A Probabilistic Radial Basis Function Approach for Uncertainty Quantification

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ABSTRACT

Interest in uncertainty quantification is rapidly increasing, since inherent physical variations cannot be neglected in computational modeling with increasing accuracy of the deterministic computations. In Computational Fluid Dynamics (CFD) many uncertainties can be present, such as flow parameters, boundary conditions, and the geometry. The amount of work, however, increases significantly when uncertainty quantification is applied. Since deterministic CFD can already be computationally intensive, an efficient uncertainty quantification method is required. When the number of uncertain parameters becomes large, a high dimensional response surface has to be computed. In this paper, radial basis functions (RBFs) are used since they are known to be efficient interpolants in high dimensional spaces. This Probabilistic Radial Basis Function (PRBF) approach is applied to several test problems with multiple uncertain parameters to gain some first insights. Combinations of different radial basis functions and sampling techniques are used to study the performance of different combinations. First the PRBF approach is applied to a mass-spring problem, for which the convergence of the RBFs and the influence of the sampling techniques is investigated. The second test case is the piston problem with an external forcing, for which the number of uncertain parameters is increased. The results of the PRBF approach are compared with those obtained from the Probabilistic Collocation method. Finally, the method is applied to an airfoil with uncertain free stream conditions and geometry. The flow is computed for a limited number of deterministic turbulent Navier-Stokes computations using a commercial CFD solver.

1.0 INTRODUCTION

Advanced algorithms and increasing computer power lead to accurate deterministic simulations. The physical systems that are modeled, often have variability which is neglected. There is interest in modelling this variability in complex systems, since it can influence the solution significantly. Uncertainty quantification is used to compute the probability distribution of the solution based on uncertain input parameters.

When uncertainty quantification is used in combination with existing flow solvers ideally the uncertainty quantification method is non-intrusive. A non-intrusive method requires several deterministic solves using the deterministic solver as a black-box. Efficient non-intrusive methods are for example the Probabilistic Collocation method [1, 10, 11, 16] and the Non-intrusive Polynomial Chaos method [8, 9], which are both based on the Polynomial Chaos method [7]. For multiple uncertain parameters the amount of deterministic computations grows rapidly. For the Probabilistic Collocation method the number of points is equal to \((p + 1)^d\), with \(p\) the order of the approximation and \(d\) the number of uncertain parameters. As an alternative sparse grid approaches [5, 6, 17] can be used to increase the
efficiency. For the Non-intrusive Polynomial Chaos method the number of coefficients is \( (d + p)!/d!p! \). Hosder, Walters, and Balch [8] showed that for a good Non-intrusive Polynomial Chaos approximation the amount of sampling points should be twice the number of coefficients. The polynomial chaos based methods use a global polynomial approximation of the response surface.

In this paper the response surface is approximated using Radial Basis Functions (RBFs) [2] through a limited number of support points. RBFs are used since they are known to be efficient interpolants in high dimensional spaces. The support points can be chosen by several sampling strategies. Here several combinations of different RBFs and sampling techniques are investigated to gain some first insight in the use of RBFs for response surface approximation. Recently, RBFs [2, 12, 14] became more popular for response surface approximation. Regis and Shoemaker [14] proposed a stochastic Radial Basis Function method for global optimization problems of expensive functions. They define expensive functions as function which take from minutes to several hours to evaluate. Here the focus is on CFD, where simulations take from hours to days or even weeks to compute, so the number of available support points is minimal.

The PRBF approach is applied to three test cases. The first test case is the mass-spring problem, with the spring stiffness and mass uncertain. A comparison of several commonly used RBFs and sampling methods is made based on the convergence of the mean and variance with respect to the number of samples. Secondly, the piston problem with a forcing boundary condition is employed with four uncertain parameters. The results are compared to those obtained from the Probabilistic Collocation method. The third test case is a turbulent Navier-Stokes computations around a NACA0012 airfoil with four uncertain parameters. The free stream Mach number and angle of attack are assumed to be uncertain, as well as the geometry of the airfoil. The NACA0012 airfoil is parametrized by the maximum camber, maximum camber location and relative thickness. Here the maximum camber and relative thickness are assumed to be uncertain.

This paper is organized as follows; first the PRBF approach is explained in section 2.0. Section 3.0 evaluates the RBFs and sampling methods using the mass-spring problem in section 3.1 and the piston problem in section 3.2. Next section 4.0 shows the application of the PRBF approach for a turbulent Navier-Stokes computation around a NACA0012 airfoil with four uncertain parameters. Section 5.0 provides the conclusions and the final section 6.0 gives input for further research.

2.0 PROBABILISTIC RADIAL BASIS FUNCTION APPROACH

This section introduces the RBF framework and some commonly used functions. Furthermore, the sampling techniques used to obtain the support points for the RBFs are discussed.

2.1 Radial Basis Functions

Consider a problem with \( d \) uncertain parameters \( a_1(\omega), a_2(\omega), \ldots, a_d(\omega) \). The randomness of the parameters is indicated by \( \omega \in \Omega \), which is a random event from the set of outcomes \( \Omega \). The probability space is given by \( (\Omega, \mathcal{F}, P) \), with \( \mathcal{F} \subset 2^\Omega \) the \( \sigma \)-algebra of events and \( P \) a probability measure. The parameter space is a \( d \)-dimensional probability space \( \mathbf{a}(\omega) = \{a_1(\omega), a_2(\omega), \ldots, a_d(\omega)\} \). The response surface \( u(\mathbf{a}(\omega)) \) is approximated by

\[
  u(\mathbf{a}(\omega)) = \sum_{i=1}^{N} \gamma_i \phi(\mathbf{r}_i(\omega)) = \sum_{i=1}^{N} \gamma_i \phi(\|\mathbf{a}(\omega) - \mathbf{a}(\omega_i)\|),
\]

where \( N \) is the number of support points or centers \( \mathbf{a}(\omega_i) \), and \( \phi \) the RBF, a function of the radius from the support point. Each support point corresponds to a set of parameter values \( \mathbf{a}(\omega_i) = \{a_1(\omega_i), a_2(\omega_i), \ldots, a_d(\omega_i)\} \). The radius to the support point \( \mathbf{a}(\omega_i) \) is for three uncertain parameters
(d=3) given by
\[ r_i(\omega) = \|a(\omega) - a(\omega_i)\| = \sqrt{(a_1(\omega) - a_1(\omega_i))^2 + (a_2(\omega) - a_2(\omega_i))^2 + (a_3(\omega) - a_3(\omega_i))^2}. \] (2)

The support points are obtained by sampling, which is treated in section 2.2. At the support points the solution \(f_i\) is known
\[ u(a(\omega_i)) = f_i, \quad i = 1, \ldots, N, \] (3)

The coefficients \(\gamma_i\) are obtained by solving the system
\[ M\gamma = f, \] (4)

which is obtained from equations 1 and 3. The vector \(\gamma\) contains all coefficients \(\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)^T\), the vector \(f\) are the solutions at the support points \(f = (f_1, f_2, \ldots, f_N)^T\), and \(M\) is the interpolation matrix with \(M_{ij} = \phi(\|a(\omega_i) - a(\omega_j)\|)\).

In this study globally supported RBFs are used, which means that each RBF influence the entire domain. The functions that are used are:

- Gaussian (G): \(\phi(r) = e^{-|r|^2}\), positive definite
- Thin plate spline (TPS): \(\phi(r) = r^2 \log(|r|)\), second order conditionally positive definite
- Multiquadric biharmonic (MQB): \(\phi(r) = \sqrt{c^2 + r^2}\), positive definite
- Inverse multiquadric biharmonic (IMQB): \(\phi(r) = \frac{1}{\sqrt{c^2 + r^2}}\), positive definite

The RBFs are shown in Figure 1, for a shape parameter \(c = 1\).

![Figure 1: The Gaussian (--), thin plate spline (---), multiquadric biharmonic (:-), and the inverse multiquadric biharmonic (· · ·) RBFs, with shape parameter \(c = 1\).](image)

When the interpolation matrix \(M\) is positive definite, the RBF is also called positive definite. When an RBF is not positive definite a polynomial is added to the right hand side of equation 1 to make the interpolation uniquely solvable:
\[ u(a(\omega)) = \sum_{i=1}^{N} \gamma_i \phi(\|a(\omega) - a(\omega_i)\|) + p(a(\omega)). \] (5)
For example the thin plate spline RBF is second order conditionally positive definite. A \( k \)th order conditionally positive definite RBF requires the addition of a polynomial of order \( k - 1 \). Therefore, the thin plate spline RBF requires an additional linear polynomial. For three uncertain parameters \((d=3)\) this is equal to

\[
p(\mathbf{a}(\omega)) = \beta_1 + \beta_2 a_1(\omega) + \beta_3 a_2(\omega) + \beta_4 a_3(\omega).
\]  

(6)

The system that has to be solved to obtain the coefficients is:

\[
\begin{bmatrix}
M & P \\
P^T & 0
\end{bmatrix}
\begin{bmatrix}
\gamma \\
\beta
\end{bmatrix}
= \begin{bmatrix}
f
\end{bmatrix},
\]  

(7)

with \( P_i = (a_1(\omega_i) \ a_2(\omega_i) \ a_3(\omega_i)) \), \( i = 1, \ldots, N \) the \( i \)-th row of \( P \) for the three dimensional case.

### 2.2 Sampling of the support points

The following sampling techniques [3, 4, 13, 15] are considered:

- Random sampling: \( N \) samples are taken randomly in \( \Omega \);
- Latin Hypercube sampling (LHS): \( N \) samples are taken random in \( N \) volumes in \( \Omega \) of equal probability;
- Centroidal voronoi tessellation (CVT): The probability space is divided into \( N \) volumes in \( \Omega \) of equal probability. The \( N \) centroids of the volume are the samples;
- Halton sampling: the \( N \) samples are the first \( N \) values of the Halton sequence;
- Hammersley sampling: the \( N \) samples are the first \( N \) values of the Hammersley sequence.

Figure 2 shows 100 samples for two random variables using the mentioned sampling techniques. It can be seen that the deterministic sampling techniques provide a more homogeneous coverage than the random sampling and latin hypercube sampling. The sampling of the support points is done in the domain \( \Omega \in [0,1]^d \), where \( d \) is the number of uncertain parameters. The corresponding parameter values for each support point \( \mathbf{a}(\omega_i) = \{a_1(\omega_i), a_2(\omega_i), \ldots, a_d(\omega_i)\} \) are obtained by the probability distribution functions of the uncertain parameters.
3.0 EVALUATION OF THE RADIAL BASIS FUNCTIONS AND SAMPLING TECHNIQUES

The PRBF approach using different combinations of RBFs and sampling techniques is discussed in the next sections using the following relatively simple test cases: the mass-spring problem with two uncertain parameters and the piston problem with four uncertain parameters.

3.1 Mass-spring problem

The mass-spring configuration indicated by Figure 3 is used to investigate the convergence of the RBFs and sampling methods. Consider a mass \( m \) mounted on a spring of length \( L \) in equilibrium with stiffness \( k \). The base movement is prescribed by the function \( x_{\text{base}}(t) \). The governing equations and initial conditions for this system are

\[
\begin{align*}
m\ddot{x} + kx &= kx_{\text{base}}, & t > 0 \\
x(0) &= x_0 \\
\dot{x}(0) &= 0,
\end{align*}
\]

where \( x_0 \) is the initial position of the mass. It is assumed that the base is harmonically excited by \( x_{\text{base}}(t) = A\sin(\omega t) \), with \( A = 0.1 \) and \( \omega = 0.5 \). For this excitation the analytical solution is given by

\[
x(t) = x_0 \cos(\omega_n t) - \frac{\omega_n \omega A}{\omega_n^2 - \omega^2} \sin(\omega_n t) + \frac{\omega_n^2 A}{\omega_n^2 - \omega^2} \sin(\omega t),
\]

in which \( \omega_n = \sqrt{\frac{k}{m}} \) is the natural frequency of the system. Figure 3(b) shows three deterministic solutions for \( \omega_n = 0.9, 1, \) and \( 1.1 \), it can be seen that a variation in \( \omega_n \) leads to significant variations of the mass position \( x \). The mass \( m \) and spring stiffness \( k \) are assumed to be uncertain, both with mean \( \mu_k = \mu_m = 1 \) and a coefficient of variation of \( CV = \mu/\sigma = 0.1 \).

![Figure 3: The mass-spring test problem with (a) the configuration and (b) deterministic solutions for values of \( \omega_n^2 = 0.9, 1 \) and 1.1.](image)

Figure 4(a) and (b) shows the convergence of the relative error of the mean and variance of the mass position at \( t = 10 \) for uniformly distributed \( m \) and \( k \) using cvt sampling. The relative error of the mean and variance are defined as

\[
\epsilon_\mu = \left| \frac{\mu_x(t) - \mu_{x_{\text{reference}}}(t)}{\mu_{x_{\text{reference}}}(t)} \right|, \quad \epsilon_\sigma^2 = \left| \frac{\sigma_x^2(t) - \sigma_{x_{\text{reference}}}(t)}{\sigma_{x_{\text{reference}}}(t)} \right|,
\]

(12)
Figure 4: Error convergence of the different RBFs for the mean (a) and variance (b) of the mass position at $t = 10$ resulting from uncertain mass $m$ and spring stiffness $k$ using cvt sampling and (c) the cumulative distribution function of the mass position at $t = 10$ obtained using the PRBF approach ($-$) using 16 CVT samples with the Gaussian RBF and Monte Carlo simulation ($- -$) with 1,000,000 samples.

Figure 5: Error convergence of the mean of the mass position at $t = 10$ resulting from uncertain mass $m$ and spring stiffness $k$ using the Gaussian RBF using random (a), latin hypercube (b), centroidal voronoi tessellation (c), Halton (d) and Hammersley (e) sampling.

with the reference solution $x_{\text{reference}}(t)$ obtained from a Monte Carlo simulation using 1,000,000 samples. The only RBF that performs worse than the others is the thin plate spline (TPS). The Gaussian RBF shows the smoothest decrease of the error compared to the (inverse) multiquadric biharmonic RBFs. The cumulative distribution function $F_x(x)$ of the mass position $x$ at $t = 10$ is given in Figure 4(c). The distribution function is obtained using 16 CVT samples and the Gaussian RBF, the result shows good correspondence with the Monte Carlo simulation using 1,000,000 samples.

Advantageous properties of sampling techniques are:

- the convergence of the mean and variance of the solution;
- the uniformity of the distribution of the samples in probability space $\Omega$;
- the ability to reuse earlier samples when the number of samples is increased.

The effect of the sampling of the support points on the convergence of the mean is shown in Figure 5. The Gaussian RBF is used for the response surface approximation. No significant difference exists between the sampling techniques. The error convergence of the variance shows similar results, only the CVT and Halton sampling show a smooth decrease of the error.

A uniform distribution of the samples in probability space is important for the stability of the RBFs. When two samples are close together the resulting system of equations for the coefficients of the RBFs becomes nearly singular. The most homogeneous distribution of the samples in probability
space is obtained by the CVT sampling, also in higher dimensions. Also the Halton and Hammersley sampling result in a uniformly covered space. Random sampling and the latin hypercube sampling result in a less homogeneous distribution.

In this paper, the Halton sampling is chosen, based on the property that samples can be reused when the accuracy is expected to be not sufficient. The convergence of the PRBF approach can be tracked by monitoring the change in mean and variance when a sample is added. Furthermore, the solution of the added sample can be predicted based on the RBF approximation of the previously computed samples. The difference between the prediction and the computed solution is an indication for the accuracy of the response surface approximation. The random sampling has the same property, however, the spread of the samples through the $d$-dimensional space is less favorable based on the uniformity of the samples in probability space. The other three sampling methods, Hammersley sampling, Latin hypercube and centroidal voronoi tessellations, recompute the sampling grid when more samples are required. This means that either earlier computed results cannot be reused or that the uniform distribution of the samples in probability space is not maintained.

3.2 The piston problem

The next test problem is the piston problem, which consist of a mass-spring system and a linearized Euler fluid domain. The results of the PRBF approach for four uncertain parameters are compared with results obtained with the Probabilistic Collocation method. The linear piston problem is used with a moving wall at the left boundary, see Figure 6. The forcing is assumed to be harmonic: $f(t) = A\sin(\omega t)$, with $A$ the amplitude and $\omega$ the frequency. At $t = 0$ the fluid and piston are at rest.

The mass, spring stiffness, forcing amplitude and forcing frequency are assumed to be uncertain, and uniformly distributed with a coefficient of variation of 10%. The mean values of the parameters are set to $\mu_k = 1$, $\mu_m = 1$, $\mu_A = 0.1$, and $\mu_\omega = 0.8$.

![Figure 6: Linear piston with an unsteady forcing $f(t)$ at the left wall.](image)

The results presented in Table 1 show the mean and standard deviation of the piston position at $t = 3$ obtained from the PRBF approach and the Probabilistic Collocation method. An advantage of the PRBF approach is the fact that the number of support points (samples) can be chosen to be any number. The Probabilistic Collocation method, on the other hand has a fixed number of samples: a first order approximation requires 16 computations or a second order approximation needs 81 computations. When the number of uncertain parameters increases the Probabilistic Collocation method rapidly requires more computations ($N_p = (p + 1)^d$), which means that often even a second order approximation is already too expensive.

Here the multiquadric biharmonic RBF is used with support points from a 4-dimensional Halton sequence, the Gaussian and inverse multiquadric biharmonic RBFs show similar results. It can be seen that the Probabilistic Collocation method has superior convergence properties, since a low order approximation results in a high accuracy. However, the amount of collocation points is fixed to 16, 81, 256, or 625. The PRBF approach is within 1% accuracy from 15 support points, which is sufficient for engineering applications. Figure 7 shows the probability density function and cumulative distribution function of the piston position $q$ at $t = 3$. 

Table 1: The mean and standard deviation of the piston position $q$ at time $t = 3$ obtained from the Probabilistic Radial Basis Function approach and the Probabilistic Collocation method.

<table>
<thead>
<tr>
<th>Number of samples</th>
<th>Mean $\mu_q$</th>
<th>Standard deviation $\sigma_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.07276 (-0.08%)</td>
<td>0.008538 (+4.16%)</td>
</tr>
<tr>
<td>15</td>
<td>0.07284 (+0.02%)</td>
<td>0.008155 (-0.59%)</td>
</tr>
<tr>
<td>25</td>
<td>0.07286 (+0.05%)</td>
<td>0.008264 (+0.74%)</td>
</tr>
<tr>
<td>50</td>
<td>0.07282</td>
<td>0.008203</td>
</tr>
<tr>
<td>100</td>
<td>0.07284 (+0.02%)</td>
<td>0.008203</td>
</tr>
<tr>
<td>150</td>
<td>0.07281 (-0.01%)</td>
<td>0.008203</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Polynomial order $p$</th>
<th>Number of points $N_p$</th>
<th>Mean $\mu_q$</th>
<th>Standard deviation $\sigma_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>0.07282</td>
<td>0.008118 (-1.04%)</td>
</tr>
<tr>
<td>2</td>
<td>81</td>
<td>0.07282</td>
<td>0.008203</td>
</tr>
<tr>
<td>3</td>
<td>256</td>
<td>0.07282</td>
<td>0.008203</td>
</tr>
<tr>
<td>4</td>
<td>625</td>
<td>0.07282</td>
<td>0.008203</td>
</tr>
</tbody>
</table>

Figure 7: The probability density function (a) and the cumulative distribution function (b) of the piston position $q$ at $t = 3$ with four uniformly distributed input uncertain parameters. The blue results are obtained using a 2nd order probabilistic collocation computation using 81 samples and the red results are from the PRBF approach using 50 samples.

4.0 APPLICATION TO A NACA0012 AIRFOIL

To demonstrate the PRBF approach for a more realistic test problem, it is applied to steady flow around a NACA0012 airfoil with uncertain Mach number, angle of attack, thickness and maximum camber using a commercial deterministic CFD code. The deterministic case is a flow around a NACA0012 airfoil at an angle of attack of 5 degrees and a free stream Mach number of 0.3. The Reynolds number is equal to $3 \cdot 10^6$. The deterministic computations are performed using the Fine\textsuperscript{TM}/Hexa solver by Numeca Int. on a grid of 80,000 cells. The grid layout is shown in Figure 8.
The flow is modeled by the Reynolds-Averaged Navier-Stokes equations using the Spalart-Allmaras turbulence model. The air properties are at 0m ISA. The uncertainties present in the free stream flow conditions are: the Mach number, angle of attack, and the geometry. All parameters are assumed to have a truncated Gaussian distribution. The geometry of the airfoil is parametrized according to the NACA 4-digit airfoil series, uncertain are the thickness and maximum camber. The free stream Mach number has a mean $\mu_M = 0.3$ and a standard deviation $\sigma_M = 0.03$, on the interval $[0.23, 0.37]$. The mean angle of attack is $\mu_\alpha = 5^\circ$ with a standard deviation $\sigma_\alpha = 0.50^\circ$, in the interval $[3.84^\circ, 6.16^\circ]$. The geometric parameters that represent the uncertainty are the thickness of the airfoil in percents with mean $\mu_t = 12\%$, a standard deviation $\sigma_t = 0.425\%$ and truncated to the interval $[11.02\%, 12.98\%]$ and the maximum camber in percents of the chord, which has mean $\mu_c = 0\%$, standard deviation $\sigma_c = 0.4472$ and is truncated in the interval $[-1.04\%, 1.04\%]$. The uncertainty is propagated using the Probabilistic Radial Basis Function approach, with 35 support points obtained with Halton sampling. The flow solver is run deterministically for every sample. Figure 9(a) shows the pressure field of the mean conditions. Figures 9(b) till (d) show the pressure fields of three samples. Figure 10 shows the convergence of the lift-over-drag ratio with respect to the number of samples for different values of the shape parameter $c$ using the Gaussian RBF. A larger shape parameters results in more localized RBFs, the figure shows that with only 35 samples a global RBF results in a better approximation of the mean $L/D$. The mean $L/D$ converges to 42.17, which is 1% lower than the deterministic $L/D$ of 42.59. The standard deviation becomes $\sigma_{L/D} = 4.051$, which results in a coefficient of variation of $CV_{L/D} = (\mu/\sigma)_{L/D} = 9.6\%$.

The pressure on the airfoil surface is presented in Figure 11. The mean pressure is shown with uncertainty bars indicating the area of plus and minus one standard deviation. It can be seen that the uncertain parameters result in the largest variation in the pressure on the upper part of the airfoil and mainly near the leading edge.

5.0 CONCLUSIONS

Radial basis function interpolation combined with sampling techniques for non-intrusive uncertainty quantification have been studied in this paper. For a low number of uncertain parameters and sufficiently smooth responses the polynomial chaos based methods are superior due to their exponential convergence. The convergence study using the mass-spring problem shows that the Gaussian and (inverse) Multiquadric biharmonic RBFs show the best convergence for the mean and variance of the mass position using CVT sampling. The sampling techniques are compared on three criteria:
Figure 9: Pressure field of the mean conditions (a) and three samples (b), (c), and (d).

(a) $M = 0.3, \alpha = 5^\circ, t = 12\%, c = 0\%$

(b) $M = 0.261, \alpha = 4.73^\circ, t = 12.7\%, c = -0.01\%$

(c) $M = 0.330, \alpha = 3.88^\circ, t = 11.9\%, c = 0.62\%$

(d) $M = 0.322, \alpha = 5.64^\circ, t = 11.4\%, c = -0.57\%$

Figure 10: The convergence of the mean lift-over-drag ratio with respect to the number of samples for varying shape parameter $c$ using the Gaussian RBF. The deterministic value is indicated by the dashed red line ($\cdots$).

Figure 11: The mean pressure ($-$) along the surface of the airfoil with the bars indicating the standard deviation of the pressure, obtained using the Gaussian RBF with shape parameter $c = 0.25$. 
(1) convergence of the error of the mean and variance of the solution; (2) uniformity of the samples distribution in probability space; (3) the ability of adding an extra sample. The investigated sampling methods show no significant differences in convergence. The CVT sampling produces the most uniformly distributed samples, also the Halton and Hammersley sampling provide a good sample distribution. The Halton sampling is chosen for the other test cases, since it has the property that samples can be added later without loosing the good spatial distribution of the samples.

From the piston problem with unsteady forcing the advantage of sampling support points for the PRBF approach becomes clear. For the Probabilistic Collocation method 16, 81, 256, or 625 deterministic solves have to be performed when four uncertain parameters are considered. The PRBF approach provides accurate results using 50 support points.

Finally, the PRBF approach is applied to a NACA0012 airfoil with uncertain camber and thickness and the freestream Mach number and angle of attack. To propagate the four uncertain parameters, 35 samples are used obtained from the 4-dimensional Halton sequence. For the approximation of the response surface the Gaussian RBF is used with different values for the shape parameter. By monitoring the change in the mean the convergence can be tracked. The shape parameter influences the width of the RBF, which means that a larger shape parameter results in a more localized RBF. For this low number of support points, a more global RBF ($c = 0.25$) results in a better approximation.

### 6.0 FURTHER RESEARCH

This preliminary study to the application of radial basis functions (RBFs) to response surface approximations shows that there is a good potential, especially when the number of uncertain parameters becomes large. However, there are some unanswered questions which need further research: (1) A systematic investigation of the RBFs applied to problems with different types of responses should be conducted in order to obtain a better understanding for which problems the RBFs will provide a good response surface approximation. Typical response surfaces that occur in engineering problems are smooth responses, responses with high gradients or responses with a bifurcation; (2) The motivation to use RBFs was the well-known efficiency of high-dimensional interpolation. More uncertain parameters should be taken into account, this study was limited to four parameters; (3) This study compared commonly used RBFs with global support. By adjusting the shape parameter the RBF is acting more globally or locally around the support point. There is also a class of compactly supported RBFs, which only act inside a support radius around the support point. It is interesting to study these RBFs for response surface approximations as well; (4) Finally, in this paper, the new samples are added to the Halton samples by computing the additional values from the corresponding Halton sequence. It is also possible to start with a certain sized sampling grid and use the available information to compute good new samples.

### 7.0 ACKNOWLEDGMENTS

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### REFERENCES


Question: (1) Have you looked at approaches for getting past the ill-conditioned matrix problem as C is reduced for smooth surfaces? (2) Have you evaluated the optimizing of C with respect to the smoothness/non-smoothness of the response surface?

Author’s Reply: (1) Literature also showed that there is an optimum for the value of C which depends on the number of support points and the function that is approximated. There is no way to avoid the system to be ill-conditioned when C becomes very low (for a Gaussian RBF). There are, however, ways of estimating a good value for C. Those methods are very cheap (computationally) compared to the CFD computation for the support points. (2) In this study it was assumed that the solution value of the samples is exact. Of course, in real-life problems the solutions suffer from all kinds of errors, e.g. numerical or modeling errors. Perhaps a least-squares approach can help here to prevent oscillatory behavior of the approximation. Interesting would be to add noise to the analytical test problems and see how the polynomial chaos method and the probabilistic radial basis function approach react on that.

Discusser’s Name: A. Cunningham

Question: Have you considered least-squares methods to reduce sensitivity of the “C” parameter for randomly spaced data in your uncertainty models (or inputs)?

Author’s Reply: I have not considered it, but it is a good idea to take it into account in further research. The presented method utilizes sampled points. By using a deterministic sampling, like CVT, you are sure that the samples are uniformly distributed in the parameter space. The sensitivity of the solution with respect to “C” is inherently present, but methods exist to obtain a good estimate for a proper value for “C”. Those methods will be included in further research.