A new inverse problem for the diffusion operator

Hikmet Koyunbakan

Fırat University, Department of Mathematics, 23119, Elazığ, Turkey

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Abstract

In this work, we have estimated nodal points and nodal lengths for the diffusion operator. Furthermore, by using these new spectral parameters, we have shown that the potential function of the diffusion operator can be established uniquely. An analogous inverse problem was solved for the Sturm–Liouville problem in recent years.

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1. Introduction

Inverse problems are studied for certain special classes of ordinary differential operators. Typically, in inverse eigenvalue problems, one measures the frequencies of a vibrating system and tries to infer some physical properties of the system. An early important result in this direction, which gave vital impetus for the further development of inverse problem theory, was obtained in [1]. Inverse Sturm–Liouville problems have various versions. Of the first version, the best known case is the one studied by Gelfand and Levitan [2], in which the potential is uniquely determined by the spectral function. For the second version, Gasymov and Levitan [3] and Marchenko [4], using two spectra \( \{ \lambda_n \} \) and \( \{ \lambda'_n \} \), determined the potential \( q(x) \) uniquely and boundary conditions. Finally, Borg [5], Levinson [6] and Hochstadt [7] have shown that when the boundary condition and one possible reduced spectrum are given, then the potential is uniquely determined.

One topic in inverse problems for spectral analysis is the transmutation operator. In [8], Gilbert examined some problems for boundary value problems by using this method.


In some recent interesting work [13,14], Hald and McLaughlin, and Browne and Sleeman have taken a new approach to inverse spectral theory for the Sturm–Liouville problem. The novelty of this work lies in the use of nodal points as the given spectral data. In later years, inverse nodal problems were studied by several authors [15–17, 14].

In this work, we are concerned with the inverse problem for the diffusion operator, using a new kind of spectral data.

E-mail address: hkoyunbakan@gmail.com.

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The problem of describing the interactions between colliding particles is of fundamental interest in physics. One is interested in collisions of two spinless particles, and it is supposed that the \( s \)-wave scattering matrix and the \( s \)-wave binding energies are exactly known from collision experiments. With a radial static potential \( V(x) \) the \( s \)-wave Schrödinger equation is written as

\[
y'' + [E - V(E, x)]y = 0,
\]

where \( V(E, x) \) is the following form for the energy dependence:

\[
V(E, x) = 2\sqrt{E}P(x) + Q(x).
\]

We note that with the additional condition \( Q(x) = -P^2(x) \) the above equation reduces to the Klein–Gordon \( s \)-wave equation for a particle of zero mass and energy \( \sqrt{E} \) [18].

Before giving the main results of this work, we mention some properties of the diffusion equation.

The diffusion operator is written as

\[
Lu = -u'' + [q(x) + 2\lambda p(x)]u, \quad x \in [0, \pi]
\]

where the function \( q(x) \in W_2^m[0, \pi] \), \( p(x) \in W_2^{m+1}[0, \pi] \) \((m \geq 0)\). Let \( \lambda_0 < \lambda_1 \ldots \) be the spectrum of the problem

\[
L[u] = \lambda^2 u, \tag{1.1}
\]

\[
u'(0) + hu(0) = 0, \tag{1.2}
\]

\[
u'(\pi, \lambda) + Hu(\pi, \lambda) = 0. \tag{1.3}
\]

It is well known that the sequence \( \{\lambda_n\} \) satisfies the classical asymptotic form [19]

\[
\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1, n}}{n}
\]

and if the solution of the problem (1.1)–(1.3) is

\[
f(x, \lambda) = \cos[\lambda x - \alpha(x)] + \int_0^x A(x, t) \cos \lambda t dt + \int_0^x B(x, t) \sin \lambda t dt,
\]

where

\[
\alpha(x) = x \cdot p(0) + 2 \int_0^x [A(\xi, \xi) \sin \alpha(\xi) - B(\xi, \xi) \cos \alpha(\xi)] d\xi,
\]

then the kernels \( A(x, t) \) and \( B(x, t) \) are the solution of the problem for \( m \geq 1 \) [19]:

\[
\frac{\partial^2 A(x, t)}{\partial x^2} - 2p(x) \frac{\partial B(x, t)}{\partial t} - q(x) A(x, t) = \frac{\partial^2 A(x, t)}{\partial t^2},
\]

\[
\frac{\partial^2 B(x, t)}{\partial x^2} + 2p(x) \frac{\partial A(x, t)}{\partial t} - q(x) B(x, t) = \frac{\partial^2 B(x, t)}{\partial t^2},
\]

\[
q(x) = -p^2(x) + 2 \frac{d}{dx} [A(x, x) \cos \alpha(x) + B(x, x) \sin \alpha(x)],
\]

\[
A(0, 0) = h, \quad B(x, 0) = 0, \quad \left. \frac{\partial A(x, t)}{\partial t} \right|_{t=0} = 0,
\]

\[
\alpha(x) = \int_0^x p(t) dt.
\]

Let \( \lambda_0 < \lambda_1 < \ldots \rightarrow \infty \) be the eigenvalues of the problem (1.1)–(1.3) and \( 0 < x_1^n < \cdots < x_j^n < \pi, j = 1, 2, \ldots, n-1 \), be nodal points of the \( n \)-th eigenfunction. It is shown that the set of all nodal points \( \{x_j^n\} \) is dense in \([0, \pi]\); in fact, a judicious choice of one nodal point \( x_j^n \) for each \( y_n, n > 1 \), also forms a dense set in \([0, \pi]\). The simplest method for choosing a dense subset of the nodes is to choose all the eigenfunctions. In addition to this, we will find that the nodes for \( y_n \) are roughly equally spaced. Hence, we can choose the first node in \([0, \frac{\pi}{2}]\) and the second in \([\frac{\pi}{2}, \pi]\). The third, fourth, fifth and sixth nodes should lie in \([0, \frac{2\pi}{3}], [\frac{2\pi}{3}, \frac{\pi}{2}], [\frac{\pi}{2}, \frac{3\pi}{4}] \) and \([\frac{3\pi}{4}, \pi]\) and so on. This
method gives a dense set of nodes and it is not even necessary to choose the next node from the next eigenfunction and any finite amount of nodes can be deleted.

2. Main results

In this section, our purpose is to develop asymptotic expressions for the points \( x_j^n \) and \( l_j^n \) (\( j = 1, 2, \ldots, n-1, n = 1, 2, \ldots \)) at which \( y_n \), the eigenfunction corresponding to the eigenvalue \( \lambda_n \) of the problem (1.1)–(1.3), vanishes.

**Theorem 2.1.** We consider the equation

\[-u'' + [q(x) + 2\lambda p(x)]u = \lambda^2 u\]  

with the initial and boundary conditions

\[u'(0) - hu(0) = 0,\]
\[u'(\pi, \lambda) + Hu(\pi, \lambda) = 0.\]

Then, the nodal points of the problem (2.1)–(2.3) are

\[x_j^n = \left( j - \frac{1}{2} \right) \frac{\pi}{n} + O \left( \frac{1}{n} \right), \quad (j = 1, 2, \ldots, n-1, n = 1, 2, \ldots) \]  

and the nodal length is

\[l_j^n = \frac{\pi}{n} + O \left( \frac{1}{n} \right).\]

**Proof.** Solutions of (2.3) for \( \lambda \) are eigenvalues of the problem (2.1)–(2.3). So, the asymptotic expression for the eigenvalues is [19]

\[\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n},\]  

where

\[c_0 = \frac{1}{\pi} \int_0^\pi p(x) dx, \quad c_1 = \frac{1}{\pi} \left[ h + H + \frac{1}{2} \int_0^\pi \left( q(x) + p^2(x) \right) dx \right]. \quad \sum_n |c_{1,n}|^2 < \infty.\]

It is obtained that if the relations

\[\int_0^\infty |p(x)| dx < \infty, \quad \int_0^\infty x[|q(x)| + |p'(x)|] dx < \infty\]

hold, then (2.1) has the Jost solutions [18,19]

\[f(x, \lambda) = e^{i\alpha(x)} e^{i\lambda x} + \int_x^\infty A(x, t) e^{i\lambda t} dt.\]

Then, we can easily get [20,18]

\[f(x, \lambda) = e^{i\alpha(x)} e^{i\lambda x} + o(1), \quad |\lambda| \to \infty,\]

or

\[f(x, \lambda) = e^{i\lambda x} [1 + o(1)].\]

Hence, we use the classical estimate

\[f(x, \lambda) = e^{i\lambda x} + O(1), \quad |f(x, \lambda) - e^{i\lambda x}| \leq M,\]

where \( M \) is a constant. Thus \( f(x, \lambda) \) will vanish in the intervals whose end points are solutions to

\[e^{i\lambda x} = \pm M.\]
This equation can also be written as
\[ \cos \lambda x + i \sin \lambda x = M. \]

After some straightforward computations, we get
\[ \cos \lambda x = M, \]
then, expanding \( \arccos(M) \), we obtain that
\[ \lambda x = \left( j - \frac{1}{2} \right) \pi + \frac{M^2 + 1}{2M} + O(1), \quad (j = 1, 2, \ldots, n - 1) \]
\[ x = \frac{\left( j - \frac{1}{2} \right) \pi}{\lambda} + O \left( \frac{1}{n} \right), \]
\[ x_j^n = \frac{\left( j - \frac{1}{2} \right) \pi}{n} + O \left( \frac{1}{n} \right), \quad (j = 1, 2, \ldots, n - 1, n = 1, 2, \ldots). \] (2.6)

The nodal length is
\[ l_j^n = x_{j+1}^n - x_j^n, \]
\[ l_j^n = \left( j + 1 - \frac{1}{2} \right) \pi - \left( j - \frac{1}{2} \right) \pi + O \left( \frac{1}{n} \right), \]
\[ l_j^n = \frac{\pi}{n} + O \left( \frac{1}{n} \right). \]

This completes the proof. \( \square \)

Now, we will give a uniqueness theorem. It says that the potential function \( q(x) \) for a diffusion operator is uniquely determined by a dense subset of the nodes. We mentioned that this theorem was given for regular Sturm–Liouville problems by McLaughlin [17], Hald and McLaughlin [13], Browne and Sleeman [14].

**Theorem 2.2.** Suppose that \( q \) is integrable. Then, \( h \) and \( q - \int_0^\pi q \) are uniquely determined by any dense set of nodal points.

**Proof.** Assume that we have two problems of the type (2.1)–(2.3) with \( h, \tilde{h} \) and \( q, \tilde{q} \). Let the nodal points \( x_j^n, \tilde{x}_j^n \) satisfying \( x_j^n = \tilde{x}_j^n \) form a dense set in \([0, \pi]\). We take solutions of (2.1)–(2.3) as \( \varphi_n \) for \( (h, q) \) and \( \tilde{\varphi}_n \) for \( (\tilde{h}, \tilde{q}) \). It follows from (2.1) that
\[ (\varphi_n' \tilde{\varphi}_n - \varphi_n \tilde{\varphi}_n')' = [q - \tilde{q} + 2p(\lambda_n - \tilde{\lambda}_n) + (\tilde{\lambda}_n^2 - \lambda_n^2)]\varphi_n \tilde{\varphi}_n. \] (2.7)

Let \( x_j^n = \tilde{x}_j^n \). To show that \( h = \tilde{h} \), we integrate both sides of (2.7) from 0 to \( x_j^n \) and using the boundary conditions (2.3) and (2.4) we obtain
\[ (h - \tilde{h})\varphi_n(0)\tilde{\varphi}_n(0) = \int_0^{x_j^n} [q - \tilde{q} + 2p(\lambda_n - \tilde{\lambda}_n) + (\tilde{\lambda}_n^2 - \lambda_n^2)]\varphi_n \tilde{\varphi}_n dx. \] (2.8)

We note that \( \tilde{\lambda}_n - \lambda_n \) are uniformly bounded in \( n \) and the \( \varphi_n \tilde{\varphi}_n \) are uniformly bounded in \( n \) and \( x \in [0, \pi] \). We now select a subsequence of nodes from the dense set. If the subsequence tends to zero, then the right side of (2.8) is equal to zero. Hence we get \( h = \tilde{h} \).

To show that \( H = \tilde{H} \), we integrate both sides of (2.7) from \( x_j^n \) to \( \pi \) and select a subsequence that tends to \( \pi \). Repeating the above processes, it is obtained that \( H = \tilde{H} \). Finally, we take a sequence \( x_j^n \) accumulating at an arbitrary \( x \in [0, \pi] \) and using the above technique, since \( h = \tilde{h} \) and \( H = \tilde{H} \),
\[ \int_0^{x_j^n} (\varphi_n' \tilde{\varphi}_n - \varphi_n \tilde{\varphi}_n') dx = \int_0^{x_j^n} [q - \tilde{q} + 2p(\lambda_n - \tilde{\lambda}_n) + (\tilde{\lambda}_n^2 - \lambda_n^2)]\varphi_n \tilde{\varphi}_n dx. \]
\[ \phi_n(x) \sim \frac{1}{\sqrt{2\pi}} e^{-i\lambda_n x} + \frac{1}{\sqrt{2\pi}} e^{i\lambda_n x}, \]

From the asymptotic forms of \( \lambda_n \) and \( \phi_n \), we have

\[ 0 = \int_0^\infty \left[ q - \tilde{q} + 2p(\lambda_n - \tilde{\lambda}_n) + (\tilde{\lambda}_n^2 - \lambda_n^2) \right] \phi_n \tilde{\phi}_n \, dx. \]

We take a sequence \( x_n \) accumulating at an arbitrary \( x \in [0, \pi] \). Hence,

\[ 0 = \int_0^x \left( q - \tilde{q} - \int_0^\pi (\tilde{q} - q) \, ds \right) \phi_n \tilde{\phi}_n \, dt, \]

and this holds for all \( x \). We can therefore conclude that \( q - \int_0^\pi q(s) \, ds \) is uniquely determined by a dense set of nodes. This completes the proof. \( \square \)

**Corollary.** For the problem (2.1)–(2.3), the potential \( q \) is uniquely determined by a dense set of nodes and the constant

\[ c_1 = \pi^{-1} \left\{ h + H + \frac{1}{2} \int_0^\pi [q(x) + p^2(x)] \, dx \right\}. \]

**Proof.** Suppose that \( c_1 = \tilde{c}_1 \). Since \( h = \tilde{h} \) and \( H = \tilde{H} \), it follows that \( \int_0^\pi q = \int_0^\pi \tilde{q} \). Hence, we can conclude from Theorem 2.2 that \( q = \tilde{q} \) almost everywhere on \([0, \pi]\). \( \square \)

**References**


