EXACT ANALYTICAL SOLUTIONS OF THE WAVE FUNCTION FOR SOME $q$-DEFORMED POTENTIALS

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We derive analytical solutions of the Schrödinger wave equation for some $q$-deformed potentials. Most of these solutions are obtained in terms of Heun functions. However, for some particular cases the solutions are obtained in terms of geometric and hypergeometric functions. Different observations are realized for each potential. However, for all the cases which we study the oscillations are the usual ones. It is also noted that all of these functions are sensitive to variation in the parameters involved. Our discussion for some particular potential has shown us that for a large value of the parameter $q$ the system tends to exhibit the energy for an infinite potential well between two points zero and $a$ in addition to a free particle.

Keywords: Schrödinger equation, analytic solutions, $q$-deformed potentials.

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1. Introduction

The Schrödinger equation [1–3] is a fundamental equation in modern science. It is used to describe atomic structures [1], nuclear structures [4], nanostructures [5], chemical processes [6, 7], new aspects of information processes [8], biological processes [9] and evolution of financial markets [10]. It is well known that solutions in closed forms of the Schrödinger wave equation are possible only for a handful of potentials and in most cases one has to resort to numerical techniques or approximation schemes. In recent years several methods and techniques were developed to approximate solutions of the Schrödinger equation such as perturbation
theory, variational method, matrix methods [11] and the JWKB method [11, 12]. A closed-form solution for the Schrödinger equation has several advantages other than giving physical insight into the system. The solution can be regarded as a part of the unperturbed part of a realistic Hamiltonian [13]. Recently approximate solutions to the stationary Schrödinger equation were developed (the ERS method [14]). Approximate solutions for 1-D [14–16] and 3-D problems [17] as well as for the Schrödinger equation with mass-dependent position [18, 19] were established. Also for the problem of a time-dependent harmonic oscillator the Schrödinger wave function has been extensively used [20, 21]. Most of the previous works depended upon the nature of potential which plays an essential role to produce either an exact or an approximate solution. This situation also fits well in the problem of the Dirac equation, see for example [22–25]. In this paper we present some closed-form solutions for the stationary Schrödinger equation with particular potentials, namely with $q$-deformed hyperbolic potentials and $q$-deformed harmonic potentials. These potentials can be used to model the atom-trapping potentials in Bose–Einstein condensates or semiconductor nanostructures (such as quantum dots, nanotubes) as well as in periodic optical lattices. Furthermore the deformed hyperbolic potential has been approached in the framework of supersymmetric quantum mechanics (SUSY QM) [26]. These solutions allow us not only to determine the wave function but also to determine the energy spectrum for the modeled systems. However, this depends upon the boundary conditions which are imposed upon the system. The paper is organized as follows: Section 2 is mainly devoted to the consideration of the wave-function for $q$-deformed hyperbolic potentials. We consider some $q$-deformed harmonic potentials in Section 3. Our conclusions are given in Section 4.

2. The wave functions for deformed hyperbolic-potentials

The 1-D stationary Schrödinger equation in dimensionless units ($\hbar = 2m = 1$) with the potential $V(x)$ is given by

$$\frac{d^2\Psi(x)}{dx^2} + (E - V(x))\Psi(x) = 0. \quad (1)$$

In this section we concentrate upon introducing a closed-form analytical solution for the Schrödinger wave function for some deformed hyperbolic potentials. The $q$-deformed hyperbolic functions were introduced by Arai [27, 28] and are defined by

$$\cosh_q(x) = \frac{e^x + qe^{-x}}{2}, \quad \sinh_q(x) = \frac{e^x - qe^{-x}}{2}, \quad (2)$$

where $q > 0$ and have the property

$$\cosh^2_q(x) - \sinh^2_q(x) = q. \quad (3)$$

Although these functions are not symmetric, they have the properties of differentiation

$$\frac{d}{dx} \cosh_q(x) = \sinh_q(x), \quad \frac{d}{dx} \sinh_q(x) = \cosh_q(x). \quad (4)$$
In fact the $q$-deformed hyperbolic potential destroys the symmetry of the system and consequently the symmetry of the solution. For $q = 1$ we have the standard hyperbolic functions.

2.1. Analytical solution for a $q$-deformed cosh-potential

We first consider the potential $V(x) = \alpha \cosh_q(\gamma x)$. In order to find the wave function for this potential, we introduce the transformation $y(x) = 1 + q \exp(-2\gamma x)$. In this case the Schrödinger equation given by (1) takes the form

$$\frac{(y - 1)^2}{y^2} \frac{d^2\Psi}{dy^2} + (y - 1) \frac{d\Psi}{dy} + \frac{1}{B} \left( A - \frac{y}{\sqrt{(y - 1)}} \right) \Psi = 0,$$

which is called biconfluent Heun equation, where $A$ and $B$ are

$$A = \frac{2E}{\alpha \sqrt{q}}, \quad B = \frac{\alpha \sqrt{q}}{8\gamma^2}. \quad (6)$$

It should be noted that the Heun equation is defined as a natural generalisation of the hypergeometric equation. Consequently the Heun function is a generalisation of the hypergeometric function and therefore we can say it is a solution of the Fuchsian differential equation, for more details see [30]. When one reverts to the original coordinates, the general solution for this equation takes the form

$$\Psi(x) = Aw_1(x) + Bw_2(x), \quad (7)$$

where $A$ and $B$ are arbitrary constants which can be determined from the boundary conditions, while $w_1(x)$ and $w_2(x)$ are two linearly independent solutions. The first solution is given by the function $w_1(x)$ which takes the form [29]

$$w_1(x) = \text{HeunD}\left(0, -\left(\frac{16\gamma}{\alpha \sqrt{q}}\right)^2 (\alpha \sqrt{q} - E), -\left(\frac{16\gamma}{\alpha \sqrt{q}}\right)^2 (\alpha \sqrt{q} + E), \frac{1 + \sqrt{q}e^{-\gamma x}}{1 - \sqrt{q}e^{-\gamma x}}\right)$$

and the second solution can be determined from the integral

$$w_2(x) = \frac{w_1(x)}{q} \int \frac{1}{w^2_1(x)} \exp(2\gamma x) dx. \quad (9)$$

Eigenvalue for the present case can be obtained if one consults [29]. Determination of the energy eigenvalue depends upon the values of the constants $A$ and $B$. In quantum mechanics, no matter whether the state is bounded or not, the energy $E$ should be higher than the global minimum of the potential energy $V$, i.e. $E \geq \text{min} V$. Therefore the minimum of $q$-deformed cosh-potential is obtained for $\sinh_q(\gamma x) = 0$, in other words $\gamma x = \frac{1}{2}(\ln q)$ and so the energy $E \geq \alpha \sqrt{q}$. Note that for the case of the $q$-deformed hyperbolic potential, the parameter $q$ is always positive and real. This would guarantee that the analyzed potentials and consequently the eigenvalues are always real. However, for the $q$-deformed sinusoidal potential, the situation would depend on the structure of the potential as we see later.
Fig. 1. The wave function $\Psi(x)$ against the independent variable $x$ for the fixed values of $\alpha = 0.5$, $\gamma = 0.3$, $E = 5$ (a) for $q = 1$ (dashed) and $q = 2$ (dark solid), (b) for $q = 3$ (long dashed) and $q = 4$ (dotted). (c) as (a) but for $E = 10$.

In this context we discuss the behaviour of the wave function depending upon the solution given by equation (8). For this reason in Fig. (1) we plot $\Psi(x)$ versus $x$ for different values of the parameter $q$ and the energy $E$, whereas we fixed the values of the other parameters involved. This would help us to observe the variations which occur in the wave function for different values of $q$ and $E$. For example, we consider $q = 1, 2$ in Fig. (1a) and $q = 3, 4$ in Fig. (1b) for $E = 5$, $\alpha = 0.5$ and $\gamma = 0.3$. From these figures it is observed that an increase in the value of $q$ leads to an increase in the amplitude of the wave function. Moreover, for
the case in which $q = 1, 2$, there is a partial exchange between the wave function of each value, but for $q = 3, 4$ the wave function for each value is oscillating monotonically. On the other hand, when we increase the value of energy, $E = 10$, the same behaviour is observed for all cases and also the wave function increases its oscillations for all values of $q$. However, for $q = 1$ the wave function also increases its amplitude and shows rapid fluctuations with interference between the patterns at the end of the interval considered, see Fig. 1c. In fact, rapid fluctuations occurring in the dashed line curve indicate that there is a superposition of high and low oscillations which is due to the nature of the HeunD function.

Note that the Schrödinger equation for $q$-deformed cosh-potential can be transformed to the Schrödinger equation for $q$-deformed cos-potential by using the analytic continuation $\gamma \rightarrow i\gamma$. In this context it is interesting to point out that, for $q = 1$, the $q$-deformed cos-potential gives us the well-known Mathieu equation. This equation was considered in [31] in which a large number of specified eigenvalues and eigenvectors, where an accurate computation of Mathieu system as a multiparameter eigenvalue problem, was given. Furthermore one can see numerical experiments concerning nonstandard, high-order and singularly perturbed eigenvalue problems as was reported in [32]. Using the same procedure we can also obtain analytic solution for the $q$-deformed sinh-potential.

2.2. Analytical solution for a $q$-deformed sinh-potential

In this case we consider the potential function $V(x)$ in the form $V(x) = \alpha \sinh_q (\gamma x)$ and consequently the wave equation takes the form

$$\frac{d^2\Psi(x)}{dx^2} + (E - \alpha \sinh_q (\gamma x)) \Psi(x) = 0. \tag{10}$$

If we set $y(x) = 1 + q \exp[-2\gamma x]$, Eq. (10) becomes

$$(y - 1)^2 \frac{d^2\Psi}{dy^2} + (y - 1) \frac{d\Psi}{dy} + \frac{1}{B} \left( A - \frac{y}{\sqrt{1-y}} \right) \Psi = 0, \tag{11}$$

where $A$ and $B$ are given by Eq. (6). Thus the general solution after we revert to the original independent variable is given by

$$\Psi(x) = Cv_1(x) + Dv_2(x), \tag{12}$$

where $C$ and $D$ are arbitrary constants and

$$v_1(x) = \text{HeunD} \left( 0, \frac{4E}{y^2}, \frac{8\sqrt{q}}{y^2}, \frac{4E}{y^2}, \frac{\sqrt{q} e^{(-\gamma x)} + 1}{\sqrt{q} e^{(-\gamma x)} - 1} \right). \tag{13}$$

The second solution $v_2(x)$ can be obtained if one uses the relation between the two solutions similar to that given by Eq. (9). Note that in the case of the $q$-deformed sinh-potential, the minimum value of the potential energy occurs at infinity and consequently there is no lower bound for the energy.
To discuss the behaviour of the wave function for this case, we have plotted Fig. 2a with different values of $q$, $q = 1$ (gray line) and $q = 2$ (black line) and for a fixed value for the energy $E = 4$, while for other parameters we choose $\gamma = 0.2$ and $\alpha = 0.5$. As one can see, the wave function fluctuates between 1 and $-1$. However, as $x$ increases, the amplitude of the wave function increases and oscillates slightly above 1 and below $-1$, whereas, when we increase the value of $q$, the wave function decreases in amplitude. Furthermore, for $q = 1$ the number of oscillations is smaller than those when $q = 2$. Additionally we have examined the behaviour of the wave function for different values of $\alpha$, namely $\alpha = 0.4$ and 0.2. In this case and when $q = 3$, the function reduces the number of its fluctuations for $\alpha = 0.4$ compared with the case for $\alpha = 0.2$. Also an increase in the amplitude of the wave function is observed when $x$ is increased, and the wave function reversed its direction within the interval considered, see Fig. 2b. This means that there is a shift in the period of the wave function and consequently we can deduce that the wave function is sensitive to variation of parameters in the potential.

### 2.3. Analytical solution for a $q$-deformed tanh-potential

In order to find the analytic solution for a $q$-deformed hyperbolic tangent, we consider $V(x) = \alpha \tanh_q (\gamma x)$ and introduce the transformation $y(x) = 1 + q \exp(2\gamma x)$. In this case the wave equation can be written in the form

$$
(y - 1)^2 \frac{d^2 \Psi}{dy^2} + (y - 1) \frac{d\Psi}{dy} + \frac{1}{2\gamma^2} \left( E_1 - \frac{1}{y} \right) \Psi = 0,
$$

(14)
where \( E_1 = (E + \alpha)/2 \). The above equation is the hypergeometric equation and consequently its general solution after we revert to the original independent variable takes the form

\[
\Psi(x) = C \exp\left(-\frac{i}{2\gamma} \sqrt{E + \alpha - 2} (\ln q + 2\gamma x)\right) \\
\times _2 F_1\left(R^{(+)}, S^{(-)}, \left(1 - i \frac{\sqrt{E + \alpha - 2}}{\gamma}\right), -q \exp(2\gamma x)\right) \\
+ D \exp\left(\frac{i}{2\gamma} \sqrt{E + \alpha - 2} (\ln q + 2\gamma x)\right) \\
\times _2 F_1\left(S^{(+)}, R^{(-)}, \left(1 + i \frac{\sqrt{E + \alpha - 2}}{\gamma}\right), -q \exp(2\gamma x)\right),
\]

(15)

where \( R^{(\pm)} \) and \( S^{(\pm)} \) are complex parameters given by

\[
R^{(\pm)} = \frac{1}{2\gamma} \left[2\gamma \pm i \left(\sqrt{E + \alpha} - \sqrt{E + \alpha - 2}\right)\right], \\
S^{(\pm)} = \frac{1}{2\gamma} \left[2\gamma \pm i \left(\sqrt{E + \alpha} + \sqrt{E + \alpha - 2}\right)\right],
\]

(16)

where \( C \) and \( D \) are arbitrary constants. From the above equations it is easy to deduce that the energy eigenvalues for the present potential should be \( E \geq -\alpha \). To discuss the behaviour of the wave function we have plotted in Fig. 3 the function \( \Psi(x) \) against the variable \( x \) for different values of the parameter \( q \). For example, in Fig. 3a we have considered the case in which \( E = 1.5, \alpha = 0.4 \) and \( \gamma = 0.5 \) for two different values of \( q \). For \( q = 1 \) the function starts from a value greater than in the case of \( q = 3 \). However, in each case the function decreases in value up to \(-0.8\) and then it begins to increase its value up to 0.1. It is noted that this is faster for \( q = 3 \) than in the case for which \( q = 1 \). On the other hand, we examined the variation resulting from changing the parametric response for the strength of the potential \( \alpha \). In this case we considered the values \( \alpha = 0.4 \) and 0.5 keeping the values of other parameters unchanged except that we set \( q = 2 \). As we can see, the function with \( \alpha = 0.4 \) starts with a value higher than in the case of \( \alpha = 0.5 \) and decreases in value up to \(-0.8\), while for \( \alpha = 0.5 \) the function decreases in value just below \(-0.1\) and turns to increase its value but more slowly than for the case of \( \alpha = 0.4 \). This means that for the case in which \( \alpha = 0.4 \) the function shows us a long interval of negativity, see Fig. 3b. Finally, we discuss what variation would occur in the function as a result of changing the value of \( \gamma \), for example \( \gamma = 0.18 \) and 0.2. In this case we use the values for the parameters as in Fig. 3a but for different values of \( \gamma \), as we have mentioned. When we consider \( \gamma = 0.18 \), the function decreases in value dramatically and reaches the value \(-4\), while for \( \gamma = 0.2 \) it just reaches the value \(-1.5\) and takes a zero value at \( x = 0 \). Also for both cases the function shows oscillatory behaviour for a period longer than all the
previous mentioned cases, see Fig. 3c. Therefore we can conclude that any slight variation in the argument of the potential function leads to dramatic variation in the behaviour of the function.

2.4. Analytical solution for a $q$-deformed $\csc h^2$-potential

Here we seek a solution for a $q$-deformed potential in the form

$$ V(x) = \gamma^2 \left( 1 + 2q \csc h_q^2 (\gamma x) \right). $$

(17)
Equation of the wave function is now

\[
\frac{d^2\Psi}{dx^2} + \left(\beta^2 - 2q\gamma^2 \csc h^2_q(\gamma x)\right)\Psi = 0,
\] (18)

where \(\beta = \sqrt{E - \gamma^2}\). After some manipulations we have

\[
\Psi(x) = \mathcal{F} \left(\beta \cos \beta x - \gamma \coth_q(\gamma x) \sin \beta x\right) + \mathcal{G} \left(\beta \sin \beta x + \gamma \coth_q(\gamma x) \cos \beta x\right),
\] (19)

where \(\mathcal{F}\) and \(\mathcal{G}\) are arbitrary constants. We now discuss the energy eigenvalue for this particular case. To do this suppose that the density probability vanishes at \(x = 0\) and \(a\), in this case the bound-states of energy can be obtained from the equation

\[
\tan(\beta a) = \frac{\gamma}{\beta} \left[\left(1 + q\right) - \left(1 - q\right) \coth_q(\gamma a)\right] \left[\left(1 - q\right) + \left(1 + q\right) \left(\frac{z}{\beta}\right)^2 \coth_q(\gamma a)\right],
\] (20)

for which we are able to discuss some limiting cases. For example, when we consider the case in which \(q \to \infty\), we find that \(\beta = \frac{n\pi}{a}, n = 0, 1, 2, \ldots\) and consequently \(E_n = \gamma^2 + \left(n\pi/a\right)^2\). This result can also be obtained as a special case if we set \(q = 0\), which means that the potential \(V(x)\) is constant and equals \(\gamma^2\). In this case we can regard the energy eigenvalue as consisting of two energies one for a free particle and the other represents an infinite potential well between zero and \(a\). As one can see, the solution obtained is over-limited due to the difficulty to deal with the above equation. Therefore we have to seek a different solution \([33]\). To find the closed-form solution with the framework of the Asymptotic Iteration Method (AIM) we let \(z = q \exp(2\gamma x)\). Then Eq. (18) becomes

\[
\frac{d^2\Psi}{dz^2} + \frac{1}{z} \frac{d\Psi}{dx} - \left(\frac{\gamma^2 - E}{4\gamma^2} + \frac{2}{z(1 - z)^2}\right)\Psi = 0.
\] (21)

If we now define \(\varepsilon^2 = \left(\gamma^2 - E\right)/4\) and let \(l(l + 1) = 2\), then after some calculations we have

\[
\Psi_{nl}(z) = N z^{\varepsilon_n} \left(1 - z\right)^{l+1} \times _2F_1(-n, 2(\varepsilon_n + l + 1) + n; (2\varepsilon_n + 1) ; z),
\] (22)

where \(N\) is the normalization constant to be determined. To do this we use orthogonality of wave functions. In this case we have

\[
N^{-2} = \int_0^\infty z^{2\varepsilon_n} \left(1 - z\right)^{2l+2} \times \left|_2F_1(-n, 2(\varepsilon_n + l + 1) + n; (2\varepsilon_n + 1) ; z)\right|^2 dz.
\] (23)

After some calculation the normalization constant takes the form

\[
N = \sqrt{n! \left(2n + 2\varepsilon_n + 2l + 3\right) \Gamma(n + 2\varepsilon_n + 2l + 3)} \left(2\varepsilon_n^{2l+3}\right)^{n/2} \Gamma(n + 2\varepsilon_n + 2l + 3)^{n/2} \Gamma(n + 2l + 3).
\] (24)
When one reverts to the original independent variable, the wave function can be written in terms of Jacobi polynomials as

\[ \Psi_{nl}(x) = N (q \exp(2\gamma x))^{\varepsilon_n} (1 - q \exp(2\gamma x))^{l+1} \times \frac{n!\Gamma(2\varepsilon_n + 2)}{\Gamma(n + 2\varepsilon_n + 2)} P_n^{(2\varepsilon_n + 1, 2l)}(1 - 2q \exp(2\gamma x)). \] (25)

Now we turn our attention to \( q \)-deformations of some trigonometric functions.

3. Wave function for deformed harmonic-potentials

In this section we introduce two \( q \)-deformed potentials in terms of circular functions, namely the secant-potential and the sine-potential.

3.1. Analytical solution for a \( q \)-deformed \( \sec^2 \)-potential

We start this subsection by introducing an example of the \( q \)-deformed potential in the form

\[ V(x) = \frac{\gamma}{2} (2q \sec^2(q\gamma x) - 1). \]

Therefore the wave function in the Schrödinger picture can be written in the form

\[ \frac{d^2\Psi}{dx^2} + \left( \mu^2 - 2q\gamma^2 \sec^2(q\gamma x) \right) \Psi = 0, \] (26)

where \( \mu = \sqrt{E + \gamma^2} \). The general solution of the above equation is given by

\[ \Psi(x) = H \left( \cos(\mu x) - i \frac{\gamma}{\mu} \tan(q\gamma x) \sin(\mu x) \right) + K \left( \sin(\mu x) + i \frac{\gamma}{\mu} \tan(q\gamma x) \cos(\mu x) \right), \] (27)

where \( H \) and \( K \) are arbitrary constants. If we now assume that \( \Psi(x) = 0 \) at \( x = 0 \) and \( x = a \), we have

\[ \tan(\mu a) = \frac{\gamma}{\mu} \left[ \frac{(1 - q) - i (1 + q) \tan(q\gamma a)}{(1 + q) + i \left( \frac{\gamma}{\mu} \right)^2 (1 - q) \tan(q\gamma a)} \right]. \] (28)

For a large value of the parameter \( q \) such that \( q \to \infty \) we find \( \mu = n\pi/a, n = 0, 1, 2, \ldots \), and consequently \( E_n = (n\pi/a)^2 - \gamma^2 \). The same result can be obtained for \( q = 0 \), which means that the energy eigenvalue for a large value of \( q \) is consistent with the case in which \( q \) is absent. The interpretation given in the previous section for \( q \)-deformed \( \csc h^2 \) can also be applied in this case. As we have done above, we turn our attention to an alternate solution for this problem. For this we introduce the transformation \( z = q \exp(-2i\gamma x) \) and define \( \varepsilon = \sqrt{\mu/2\gamma} \) and \( l(l + 1) = 2 \), from which we have \( l = 1 \) or \(-2\). A straightforward calculation gives us

\[ \frac{d^2\Psi}{dz^2} + \left( \frac{l(l + 1)}{z(z + 1)^2} - \frac{\varepsilon^2}{z^2} \right) \Psi = 0, \] (29)
and consequently after some manipulations we have the general solution in the form
\[
\Psi_{nl}(x) = (q \exp(-2i\gamma x))^n (1 + q \exp(-2i\gamma x))^{l+1/2} \times _2F_1 \left[ -n, 2(\epsilon_n + l + 1) + n; (2\epsilon_n + 1); -q e^{-2i\gamma x} \right]
\]
(30)
while the energy eigenvalue is given by
\[
E_n = \gamma^2 (l + n)(l + n + 2), \quad n = 0, 1, 2, \ldots
\]
(31)

To make a comparison between our result for the \(q\)-deformed sec²—potential and that of [34] we have to set in Eq. (18) ([34]) \(C = 1, V_0 = 0, E = \hbar^2 \mu^2 / 2m, \alpha = -2i \gamma \) and \(V_1 = -4\hbar^2 q \gamma^2 / m\). In this case we can reach Eq. (26) of the present paper. Consequently, the discrete energy eigenvalues \(E_{nq}\) given by Eq. (22) of [34] becomes
\[
E_{nq} = \frac{\hbar^2 \gamma^2}{2m} (n^2 - 2n + 1),
\]
(32)
which is consistent with Eq. (31) provided \(\mu = \sqrt{\gamma^2 + E} \) and \(l = -2\). It should be noted that the existence of the \(q\)-parameter in the amplitude of the potential in Eq. (26) leads to removing this parameter from Eq. (31) and consequently to have real energy eigenvalues. Therefore we can say that the structure of the potential function plays a role for obtaining either real or complex eigenvalues.

In the following subsection we introduce the final potential in this work, which is the \(q\)-deformed sine-potential.

3.2. Analytical solution for a \(q\)-deformed sine-potential

As another example we introduce as a \(q\)-deformed potential a sinusoidal function for which the analytical solution can be obtained. This is the potential function \(V(x) = \alpha \sin_q(\gamma x) = \alpha (e^{i\gamma x} - q e^{-i\gamma x}) / 2i\). The Schrödinger wave equation in this case takes the form
\[
\frac{d^2\Psi(x)}{dx^2} + (E - \alpha \sin_q(\gamma x))\Psi(x) = 0.
\]
(33)

This equation can be obtained from Eq. (18) by using the analytic continuation \(\gamma \rightarrow i\tilde{\gamma} \) and \(\alpha \rightarrow -i\tilde{\alpha}\). Therefore the general solution can also be obtained from Eq. (13). In this case we have
\[
f_1(x) = \text{HeunD} \left( 0, -4E, \frac{8i\alpha \sqrt{q}}{\gamma^2}, \frac{4E}{\gamma^2}, \frac{\sqrt{q} e^{-i\gamma x} + 1}{\sqrt{q} e^{-i\gamma x} - 1} \right),
\]
(34)
where \(f_1(x)\) is one of the linearly independent solutions for Eq. (33). The other solution can be obtained if we use the relation between two solutions similar to that given by Eq. (3). It should be noted that in order to derive the above solution it is essential to use a transformation similar to that used for the previous cases. To obtain other \(q\)-deformed sinusoidal functions we have to use analytic continuation as in the above example. However, we emphasise that the behaviour for each function is different and this depends upon the nature of the potential itself.
4. Conclusion

In the previous sections of this work we have considered the problem of $q$-deformed potential. In this context we introduced closed-form solutions of the Schrödinger wave function for some particular cases. Some solutions for these cases are given in terms of the Heun function, while the one of $q$-deformed tanh is given in terms of the hypergeometric function as should be expected. The others are given in terms of the geometric and hyperbolic functions. By using the results obtained we have managed to discuss the behaviour of the wave function for most of the potentials considered. For instance, we have considered the $q$-deformed cosh and sinh potentials where the oscillation behaviour was characterized between both cases. However, we have observed that, as the parameter $q$ increased, the amplitude of the wave function also increased for the $q$-deformed cosh potential and decreased for the $q$-deformed sinh potential. This is consistent with the definition of the $q$-deformed hyperbolic function, see Eq. (3). Furthermore these two function fluctuate between 1 and $-1$ within the interval in which the Heun function is convergent. For the $q$-deformed tanh potential we have realized that the function is sensitive to the variation in the parameters involved. On the other hand we found that for a large value of $q$-parameter the energy eigenvalues for the $q$-deformed $\csc^2$ potential and the $q$-deformed $\sec^2$ potential tend to possess energy for an infinite potential well between zero and $a$ in addition to that for a free particle. Finally we emphasise that all the solutions which are obtained in this paper are solutions in closed form. However, some of these solutions are introduced in different forms. This depended upon the nature of the transformation which we used in order to cover certain discussions.

In fact, the consideration of different kinds of potential is one of the interesting problems in the field of quantum mechanics as we have mentioned earlier. This can be seen in the work by the authors of ref. [35] where some of the potentials such as trigonometric and hyperbolic Pöschl–Teller potentials, the Scarf potential, the Eckart potential, the Manning–Rosen potential and the Rosen–Morse potential have been considered. In the meantime, the $q$-deformed potential can be regarded as a generalization of such kind of problems. Consequently the generalization of these potentials in the light of $q$-deformed potentials is worth considering. This will be one of our future task.

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