just before the second Hadamard gate degrade the fidelity at the end of the unit. Thus,

$$1 - P = 1 - 4\left\{\frac{1}{3}p(1-p)\right\} \approx 1 - \frac{4}{3}p$$

. Hence, fidelity is calculated as $\frac{1+(1-\frac{4}{3}p)^n}{2}$.

Let us consider the encoded case. First, to make the analysis simpler, we deal with the case where syndrome measurement and recovery operations are not performed. In this case, we apply only the encoded Hadamard gates. We consider two consecutive encoded Hadamard gates as the unit. We assume that there are at most one error in the unit. We generate one error in the unit on purpose and we check which error really degrades the fidelity at the end of the unit. Through simulations, we verify that the number of such errors are 12 in this case. Thus,

$$1 - P = 1 - 12\left\{\frac{1}{3}p(1-p)^{13}\right\} \approx 1 - 4p.$$

Therefore, fidelity is calculated as $\frac{1+(1-4p)^n}{2}$. This approximation is consistent with the simulation results as shown in Figure 9. We set the number of computation 2000, Therefore, "QECC/2000" means that syndrome measurement and recovery operations are not performed.

Second, suppose that QEC operations are performed at every y main (i.e., Hadamard) gate. We consider y consecutive encoded Hadamard gates, the QEC circuit, y consecutive encoded Hadamard gates and the QEC circuit as the unit shown in Figure 10.
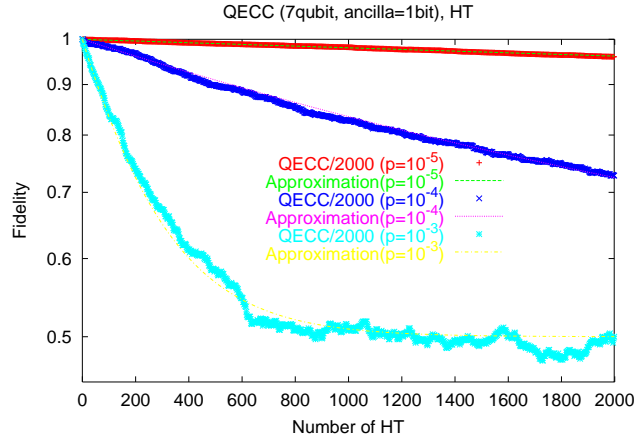
7

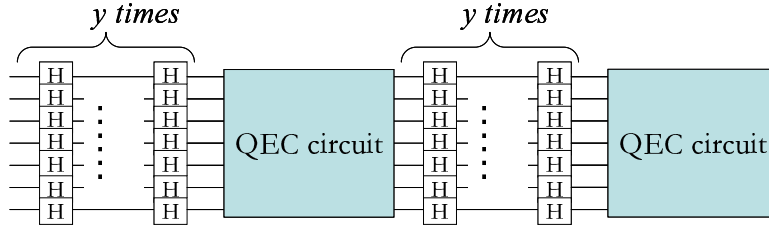Figure 9: Comparison between approximation and simulation for the 7 qubit code (1).

Figure 10: The unit for the general case

Again, we assume that there are at most one error in the unit. We generate one error in the unit on purpose and we check which error really degrades the fidelity at the end of the unit. Through simulations, we verify that the number of such errors are 272 and it is independent of y. Thus,

$$1 - P = 1 - 272 \left\{ \frac{1}{3} p (1-p)^6 \cdot (1-p)^{((2 \cdot 32 + 2y) - 1)) \cdot 7} \right\} = 1 - \frac{272}{3} p (1-p)^{447 + 14y} \approx 1 - \frac{272}{3} p (1 - (447 + 14y)p).$$

Therefore, fidelity is calculated as $\frac{1 + (1-P)^{\frac{n}{2 \cdot y}}}{2}$. Of course, this approximation holds if p is sufficiently small and y is not so large. Figure 11 shows the fidelity values obtained from not only simulation results but also this approximation. We can see that this approximation is consistent with the simulation results.
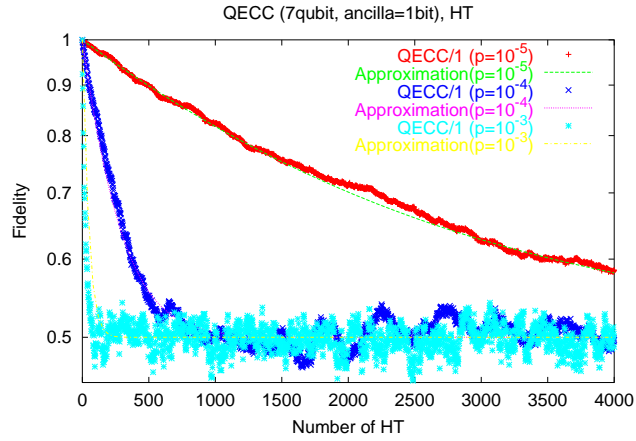

Figure 11: Comparison between approximation and simulation for the 7 qubit code (2).

Five Qubit Code Figure 12 shows how the frequency of the error-correction operation affects the fidelity for decoherence (10^{-5}, 10^{-4}, 10^{-3}) in the five qubit code case.
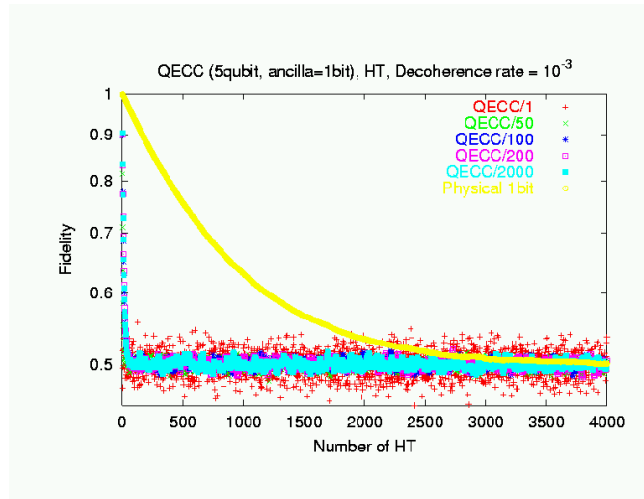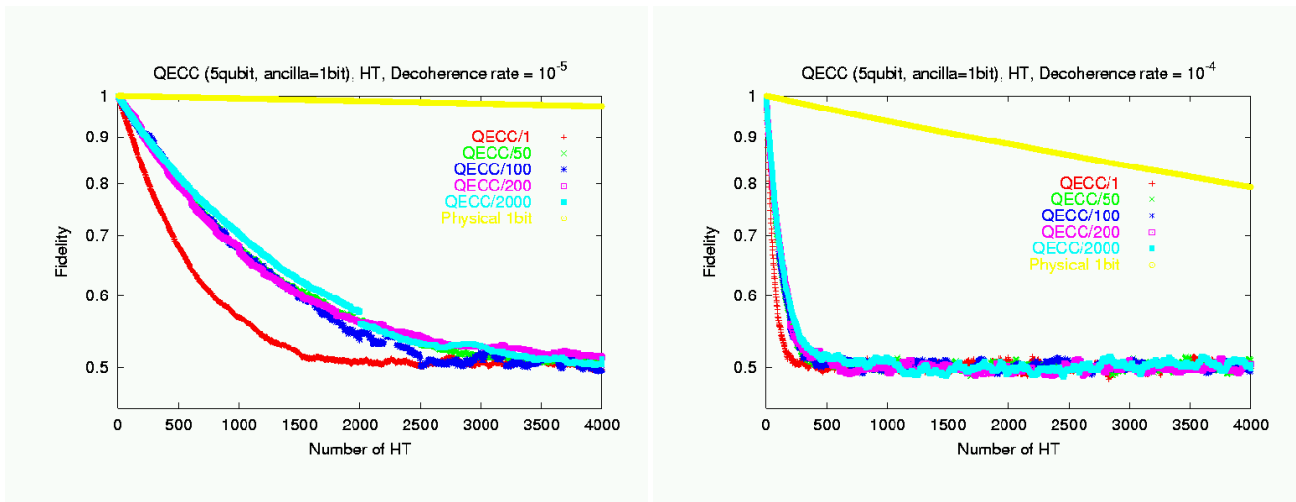
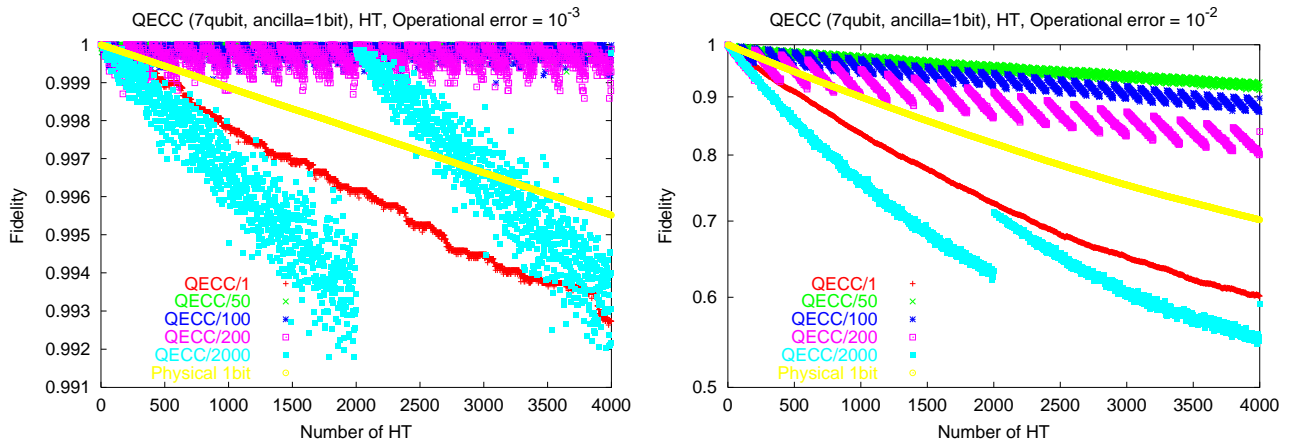Figure 12: Fidelity with decoherence errors (10^{-5}, 10^{-4} and 10^{-3}) for the five qubit code.

Figure 13: Fidelity with operational errors ($\sigma = 10^{-3}$ and 10^{-2}) for the seven qubit code.

The result clearly shows that computation with error-correction is much worse than computation without it. The comparison between Figure 8 and Figure 12 shows that the fidelity for the seven qubit code produces better results than that of the five qubit code. The results of nine qubit code is described in the Section 4.6.

4.4   Operational Errors

Figure 13 shows how fidelity is affected by performing QEC operations in the presence of operational errors. We should report that we cannot detect fidelity decrease when the standard deviation σ is no more than 10^{-4}.

It is found from the results that QEC scheme is effective for operational errors. This clearly shows that the apparent continuum of errors can be corrected by QEC process, that is, by correcting only a discrete subset of those errors [14, 16].

This is because Pauli matrices span the space of 2×2 matrices. Let $|\phi\rangle$ be the state of encoded 1 qubit. To make the analysis easy, we assume that operational error occurs on the i-th qubit only. As an error operator on i-th qubit E_i can be expanded:

$$E_i = e_{i0}I + e_{i1}X + e_{i2}Z + e_{i3}Y$$

The quantum state $E_i|\phi\rangle$ can be expressed as superposition of $|\phi\rangle$, $X|\phi\rangle$, $Z|\phi\rangle$, $Y|\phi\rangle$. The syndrome measurement process collapse the state into one of the four states: $|\phi\rangle$, $X|\phi\rangle$, $Z|\phi\rangle$, $Y|\phi\rangle$. The recovery process performs the inversion operation and hence the state recovers.
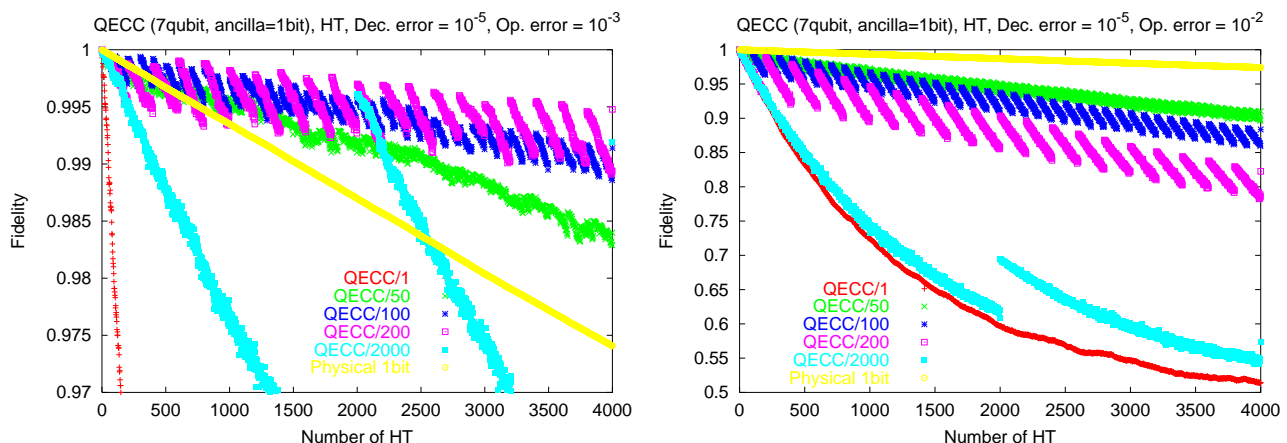
4.5   Both Decoherence and Operational Errors

Figure 14: Combined effects for the seven-qubit error-correction (($p = 10^{-5}$, $\sigma = 10^{-3}$) and ($p = 10^{-5}$, $\sigma = 10^{-2}$)).

Now, we take account of not only decoherence errors but also operational errors. It is difficult to analyze theoretically the effects of QECC when both errors exist. Therefore, numerical simulations are needed.

We use the seven qubit code because we have confirmed that it is effective at least for each error. Figure 14 and 15 show the combined effects of both errors. In this experiment, "Physical 1 qubit" represents the fidelity of physical 1 qubit with only decoherence errors. The combined effect of two factors may be worse than each factor alone, that is to say, the effect seems to be the product of each factor. As these Figures indicate, computation with QEC scheme is worse than computation without QEC scheme when the standard deviation of operational errors is 10^{-2}. Therefore, it is reasonable to suppose that the QEC scheme using the seven qubit code is still effective even if both errors exist, as long as the decoherence rate is not more than 10^{-4} and the standard deviation of operational errors is not more than 10^{-3} and the frequency of QEC operations are appropriate (i.e., at every $50 \sim 2000$ main gate).

4.6   Code Comparison

Figure 16 shows the fidelity with decoherence errors (10^{-5}, 10^{-4} and 10^{-3}) for the QEC circuit of nine qubit code described in Section 2. As stated above, the error-correction operation for nine qubit code scheme must be performed at every 1 main gate. For comparison, Figure 16 also shows the best fidelity for five qubit code and that for seven qubit code.
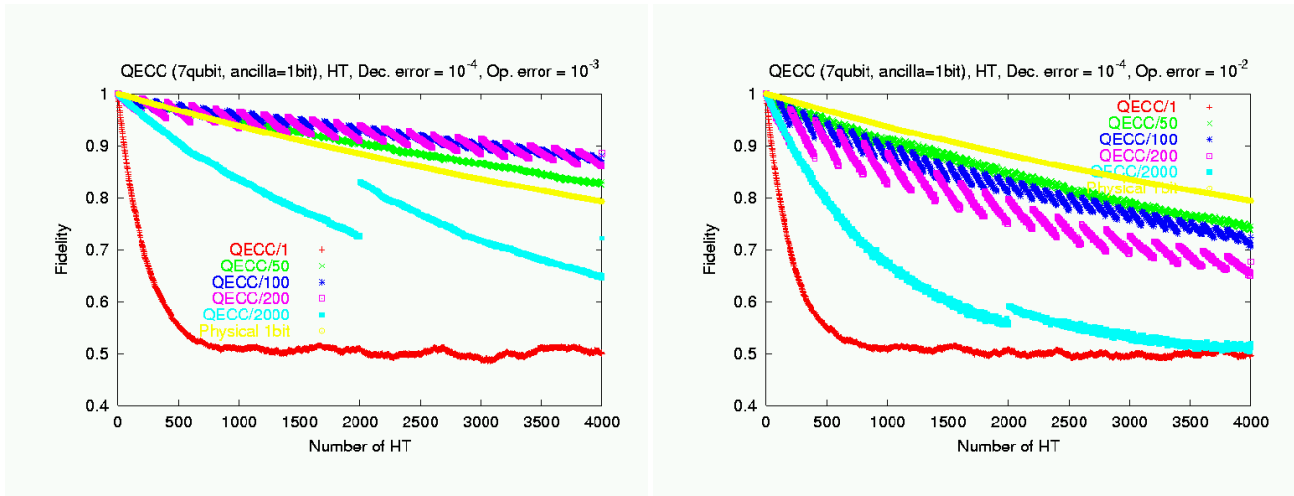
Figure 15: Combined effects for the seven-qubit error-correction (($p = 10^{-4}$, $\sigma = 10^{-3}$) and ($p = 10^{-4}$, $\sigma = 10^{-2}$)).

From the Figure 16, we can see that seven qubit code is best. As stated in Section 4.1, the five qubit code scheme and the nine qubit code scheme always require the encoding and decoding process per 1 main computation step. From the Table 1, we find that the five qubit scheme requires at least 21 steps to perform 1 main computation step and the nine qubit scheme requires at least 13 steps, whereas the seven qubit code scheme requires at least 1 step. This difference leads to the fidelity difference shown in Figure 16.

We can also find that the fidelity for the nine qubit code is higher than that for the five qubit code. We consider this is partly because the QEC circuit of the nine qubit code scheme is simpler and hence induces less errors. When the decoherence rate is 10^{-3}, the fidelity for the nine qubit code is best during the first 200 times computation, which is contrary to our intention.

5 Related Work

There are several numerical studies paying attention to the realistic case in which errors occur during the error-correction operations. In Ref[10], the seven qubit code is simulated in the Obenland's simulator based on an actual physical experimental realization. Its error model is the same as our model.

In Ref[11], the seven qubit code is used and the ability of Steane's and Shor's ancilla are tested by Monte Carlo simulations. In this paper, on the other hand, not only the seven qubit code but also the five qubit code and the nine qubit code are dealt with and their effects are simulated and compared.

In Ref[18, 19], more various CSS codes (such as Golay code [[23, 1, 7]]) are dealt with. In these studies, error models are different from ours. They use gate errors instead of operational errors. Every gate is modeled by a failure followed by a perfect operation of the gate. The failure for a single-qubit gate is the same as the decoherence error and the failure for two-qubit gate is modeled as follows. With probability $1 - q$ there are no changes and with equal probabilities $\frac{q}{15}$, 15 possible one or two qubit errors occur $\{IX, IY, IZ, XI, XX, XY, XZ, YI, YX, YY, YZ, ZI, ZX, ZY, ZZ\}$. They keep track of not the quantum states but the error operators (X, Y, Z).

6 Summary

It is necessary to investigate the effects of quantum error-correction schemes in the realistic case. We have investigated the effects of QEC scheme by simulations. We confirmed that the seven qubit code scheme is really effective when decoherence rate is not more than 10^{-4} and the standard deviation of operational error is not more than 10^{-3} and the error-correction process is performed at every $50 \sim 200$ main gate even if both errors exist. We also have confirmed that transversal operations are really important. However, we have found that we cannot say that the four qubit ancilla system is better than the one qubit ancilla system.

For the future work, we will implement transversal gate for the five qubit code scheme and the nine qubit code scheme. We will investigate other CSS codes. We will adopt more complex computation as main computation,
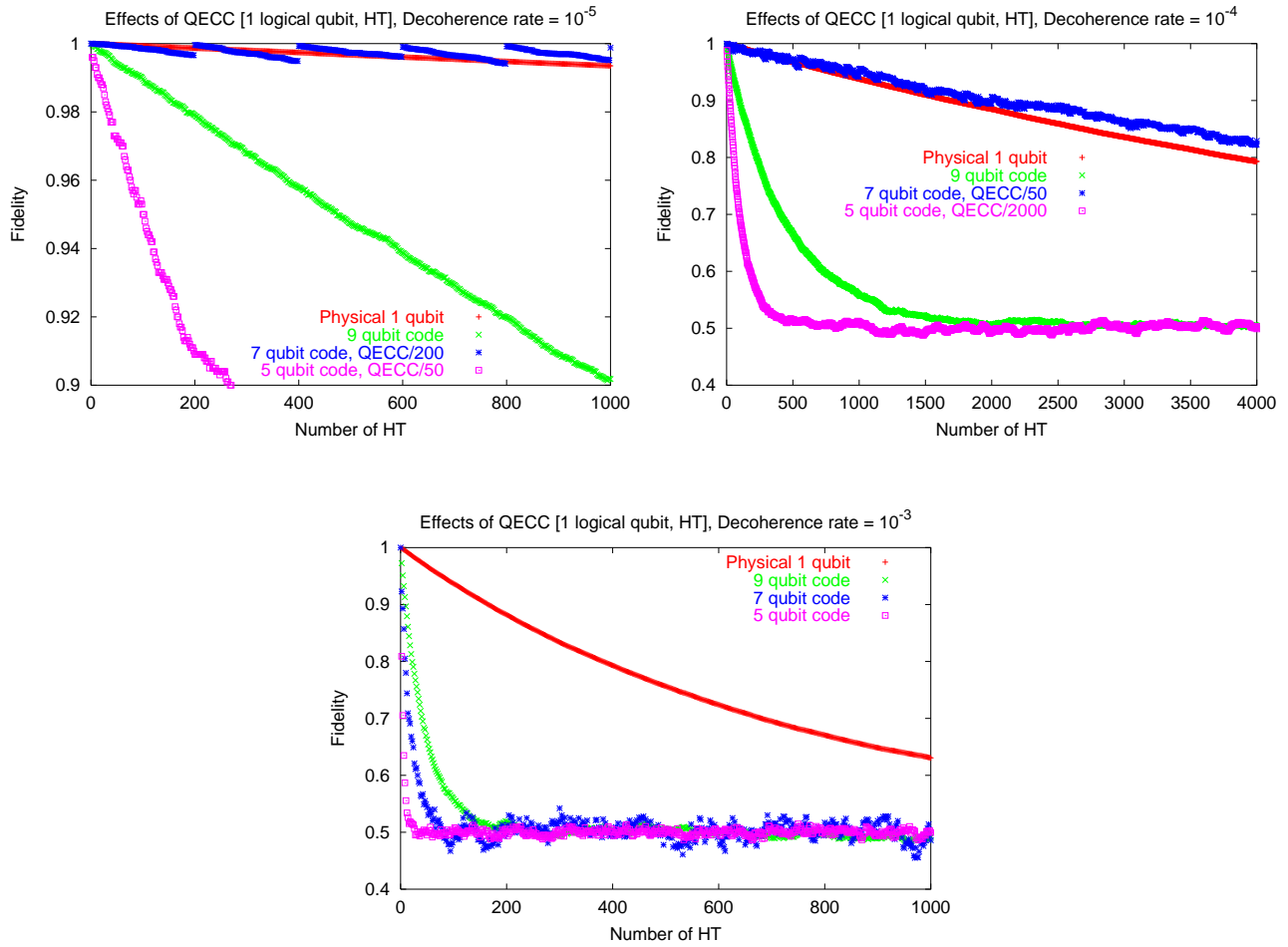
Figure 16: Fidelity with decoherence errors (10^{-5}, 10^{-4}, 10^{-3}) for nine qubit code, seven qubit code and five qubit code.

such as QFT and Grover circuit. Further investigations are required for fault-tolerant measurements with which we are not concerned in this paper and the speed of gate operations should be considered.

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