Finite trees as ordinals

Herman Ruge Jervell

University of Oslo

Abstract. Finite trees are given a well ordering in such a way that there is a 1-1 correspondence between finite trees and an initial segment of the ordinals. The ordinal ϵ_0 is the supremum of all binary trees. We get the (fixpoint free) n-ary Veblen hierarchy as tree functions and the supremum of all trees is the small Veblen ordinal $\phi_{\Omega^\omega}(0)$.

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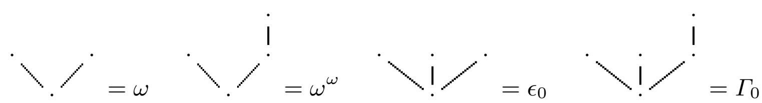
We work with ordinary finite trees with immediate subtrees ordered sequentially from left to right and with no labels at the nodes. Here we just call them trees. The smallest tree is $\cdot$ and we draw the trees with the root at the bottom. Here are four examples where we have indicated the corresponding ordinals in the ordering defined below



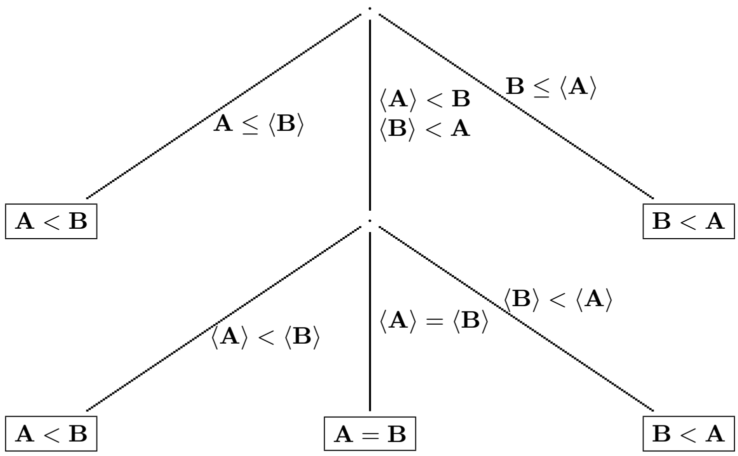
We write $\langle \mathbf{A} \rangle$ for the finite sequence of immediate subtrees of the tree $\mathbf{A}$, and $\langle \cdot \rangle$ is the empty sequence. Equality between trees is the usual equality. Given that we already know the ordering of some trees we let

- $\mathbf{A} \leq \langle \mathbf{B} \rangle$ **:** There is an immediate subtree $\mathbf{B}_i$ of $\mathbf{B}$ such that either $\mathbf{A} < \mathbf{B}_i$ or $\mathbf{A} = \mathbf{B}_i$
- $\langle \mathbf{A} \rangle < \mathbf{B}$ **:** For all immediate subtrees $\mathbf{A}_j$ of $\mathbf{A}$ we have $\mathbf{A}_j < \mathbf{B}$
- $\langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle$ **:** The inverse lexicographical ordering of the immediate subtrees — we first check which sequence have smallest length, and if they have equal length we look at the rightmost immediate subtree where they differ

We are now ready to define the ordering of trees by recursion over the immediate subtrees.

$$\boxed{\mathbf{A} < \mathbf{B} \;\Leftrightarrow\; \mathbf{A} \leq \langle \mathbf{B} \rangle \;\vee\; (\langle \mathbf{A} \rangle < \mathbf{B} \wedge \langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle)}$$

We must prove that this defines an ordering. The following decision tree shows (by induction) that we have a total relation



To get that this total relation is a total ordering we prove that it is transitive. As before the argument is by induction over the heights of the trees. So assume we have

$$\mathbf{A} < \mathbf{B} < \mathbf{C}$$

and want to prove by induction over the sum of heights of the three trees that we get $\mathbf{A} < \mathbf{C}$. This is done by cases

- $\mathbf{B} \leq \langle\mathbf{C}\rangle$: Then $\mathbf{A} < \mathbf{B} \leq \langle\mathbf{C}\rangle$, and by induction $\mathbf{A} \leq \langle\mathbf{C}\rangle$ and $\mathbf{A} < \mathbf{C}$
- $\langle\mathbf{B}\rangle < \langle\mathbf{C}\rangle$ and $\langle\mathbf{B}\rangle < \mathbf{C}$: Then
  - $\mathbf{A} \leq \langle\mathbf{B}\rangle$: Then $\mathbf{A} \leq \langle\mathbf{B}\rangle < \mathbf{C}$ and by induction $\mathbf{A} < \mathbf{C}$
  - $\langle\mathbf{A}\rangle < \langle\mathbf{B}\rangle$ and $\langle\mathbf{A}\rangle < \mathbf{B}$: Then we have $\mathbf{A} < \mathbf{C}$ from
    - $\langle\mathbf{A}\rangle < \langle\mathbf{B}\rangle < \langle\mathbf{C}\rangle$ which gives by induction $\langle\mathbf{A}\rangle < \langle\mathbf{C}\rangle$
    - $\langle\mathbf{A}\rangle < \mathbf{B} < \mathbf{C}$ which gives by induction $\langle\mathbf{A}\rangle < \mathbf{C}$

Theorem 1. *The ordering between finite trees is a total ordering where the equality is the usual equality between trees.*

By induction over the build up of trees we prove

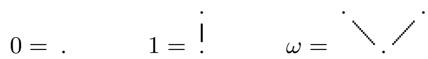
Theorem 2. *Let $\boldsymbol{T}(x)$ be a tree where x indicates a place where we can substitute trees. Then*

$$\boldsymbol{A} < \boldsymbol{B} \ \Leftrightarrow\ \boldsymbol{T}(\boldsymbol{A}) < \boldsymbol{T}(\boldsymbol{B})$$

Furthermore by induction over the build up

Theorem 3. *If tree $\boldsymbol{S}$ can be embedded in tree $\boldsymbol{T}$, then $\boldsymbol{S} \leq \boldsymbol{T}$.*

We note here that we have the following:



But we need to develop more of the theory to come to these and other calculations. First we give approximations of a tree from below. Given a tree **A** with immediate subtrees

The immediate subtrees a_i of **A** are smaller.
Now assume

- $b_l < a_l$
- $c_i <$ **A** for all $i < l$

Then the following tree is less than **A**

This can be rephrased that for $b_l < a_l$ the function which to $x_0, \ldots, x_{\ell-1}$ gives

is closed under **A**. We also get that for $s < p$ that **A** is closed under the function which to $x_0, \ldots, x_s$ gives

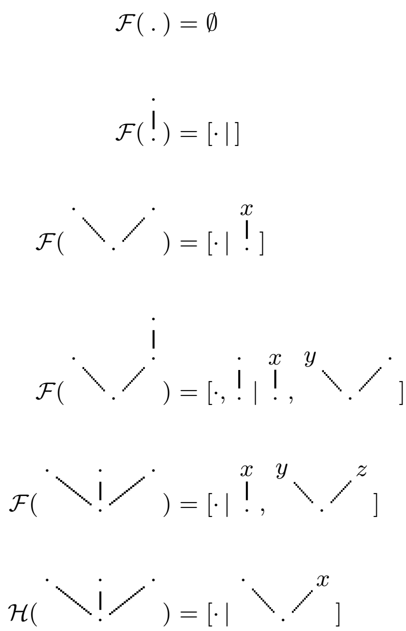
In the usual theory of ordinal notations we use fundamental sequences as a way to approach ordinals from below. [2] For a given tree **A** we call

Fundamental subtrees of A : The immediate subtrees of **A**.
Fundamental functions of A : The two types of functions above.
Fundamental set of A : The set of trees generated by the fundamental functions starting with the fundamental subtrees.
Elementary fundamental function of A : We first get unary functions by letting all variables except the rightmost be 0. Then use all such unary functions of the first type. If there are no functions of the first type use the one of the second type with largest branching.
Elementary fundamental set of A : The set of trees generated by the elementary fundamental functions starting with the fundamental subtrees.

We denote the fundamental set of $\mathbf{A}$ with $\mathcal{F}(\mathbf{A})$ and we shall write it as

$$[S, \ldots, T | F, \ldots, G]$$

where we have displayed the fundamental subtrees $S, \ldots, T$ and the fundamental functions $F, \ldots, G$. Similarly for the elementary fundamental set $\mathcal{H}(\mathbf{A})$
We have the following:



Here x, y, z are variables used for describing fundamental functions.
The following theorem shows the importance of the fundamental sets.

Theorem 4. *For any tree $\boldsymbol{A}$:*

$$\boldsymbol{B} < \boldsymbol{A} \Leftrightarrow \exists \boldsymbol{C} \in \mathcal{F}(\boldsymbol{A}). \boldsymbol{C} \geq \boldsymbol{B}$$

We prove this by induction over the height of $\mathbf{B}$. It is trivial for height 0. So assume it proved for smaller heights than the height of $\mathbf{B}$. The direction $\Leftarrow$ is obvious. We assume $\mathbf{B} < \mathbf{A}$ and divide up into cases:

$\mathbf{B} \leq \langle \mathbf{A} \rangle$ **:** But then $\mathbf{B}$ is less than or equal to one of the fundamental subtrees of $\mathbf{A}$.

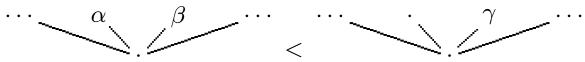
$\langle \mathbf{B} \rangle < \mathbf{A} \wedge \langle \mathbf{B} \rangle < \langle \mathbf{A} \rangle$ **:** By induction — to each immediate subtree $\mathbf{B}_i$ there is an $\mathbf{C}_i \in \mathcal{F}(\mathbf{A})$ with $\mathbf{C}_i \geq \mathbf{B}_i$. Depending on how we prove $\langle \mathbf{B} \rangle < \langle \mathbf{A} \rangle$ we get a fundamental function which we can apply to some of the $\mathbf{C}_i$'s to get a $\mathbf{C} \in \mathcal{F}(\mathbf{A})$ with $\mathbf{C} \geq \mathbf{B}$

And the theorem is proved. We can also use the elementary fundamental set

Theorem 5. *For any tree* $\boldsymbol{A}$*:*

$$\boldsymbol{B} < \boldsymbol{A} \Leftrightarrow \exists \boldsymbol{C} \in \mathcal{H}(\boldsymbol{A}).\boldsymbol{C} \geq \boldsymbol{B}$$

We only need to note that



where $\gamma > \max(\alpha, \beta)$ and that the result of of an application of the second type of fundamental function can be embedded into an application of the first type.
We have also

Theorem 6. *Assume that all trees less than or equal to the fundamental subtrees of* $\boldsymbol{A}$ *is contained in* $\mathcal{F}(\boldsymbol{A})$*, then*

$$\boldsymbol{B} < \boldsymbol{A} \Leftrightarrow \boldsymbol{B} \in \mathcal{F}(\boldsymbol{A})$$

The proof follows the lines above. We have induction over the height of $\mathbf{B}$ and get to the cases

$\mathbf{B} \leq \langle \mathbf{A} \rangle$ **:** Then by assumption $\mathbf{B} \in \mathcal{F}(\mathbf{A})$.

$\langle \mathbf{B} \rangle < \mathbf{A} \wedge \langle \mathbf{B} \rangle < \langle \mathbf{A} \rangle$ **:** By induction — for each immediate subtree $\mathbf{B}_i$ we have $\mathbf{B}_i \in \mathcal{F}(\mathbf{A})$. Depending on how we prove $\langle \mathbf{B} \rangle < \langle \mathbf{A} \rangle$ we get a fundamental function which we can apply to the $\mathbf{B}_i$'s to get $\mathbf{B} \in \mathcal{F}(\mathbf{A})$.

We call a fundamental set which is an initial segment of the ordinals for *full*. The fundamental sets mentioned above are full.

Theorem 7. $\overset{\alpha}{\underset{.}{\mid}} = \alpha + 1$

We can prove this by simple induction over trees. We prove

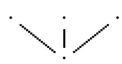
$$\beta < \overset{\alpha}{\underset{.}{\mid}} \Leftrightarrow \beta \leq \alpha$$

or we can use that the ordering is a wellordering and then induction over α noting that

$$\mathcal{F}(\overset{\alpha}{\underset{.}{\mid}}) = [\alpha \,|\, \overset{\alpha-}{\underset{.}{\mid}}\ ]$$

We are now getting a clearer picture of the ordering. The trees can be divided into layers — we let $\mathcal{T}_i$ be the trees with at most i-branchings. We then get that $\mathcal{T}_1$ is majorised by

and this tree is the least in $\mathcal{T} - \mathcal{T}_1$. The $\mathcal{T}_2$ is majorised by



and this tree is the least in $\mathcal{T} - \mathcal{T}_2$. The $\mathcal{T}_3$ are majorised by

and this tree is the least in $\mathcal{T} - \mathcal{T}_3$. And so on.

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So far the properties could be proved by simple inductions over the heights. Below we shall prove that the trees are well ordered. This requires a stronger method of proof. Let $\mathcal{T}$ be the initial segment of trees which are well ordered. We then prove

Theorem 8. *Assume $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_k$ are well ordered. Then so is $\boldsymbol{S}$ given by*

Let $\mathbf{A} < \mathbf{S}$ be as low as possible and not well ordered. We shall show that $\mathbf{A}$ is well ordered. This is done by the inverse lexicographical ordering of $\mathbf{S}_1, \ldots, \mathbf{S}_k$ where the field of the inverse lexicographical ordering is the set $\mathcal{T}$ of well ordered trees. We now have the following cases

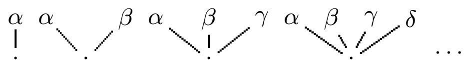
$\mathbf{A} \leq \langle \mathbf{S} \rangle$ **:** Then $\mathbf{A} \leq \mathbf{S}_i$ and is therefore well ordered.

$\langle \mathbf{A} \rangle < \mathbf{S} \wedge \langle \mathbf{A} \rangle < \langle \mathbf{S} \rangle$ **:** Then all immediate subtrees of $\mathbf{A}$ are less than $\mathbf{S}$ and since $\mathbf{A}$ is lowest, then all the immediate subtrees of $\mathbf{A}$ are well ordered. But the immediate subtrees of $\mathbf{A}$ comes before the immediate subtrees of $\mathbf{S}$ in the inverse lexicographical ordering of well ordered sets and with field $\mathcal{T}$. We conclude that $\mathbf{A}$ is well ordered.

The proof is finished. Observe that this proof can be formalized and proved within the theory of inductive definitions. This means — in proof theoretical terms — that the trees give an initial segment of the ordinals, and this segment is below the Howard ordinal. Using that the ordering respects embedding we can lower this estimate considerably to the small Veblen ordinal. [3]

Theorem 9. *The ordering is a well order.*

For the rest of the paper when we talk about ordinals, we mean ordinals from the initial segment given by the (finite) trees. With our orderings on the trees we have obtained an easy translation between ordinals and trees. Note that this is an improvement over the usual theories of ordinal notations. Then we can have multiple representations of an ordinal and must worry how to pick a notation for an ordinal.
The function below are well defined:



where α, β, γ, δ, $\ldots$ are ordinals corresponding to (finite) trees. The first function is the successor, but the other functions have not been characterized so far.

Here we want to describe the trees $\mathcal{T}_2$ with at most binary branching. We know that the supremum of it is

We shall prove that this is in fact the ordinal ϵ_0. We can describe $\mathcal{T}_2$ as the set of ordinals/trees given by

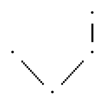
Start: $\cdot$

Successor: The function f_+ given by $x \mapsto$ 

α-jump: The function f_α given by $x \mapsto$ 

where α is in $\mathcal{T}_2$

Furthermore we have a 1-1 correspondence between the terms given above and the ordinals in $\mathcal{T}_2$. This can be used to describe the ordinals less than

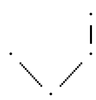


The ordinals are given by terms built up from f_+ and f_0. For simplicity we just write down the sequence of indices. So we have a finite sequence of + and 0. The empty sequence correspond to 0, and the finite ordinals are +,++, + + +, + + ++, ... The tree ordering correspond exactly to the lexicographical ordering of the sequences where we let the 0's be signs separating the +'s. The sequence 0++0+00+++0++ correspond to the sequence $\langle 0, 2, 1, 0, 3, 2\rangle$ and this is again given by the ordinal

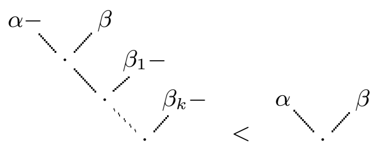
$$\omega^5 + \omega^4 \cdot 2 + \omega^3 \cdot 1 + \omega^2 \cdot 0 + \omega^1 \cdot 3 + 2$$

We get

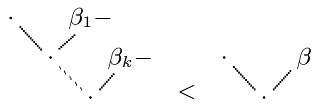
Theorem 10.

$\omega^\omega =$ 

It is no surprise that we get connections to the lexicographical ordering in the ordering of binary trees. Let us note that we have



and



where $\alpha- < \alpha$ and $\beta_i- < \beta$. These are the crucial properties for the lexicographical ordering.
We now want to characterize the function

$$g(\alpha) = \begin{array}{ccc} \cdot & & \alpha \\ & \diagdown \cdot \diagup & \end{array}$$

The fundamental set is

$$[\cdot, \alpha | x \mapsto x + 1, x \mapsto \begin{array}{ccc} x & & \alpha- \\ & \diagdown \cdot \diagup & \end{array}]$$

where $\alpha-$ runs over the ordinals $< \alpha$. Furthermore the fundamental set gives all the ordinals less than $g(\alpha)$. We get that the ordinals less than $g(\alpha)$ are those that are built up from f_+ and the $f_{\alpha-}$ where $\alpha- < \alpha$. We can write them as finite sequences of $+$ and the ordinals $\alpha-$. As before we have a lexicographical ordering where we have as separating sign the largest ordinal in the finite sequence and $+$ is the least element. Hence

$$g(\alpha + 1) = g(\alpha)^{\omega}$$

We get a recursion equation for $g(\alpha)$. We have the same recursion equation in

$$\omega^{(\omega^{\alpha+1})} = (\omega^{\omega^{\alpha}})^{\omega}$$

and both function behaves continuously at limits and have the same start. Therefore

Theorem 11. *For all $\alpha \in \mathcal{T}_2$*

$$\begin{array}{ccc} \cdot & & \alpha \\ & \diagdown \cdot \diagup & \end{array} = \omega^{(\omega^{\alpha})}$$

The ordinal $g(\alpha)$ majorises all ordinals built up from f_+ and $f_{\alpha-}$ where $\alpha- < \alpha$. Hence

Theorem 12.

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \diagdown & | & \diagup \\ & \cdot & \end{array} = \epsilon_0$$

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Consider now the function

$$\begin{array}{ccc} \alpha & & \beta \\ & \diagdown \cdot \diagup & \end{array} = \psi(\alpha, \beta)$$

It has as fundamental set

$$[\alpha, \beta \,|\, \begin{array}{ccc} \alpha- & & \beta \\ & \diagdown \cdot \diagup & \end{array} , \begin{array}{ccc} x & & \beta- \\ & \diagdown \cdot \diagup & \end{array} , \begin{array}{c} y \\ | \\ \cdot \end{array}]$$

where $\alpha-$ runs over the ordinals $< \alpha$, $\beta-$ runs over the ordinals $< \beta$ and x and y are variables indicating functions. The fundamental set gives a recursion equation for the function $\psi(\alpha, \beta)$. We note that it is the least ordinal such that

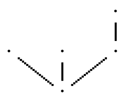
- $\psi(\alpha, \beta) > \psi(\alpha-, \beta)$
- $\psi(\alpha, \beta) > \alpha$
- $\psi(\alpha, \beta) > \beta$
- $\psi(\alpha, \beta)$ is a limit ordinal
- $\psi(\alpha, \beta)$ is a fixpoint for all $x \mapsto \psi(x, \beta-)$

But this is the fix point free Veblen hierarchy starting with $x \mapsto \omega \cdot x$. This hierarchy does not grow so fast. The first critical point is ϵ_0. Consider now the function

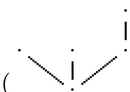
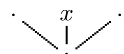
$= \phi(\alpha, \beta)$

We first observe that $\phi(\alpha, 0)$ gives a fixpoint free enumeration of the ϵ-numbers. This is immediately seen from its fundamental set. But then we get $\phi(\alpha, \beta)$ is the fixpoint free Veblen hierarchy starting with the ϵ-numbers. This is almost the same as the usual Veblen hierarchy [2] where the start is the function ω^α enumerating the multiplicative principal numbers. The first critical number is of course the fixpoint of $x \mapsto \phi(0, x)$ which gives

Theorem 13.


$= \Gamma_0$

Its elementary fundamental set is

$\mathcal{H}($  $) = [0, 0, 1 \mid$ 

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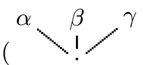
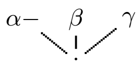
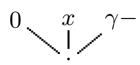
To go further along this line we need the Veblen hierarchy generalized to n-ary functions as defined by Kurt Schütte [4] based on work by Wilhelm Ackermann[1]. Assume we have the ordinary binary Veblen function $\phi(\alpha, \beta)$. We get the ternary Veblen function by

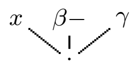
$$\begin{aligned} \phi(\alpha, \beta, 0) &= \phi(\alpha, \beta) \\ \gamma > 0 : \phi(\alpha, 0, \gamma) &= \text{ the } \alpha \text{ common fixpoint of all } \phi(0, x, \gamma-) \\ \beta, \gamma > 0 : \phi(\alpha, \beta, \gamma) &= \text{ the } \alpha \text{ common fixpoint of all } \phi(x, \beta-, \gamma) \end{aligned}$$

And we recognize these cases in the elementary fundamental set (for the case when one of the subtrees is different from 0)

$\mathcal{H}($ 

$) = [\alpha, \beta, \gamma \mid$ 

, 

, 

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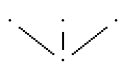
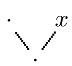
and

$\mathcal{H}($  $) = [0 \mid$ 

$]$

So here we have a fixpoint free version of the ternary Veblen hierarchy.

Theorem 14. *The ordinal*  *is the ordinal of the ternary Veblen hierarchy.*

The same argument is lifted up.

Theorem 15. *The ordinal of the trees with n-ary branching where $n > 2$ is the ordinal of the n-ary Veblen hierarchy.*

Theorem 16. *The ordinal of all finite trees is the small Veblen ordinal $\phi_{\Omega^\omega}0$.*

The small Veblen ordinal $\phi_{\Omega^\omega}0$ is the ordinal of finitary Veblen functions. It is also the ordinal connected with Kruskals theorem. [3]

References

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