\section{Introduction}

We consider the solution of linear inverse problems of the form
\begin{equation}
Tx = y^\delta,
\end{equation}
where $T \in L(X,Y)$ is a linear bounded injective operator between Hilbert spaces $X$ and $Y$; without loss of generality we assume $\|T\| \leq 1$. Such problems arise in a variety of applications; cf., e.g., [2, 3, 5, 8, 15] and the references therein. In practice, one tries to reconstruct the solution from perturbed measurements $y^\delta$, and we assume that at least a bound on the data error
\begin{equation}
\|y - y^\delta\| \leq \delta
\end{equation}
is available, where $y = Tx^\dagger \in R(T)$ denotes the unperturbed data corresponding to the true solution $x^\dagger$. It is well known (cf., e.g., [8, 11]) that in case $R(T)$ is not closed in $Y$, problem (1.1) is ill-posed, i.e., (1.1) might not admit a unique solution, and more severely, a solution is usually unstable with respect to data noise $y - y^\delta$. In order to construct a stable approximate solution of (1.1) we apply regularization methods of the form
\begin{equation}
x^\alpha_\delta := R_\alpha y^\delta := g_\alpha(T^*T)y^\delta,
\end{equation}
generated by a family of piecewise continuous (filter-) functions $\{g_\alpha\}_{\alpha \in \mathbb{R}_+}$ with the property $g_\alpha(\lambda) \rightarrow 1/\lambda$ as $\alpha \rightarrow 0$ pointwise for all $\lambda > 0$. For a convergence (rate)
analysis in case of perturbed data, the following assumptions are usually made; cf. [8]:

\begin{align}
(1.4) \quad \sup_{0 \leq \lambda \leq 1} |g_{\alpha}(\lambda)| & \leq c_{\alpha} \alpha^{-1}, \\
(1.5) \quad \sup_{0 \leq \lambda \leq 1} \lambda^{\mu} |r_{\alpha}(\lambda)| & \leq c_{\mu} \alpha^{\mu}, \quad 0 \leq \mu \leq \mu_{0},
\end{align}

where \( r_{\alpha}(\lambda) := 1 - \lambda g_{\alpha}(\lambda) \), and \( c_{\alpha}, c_{\mu}, c_{\ast} \) are positive constants independent of \( \alpha \in \mathbb{R}_{+} \). A maximal constant \( \mu_{0} \) for which \((1.5)\) holds is called \emph{qualification} of the method. In case of iterative regularization methods, we use the notation \( g_{k} = g_{\alpha} \) and replace \( \alpha \) in \((1.4), (1.5)\) by adequate functions of \( k \). If \( g_{k} \) is a polynomial, then condition \((1.4)\) already follows from \((1.5)\) by Markov’s inequality.

For illustration and later reference, we mention some important examples of regularization methods of the form \((1.3)\).

1. \textit{Tikhonov regularization} [9, 32]. The approximate solutions are defined by
\begin{equation}
(1.6) \quad x_{\alpha}^{k} = (T^{*}T + \alpha)^{-1}T^{*}y^{k}.
\end{equation}
Here, \( g_{\alpha} \) is defined by \( g_{\alpha}(\lambda) := (\lambda + \alpha)^{-1} \), and \( (1.5) \) holds for all \( \mu \leq 1 \); i.e., the qualification of Tikhonov regularization is \( \mu_{0} = 1 \).

2. \textit{Landweber iteration} [13, 18]. The approximations \( x_{k}^{\delta} \) are defined by the iteration
\begin{equation}
(1.7) \quad x_{k+1}^{\delta} = x_{k}^{\delta} + T^{*}(y^{k} - Tx_{k}^{\delta}), \quad x_{0}^{\delta} = 0.
\end{equation}
Landweber iteration has the form \((1.3)\) with \( g_{k}(\lambda) = \sum_{i=0}^{k-1} (1 - \lambda)^{i} \) and \( r_{k}(\lambda) = (1 - \lambda)^{k} \), respectively, and satisfies \((1.5)\) for all \( \mu \geq 0 \) with \( \alpha \) replaced by \( \frac{1}{k} \) and \( g_{\alpha} = g_{k} \).

3. \textit{The \( \nu \)-methods by Brakhage} [4, 12]. Semi-iterative methods whose residual polynomials \( r_{k} \) form an orthogonal sequence with respect to some positive weight function satisfy a three-term recurrence, which also carries over to the iterates; i.e., there exist sequences \( \mu_{k} \) and \( \omega_{k} \) such that
\begin{equation}
x_{k}^{\delta} = x_{k-1}^{\delta} + \mu_{k}(x_{k-1}^{\delta} - x_{k-2}^{\delta}) + \omega_{k}T^{*}(y^{k} - Tx_{k-1}^{\delta}), \quad k \geq 1,
\end{equation}
with \( x_{0}^{\delta} = x_{-1}^{\delta} = x_{0} \). The choice \( \mu_{1} = 0, \omega_{1} = (4 \nu + 2)/(4 \nu + 1) \), and
\begin{equation}
\mu_{k} = \frac{(k - 1)(2k - 2)(2k + 2 \nu - 1)}{(k + 2 \nu - 1)(2k + 4 \nu - 1)(2k + 2 \nu - 3)},
\end{equation}
\begin{equation}
\omega_{k} = 4 \frac{(2k + 2 \nu - 1)(k + \nu - 1)}{(k + 2 \nu - 1)(2k + 2 \nu - 1)}, \quad k > 1,
\end{equation}
yields the \( \nu \)-methods by Brakhage. Each \( \nu \)-method satisfies \((1.5)\) with \( \alpha \) replaced by \( \frac{1}{k^{\nu}} \) for \( 0 \leq \mu \leq \nu \).

Under condition \((1.4)\) the regularized solutions \( x_{\alpha}^{\delta} \) converge to \( x^{\dagger} \) with \( \delta \rightarrow 0 \) if \( \alpha \rightarrow 0 \) such that \( \delta^{2}/\alpha \rightarrow 0 \). In general, this convergence will be arbitrarily slow [28], and convergence rates can be obtained if and only if the true solution satisfies certain smoothness requirements, so-called \emph{source conditions}. For
\begin{equation}
x^{\dagger} \in \mathcal{R}(T^{*}T)^{\mu},
\end{equation}
for some \( \mu \leq \mu_{0} \), the estimate
\begin{equation}
\|x_{\alpha}^{\delta} - x^{\dagger}\| = O(\delta^{2/\mu+\tau})
\end{equation}
can be obtained for the parameter choice
\begin{equation}
\alpha = \alpha(\delta) \sim \delta^{\frac{2m}{2q+2}}, \tag{1.9}
\end{equation}
and one can show that the rate (1.8) is optimal under this source condition. Such a source condition is not only sufficient but also (almost) necessary in order to obtain convergence at a rate (1.8); cf., e.g., [8]. While in general, the meaning of (1.7) is difficult to understand, sufficient conditions for validity of (1.7) can be given for certain problems by a problem adapted analysis; cf. [10] for a nonlinear problem in parameter identification of a differential operator; see Assumption 3.1 below.

If the source condition (1.7) holds for some \( \mu > \mu_0 \), still only a rate of \( \delta^{\frac{2m}{2q+2}} \) can be expected; this phenomenon is called saturation. Tikhonov regularization, for instance, does not provide a better rate than \( \delta^{\frac{2}{2}} \), while for Landweber iteration the rate may be arbitrarily close to \( \delta \) if \( \mu \) is sufficiently large. For an iterative method like Landweber iteration the number of iterations yielding optimal convergence rates is \( n_\ast = O(\delta^{-\frac{2q}{2q+2}}) \) by (1.9); i.e., for small \( \mu \) (nonsmooth solution \( x^\dagger \)) one has \( n_\ast \sim O(\delta^{-2}) \). Hence, for \( \delta = 10^{-3} \) one would need about \( 10^6 \) iterations to stay within the regime of optimal convergence.

One way to overcome saturation and/or reduce the number of iterations and to relate the source condition (1.7) to standard differentiability assumptions, is provided by regularization in Hilbert scales introduced by Natterer [23] for Tikhonov regularization, and later analyzed for more general regularization methods in [8, 16, 30] and also for nonlinear problems [26, 27, 31]. We will recall the definition of a Hilbert scale and the most important results on regularization in Hilbert scales in section 2. The general idea is the following: We assume that the smoothing properties of the operator \( T \) can be characterized in terms of a simple, e.g., differential, operator \( L \). The standard analysis of regularization in Hilbert scales is then based on the condition
\begin{equation}
\|m\|_Y \leq \|L^aT^*y\| \leq \|m\|_Y \quad \text{for all } y \in Y \tag{1.10}
\end{equation}
for some \( a, m, \overline{m} > 0 \). For illustration we give a short example.

**Example 1.1.** Let \( T : L_2(0,1) \to L_2(0,1) \) denote the solution operator to
\begin{equation}
-qy'' = x \quad \text{in } [0,1]; \quad u(0) = u(1) = 0, \tag{1.11}
\end{equation}
i.e., \( Tx = y \). We assume that \( q \in L_2(0,1) \) satisfies \( 0 < q_0 \leq q \leq q_1 \) for some positive constants \( q_0, q_1 \in \mathbb{R} \). One can show that \( \mathcal{R}(T) = H^2(0,1) \cap H^0_0(0,1) \), and by the closed range theorem \( T \) is a homeomorphism between the spaces \( L_2(0,1) \) and \( H^2(0,1) \cap H^0_0(0,1) \).

Now let \( L : \mathcal{D}(L) = H^2(0,1) \cap H^0_0(0,1) \to L_2(0,1) \) be defined by \( Lx = -x'' \). Since \( L \) is self-adjoint and positive definite, one can define fractional powers of \( L \); then \( \|x\|_\alpha := \|L^\alpha x\| \) are the natural norms on a scale of (Sobolev) spaces (cf. Definition 2.1). If \( q \) is sufficiently smooth, e.g., \( q \in H^2(0,1) \), then (1.10) holds for \( a = 1 \).

Instead of (1.3) the regularized solutions can now be defined by the modified method
\begin{equation}
x^\delta_\alpha := g_\alpha(L^{-2s}T^*T)L^{-2s}T^*y^\delta, \quad s \geq -a/2. \tag{1.12}
\end{equation}
As shown in [8, 23, 30], saturation effects of a method \( g_\alpha \) due to finite qualification \( \mu_0 \) can be overcome by choosing \( s > 0 \) in (1.12); in this case \( L^{-2s} \) is smoothing. Conversely, for \( s < 0 \) the operator \( L^{-2s} \) acts as a preconditioner, and in this way
iterative regularization methods can be accelerated; cf. [6, 7]. Note that for $s < 0$ the right inequality in (1.10) is already needed to ensure well-definedness of the method (1.12).

Condition (1.10) characterizes the smoothing properties of the operator $T^*$ in terms of the operator $L$, respectively, the scale of spaces over the preimage space $\mathcal{X}$ generated by it. Regularization in Hilbert scales is applicable if the range of $T^*$ consists of sufficiently smooth functions. Note that in Example 1.1 condition (1.10) cannot hold for any $a \geq 1/4$ if $q$ has jumps. Nevertheless, the condition

\begin{equation}
\|x\| \leq \|L^s T x\| \leq \|x\| \quad \text{for all} \quad x \in L^2(0,1)
\end{equation}

holds with $a = 1$ also for nonsmooth parameters $q$. In view of (1.13) one might call $T$ smoothing like two times integration, while (1.10) would only suggest smoothing properties like integration of power $1/2$ at most. Similar examples, where (1.13) characterizes the smoothing properties of $T$ more precisely than (1.10), arise from integral operators $(Tx)(s) := \int k(s,t)x(t)dt$ with kernels $k(s,t)$ that are differentiable with respect to $s$ but not with respect to $t$; see section 5 for a detailed example.

In this paper, we analyze the solution of the inverse problem (1.1) by modified regularization methods of the form

\begin{equation}
x_\alpha := g_\alpha(T^* L^{-2s} y) T^* L^{-2s} y, \quad s \geq -a/2,
\end{equation}

under condition (1.13) or relaxed versions of it. In this way, the advantages of Hilbert scale regularization—overcoming saturation by choosing $s > 0$, respectively, by preconditioning by setting $s < 0$—can be made accessible for a new class of problems. Note that for methods (1.14) the operator $L$ acts on the image space $\mathcal{Y}$ and generates a scale of spaces over $\mathcal{Y}$. For this reason we call the methods of the form (1.14) $\mathcal{Y}$-scale regularization.

The outline of the paper is as follows. We start by recalling the basic results on regularization in Hilbert scales. In section 3, we discuss regularization methods of the form (1.14) and derive the main error estimates. Section 4 contains the analysis of $\mathcal{Y}$-scale regularization methods under relaxed assumptions with some emphasis on the preconditioning of iterative regularization methods and the design of an order optimal a posteriori stopping rule. We conclude with presenting several examples and numerical tests illustrating the theoretical results.

2. Regularization in Hilbert scales. Before we quote the most important results concerning regularization in Hilbert scales, we shortly recall the definition of a Hilbert scale; see [17] or [8, section 8.4] for details.

**Definition 2.1.** Let $L$ be a densely defined, unbounded, self-adjoint, and strictly positive operator in a Hilbert space $Z$, and let $\mathcal{M} := \bigcap_{k=0}^{\infty} D(L^k)$. By $Z_s$ we denote the completion of $\mathcal{M}$ with respect to the norm $\|z\|_s := \|L^s z\|_Z$; $(Z_s)_{s \in \mathbb{R}}$ is called a Hilbert scale (induced by $L$).

Note that obviously $\|z\|_0 = \|z\|_Z$. Two implications of this construction are that $Z_{-s} = Z'_s$ and that the following interpolation inequality holds, cf., e.g., [8, section 8.4]:

\begin{equation}
\|z\|_r \leq \|z\|_{\frac{r-q}{q}}^{\frac{r-q}{q}} \|z\|_{\frac{r-q}{s}}^{\frac{r-q}{s}}
\end{equation}

for $-\infty < q < r < s < \infty$ and $z \in Z_s$.

**Remark 2.2.** Like in Example 1.1, $L$ is chosen to be a (simple) differential operator in most situations and the spaces $Z_s$ coincide with standard Sobolev spaces for $a$
certain range of values of $s$. Note that the Sobolev spaces $H^s$ do not form a Hilbert scale in general; cf. [25]. Fractional powers of $L$ can then be efficiently implemented by Fourier transform or multilevel techniques.

Another choice of $L$ always yielding (1.10) with $a = 1$ is $L := (T^*T)^{-\frac{1}{2}}$. At least for preconditioning, e.g., for $s = -a/2$, such a choice is, however, impractical since applying $(T^*T)^{-\frac{1}{2}}$ in (1.12) or (1.14) is at least as difficult as solving the original problem (1.1). If $T$ is self-adjoint, then $L^{-2s} = (T^*T)^{-\frac{1}{2}}$ can be applied without any effort by skipping the terms $L^{-2s}T^* = I$, and the method (1.3) simplifies to

$$x^\alpha = g_\alpha(T) y^\delta,$$

which would be the default formulation of a regularization method for symmetric problems.

The standard assumption for regularization in Hilbert scales is the following norm equivalence condition:

$$m \|x\|_{-a} \leq \|Tx\| \leq M \|x\|_{-a}$$

for some $m, M, a > 0$. This condition is equivalent to (1.10) (cf. [7]; a similar result is derived in Proposition 3.3). If $x^\dagger \in \mathcal{X}_u$ for some $0 < u \leq a + 2s$ and the operator $T$ satisfies (2.2), then the regularization method (1.12) yields order optimal convergence rates

$$\|x^\alpha_{\alpha_s} - x^\dagger\| = O(\delta^{\frac{n}{2+a}}) \quad \text{for} \quad \alpha_s \sim \delta^{\frac{2(n+1)}{n+2}};$$

cf. [8, 30]. By setting $u = 2a\mu$, one can see that the rate (2.3) coincides with the standard results (1.8), and since the rate holds for $u \leq a + 2s$, saturation can be overcome by choosing $s$ sufficiently large. Note that under assumption (2.2) the source condition $x^\dagger \in \mathcal{X}_u$ is equivalent to $x^\dagger \in \mathcal{R}((T^*T)^{\frac{3}{2}})$ for $u \leq a$.

A different view on regularization in Hilbert scale has been taken in [6, 7]: If an iterative regularization method is used for the solution of (1.1), then (2.3) yields that the number of iterations needed to obtain the optimal convergence rates increases with $s$. Thus, from a numerical point of view (in particular for nonsmooth solutions) it is advantageous to choose $s$ as small as possible, eventually $s < 0$. In this case the operator $L^{-2s}$ in (1.12) acts as a preconditioner, and by setting $s = -a/2$, the number of iterations needed to guarantee the optimal rates (2.3) can be reduced to the square root, e.g., $n(\delta) = O(\delta^{-\frac{n}{n+2}})$ for Landweber iteration. Additionally, the analysis can be carried out under the weaker condition

$$\|Tx\| \leq m\|x\|_{-a}, \quad \text{respectively,} \quad \|T^*y\|_a \leq M\|y\|_Y,$$

which allows us to apply the approach for a significantly wider class of problems.

**Theorem 2.3** (cf. Theorem 1 in [7]). Let condition (2.4) hold, $g_\alpha$ satisfy (1.4), (1.5), and $-a/2 \leq s \leq 0$. Assume that $x^\dagger$ satisfies the source condition

$$x^\dagger = L^{-s}(B^*B)^{\frac{n}{2+2s}} w, \quad B := TL^{-s},$$

for some $w \in \mathcal{X}$, and let $x^\alpha_{\alpha_s}$ be defined by (1.12). Then the rate (2.3) holds.

If the stronger condition (1.10) is valid, then the source condition (2.5) is $u \leq a + 2s$ equivalent to $x^\dagger \in \mathcal{X}_u$ usually used in regularization in Hilbert scales; cf. [6, 7] for a detailed discussion. We turn now to the analysis of $\mathcal{Y}$-scale regularization methods and prove the convergence rates (2.3) for methods (1.14) under condition (1.13).
3. \( \mathcal{Y} \)-scale regularization. Let \( \{ \mathcal{Y}_s \}_{s \in \mathbb{R}} \) be a Hilbert scale induced by some densely defined self-adjoint strictly positive operator \( L : \mathcal{D}(L) \subset \mathcal{Y} \to \mathcal{Y} \) (cf. Definition 2.1), and let \( \| \cdot \|_r \) denote the norm of \( \mathcal{Y}_r \), i.e., \( \| y \|_r = \| L^r y \| \). In what follows we require that the operator \( T \) satisfies the following assumption.

**Assumption 3.1.** There exist positive real constants \( \underline{m}, \overline{m}, \) and \( a \) such that
\[
\underline{m} \| x \| \leq \|Tx\|_a \leq \overline{m} \| x \|
\]
holds uniformly for all \( x \in X \).

Note that Assumption 3.1 already implies infectivity of the operator \( T \). For the stable solution of the inverse problem (1.1) we consider modified regularization methods of the form
\[
(3.1) \quad x_\alpha := g_\alpha(T^* L^{-2s} T) y^\delta = g_\alpha(B^* B) L^{-s} y^\delta \quad \text{with} \quad B := L^{-s} T.
\]

Here and below we assume that the functions \( g_\alpha \) satisfy the standard assumptions (1.4), (1.5); \( s \in \mathbb{R} \) will be specified later. Throughout, \( A^* \) denotes the adjoint of a linear operator \( A \) with respect to the spaces \( X \) and \( \mathcal{Y} \). Under Assumption 3.1, the operator \( B \) is bounded as operator from \( X \) to \( \mathcal{Y} \) for \( s \geq -a \). Without loss of generality, we will assume \( \| B \| \leq 1 \), which can always be achieved by proper scaling.

**Remark 3.2.** Problem (1.1) is equivalent to
\[
(3.3) \quad B x = L^{-s} y^\delta,
\]
where \( B \) has to be understood as an operator between \( X \) and \( \mathcal{Y} \), respectively, \( T \) in (1.1) as an operator from \( X \) to \( \mathcal{Y}_s \). Note that \( T^* L^{-2s} \) is the adjoint of \( T \) with respect to these spaces. The \( \mathcal{Y} \)-scale method (3.2) hence corresponds to applying the regularization method defined by \( g_\alpha \) to the solution of (3.3). Note that in case \( s < 0 \), the right-hand side \( L^{-s} y^\delta \) may not be an element of \( \mathcal{Y} \). In any case the noise level estimate (1.2) is not given in the appropriate norm if \( s \neq 0 \).

For the subsequent error analysis, we need some preliminary results.

**PROPOSITION 3.3.** Assumption 3.1 is equivalent to
\[
(3.4) \quad \underline{m} \| y \|_{-a} \leq \| T^* y \| \leq \overline{m} \| y \|_{-a}.
\]

**Proof.** First assume that (3.1) holds, let \( y \in \mathcal{Y} \), and observe that
\[
\sup_{y \in \mathcal{Y}} \| T^* y \| = \sup_{y \in \mathcal{Y}} \| T^* y, x \| \| y \|_{-a} = \sup_{x \in X} \sup_{y \in \mathcal{Y}} \| T^* y, x \| \| y \|_{-a} \leq \sup_{x \in X} \| T x \| \leq \overline{m}.
\]
Hence, \( T^* \) is extendable as a bounded linear operator to \( \mathcal{Y}_{-a} \). The result then follows similar to the above by noting that
\[
\sup_{y \in \mathcal{Y}_{-a}} \| T^* y \| = \sup_{x \in X} \| T x \|.
\]
The other implication follows similarly. \( \square \)

**PROPOSITION 3.4.** Let Assumption 3.1 hold and let \( s \geq -a \). Then for \( |\nu| \leq 1 \)
\[
\mathcal{E}(\nu) \| y \|_{-\nu(a+s)} \leq \| (BB^*)^\frac{\nu}{2} y \| \leq \mathcal{E}(\nu) \| y \|_{-\nu(a+s)}
\]
holds for \( y \in \mathcal{D}((BB^*)^\frac{\nu}{2}) \) with \( \mathcal{E}(\nu) = \min(\underline{m}^{\nu}, \overline{m}^{\nu}) \) and \( \nu(\nu) = \max(\underline{m}^{\nu}, \overline{m}^{\nu}) \).

Moreover,
\[
\mathcal{D}((BB^*)^{-\frac{\nu}{2}}) = \mathcal{R}((BB^*)^\frac{\nu}{2}) = \mathcal{Y}_{\nu(a+s)},
\]

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where in the case \( \nu > 0 \), \((BB^*)^\frac{\nu}{2}\) is identified with its extension to \( \mathcal{Y} \).

**Proof.** The result follows by the inequality of Heinz; see, e.g., [8, Corollary 8.22] for details. \[ \square \]

It is well known that in order to obtain convergence rates, the true solution \( x^* \) has to satisfy some smoothness assumptions; cf. [28]. As will become clear from our analysis the appropriate source condition for \( \mathcal{Y} \)-scale regularization has to be stated in terms of the operator \( B \) (see also Remark 3.2 and Theorem 2.3).

**Assumption 3.5 (source condition).** Let \( s > -a \). There exist a real number \( u > 0 \) and an element \( \omega \in \mathcal{X} \) such that

\[
(3.5) \quad x^* = (B^*B)^{-\frac{u}{a+u}} \omega.
\]

The total error \( \|x^\delta - x^*\| \) can be decomposed into the two main components, namely, the approximation error \( \|x^\alpha - x^*\| \) and the propagated data error \( \|x^\delta - x^\alpha\| \), where \( x^\alpha \) denotes the approximate solution defined by (3.2) with \( y^\delta \) replaced by the correct data \( y = Ty^\dagger \). We will now derive estimates for the two error components in terms of \( \alpha \) and \( \delta \).

**Theorem 3.6.** Let Assumption 3.1 hold, let \( y^\delta \) satisfy (1.2), and let \( x^\delta, x^\alpha \) be defined by the \( \mathcal{Y} \)-scale method (3.2) with some \( s \geq -a/2 \). Furthermore, let \( g_\alpha \) satisfy (1.4), (1.5) for some \( \mu_0 > 0 \) and let \( x^\dagger \) satisfy Assumption 3.5 for \( u \leq 2(a + s)\mu_0 \). Then the following a priori estimates hold:

\[
(3.6) \quad \|x^\alpha - x^\dagger\| \leq C_1 \alpha^{\frac{u}{a+u}} \|\omega\|,
\]

\[
(3.7) \quad \|x^\delta - x^\alpha\| \leq C_2 \alpha^{-\frac{u}{a+u}} \delta,
\]

where \( C_1, C_2 \) denote generic constants depending only on \( \mu_0, c_*, \underline{m}, \) and \( \overline{m} \). As a consequence

\[
(3.8) \quad \|x^\dagger - x^\delta\| = O(\delta^{\frac{u}{a+u}}) \quad \text{for} \quad \delta \sim \alpha^{\frac{u}{a+u}}.
\]

**Proof.** We start by estimating the approximation error by using (1.5) and (3.5),

\[
(3.9) \quad s\|x^\alpha - x^\dagger\| = \|g_\alpha(B^*B)B^*L^{-s}y - x^\dagger\|
\]

\[
(3.10) \quad = \|r_\alpha(B^*B)(B^*B)^{s}\| w\|
\]

\[
(3.11) \quad \leq c_\alpha \alpha^{-\frac{u}{a+u}} \|\omega\|.
\]

Now consider the propagated data error. By Proposition 3.4 we have \( \mathcal{R}(L^{-s}) = \mathcal{Y} = \mathcal{R}((BB^*)^{\frac{u}{a+u}}) \), hence

\[
\quad x^\delta - x^\alpha = g_\alpha(B^*B)B^*L^{-s}(y^\delta - y) = g_\alpha(B^*B)(BB^*)^{\frac{s}{a+u}} v^\delta
\]

for some \( v^\delta \in \mathcal{Y} \) with \( \|v^\delta\| \leq c_1 \|y - y^\delta\| \). Thus, we can further estimate

\[
(3.12) \quad \|x^\delta - x^\alpha\| = \|g_\alpha(B^*B)(BB^*)^{\frac{s}{a+u}} v^\delta\|
\]

\[
(3.13) \quad \leq c_2 \|g_\alpha(B^*B)(B^*B)^{\frac{u+2}{2(a+u)}} y - y^\delta\|
\]

\[
(3.14) \quad \leq c_3 \alpha^{-\frac{u}{a+u}} \delta,
\]

where for the last estimate we utilized that

\[
\sup_{\lambda \in [0,1]} |g_\alpha(\lambda)\lambda^\mu| \leq C_\mu \alpha^{\mu-1}, \quad 0 \leq \mu \leq 1,
\]
which follows readily from (1.4) and (1.5). Equation (3.8) is implied by (3.6) and (3.7). □

Remark 3.7. In contrast to standard regularization in Hilbert scales, the source condition (3.5) cannot be interpreted in terms of the spaces $\mathcal{Y}$ directly. However, the following interpretation is possible. Define $z$ by $x = B^*z$. Then problem (1.1) is equivalent to

$$BB^*z = L^{-s}y^\delta, \quad x = B^*z.$$ 

The source condition (3.5) can be restated for $z$ by noting

$$B^*z^\delta = x^\dagger = (B^*B)^{-\frac{s}{n+s}}w = B^*(BB^*)^{-\frac{s}{n+s}}v,$$

hence $z^\delta \in \mathcal{R}((BB^*)^{-\frac{s}{n+s}}) = \mathcal{Y}_{u-a-s}$. The last identity is valid only for $u \leq a + s$. A similar restriction also holds for standard regularization in Hilbert scales, where the source condition $x^\dagger \in \mathcal{X}$ guarantees optimal convergence rates only for $u \leq a + 2s$; cf. [8, Remark 8.24]. Finally, let Assumption 3.1 hold and let $x^\dagger$ satisfy a standard source condition $x^\dagger = T^*w$ for some $w \in \mathcal{Y}$. Then we obtain by Proposition 3.4 that

$$x^\dagger = T^*w = B^*L^s w = B^*(BB^*)^{-\frac{s}{n+s}}w = (B^*B)^{-\frac{s}{n+s}}w$$

holds for some $\hat{w} \in \mathcal{X}$ with $\|\hat{w}\|_X \sim \|w\|_Y$. Hence by the inequality of Heinz we at least have $\mathcal{R}(T^*T) = \mathcal{R}((B^*B)^{-\frac{n}{n+s}})$ for $0 \leq \mu \leq \frac{n}{2}$.

Since in practice the smoothness of the solution $x^\dagger$ is not known a priori, an optimal parameter choice (3.8) is not possible in general. Therefore, we next consider an a posteriori parameter selection criterion based on the discrepancy principle (cf. [8, 22]), i.e.,

$$(3.15) \quad \alpha_* := \sup \{ \alpha : \|Tx_\alpha - y^\delta\| \leq \tau\delta \}.$$ 

The advantage of such a criterion is that it requires no a priori information about the smoothness of $x^\dagger$. Moreover, the residual $r^\delta_k := Tx_\alpha - y^\delta$ usually has to be calculated anyway, in particular for iterative regularization methods.

Theorem 3.8. Let the assumptions of Theorem 3.6 hold and let $\alpha_* = \alpha_*(\delta)$ be determined by the discrepancy principle (3.15). Then the rates (3.8) hold for $0 < u \leq 2(a + s)\mu_0 - a$.

Proof. We have to estimate the residual $\|Tx_\alpha - y^\delta\|$, 

$$(3.16) \quad s\|Tx_\alpha - y^\delta\| = \|L^s(Bg_\alpha(B^*B)B^* - I)L^{-s}y^\delta\|$$

$$\leq c_1 \|r_\alpha(BB^*)B(B^*B)^{-\frac{s}{n+s}}\|w\| + c_2 \delta$$

$$(3.17) \leq c_3 \frac{u+a}{n+s} \alpha^{\frac{n+s}{n+s}} \|w\| + c_2 \delta,$$

where we used (1.5) and $u \leq 2(a + s)\mu_0 - a$. Hence,

$$(3.18) \quad \alpha_* \geq C\delta^{\frac{2(a+s)}{n+s}}$$

as long as $\tau > c_2$, which yields a sufficient bound for the propagated data error by (3.7). It remains to consider the approximation error. First note that we have

$$(3.19) \quad \|T(x_\alpha - x^\dagger)\| \leq \|Tx_\alpha - y^\delta\| + \|T(x_\alpha - x_\alpha^\dagger)\| + \delta \leq C\delta.$$
It then follows by the interpolation inequality and (3.5) that
\[
\|x_\alpha - x^\dagger\| \leq c_3 \|T(x_\alpha - x^\dagger)\|^\frac{2}{\mu_2} \|u\|^\frac{2}{\mu_2} \leq c_4 \delta_\mu \|u\|^\frac{2}{\mu_2},
\]
which completes the proof. \(\square\)

**Remark 3.9.** For \(s = 0\) the above results coincide with the standard regularization theory by noting that \(u = 2a\mu\). The restriction \(u \leq 2(a + s)\mu_0 - a\) in Theorem 3.8 then amounts to \(\mu \leq \mu_0 - 1/2\), which is the well-known saturation for the discrepancy principle. As for regularization in Hilbert scales, saturation effects of the applied method can be overcome by choosing \(s\) large enough.

4. \(\mathcal{Y}\)-scale regularization under relaxed assumptions—preconditioning.

Probably the most severe drawback of \(\mathcal{Y}\)-scale regularization as presented in the previous section is the restrictive assumptions (3.1) (corresponding to (1.10) for standard regularization in Hilbert scales). As we will show now, most of the results remain valid if only one of the estimates in (3.1) is available. We will treat in detail the case where the right inequality holds and \(s \leq 0\) has to be chosen; the corresponding results for the other inequality and the choice \(s \geq 0\) are indicated in remarks. As can be seen from (3.8), the (optimal) regularization parameter \(\alpha_\ast\) can be increased by choosing \(s < 0\). For an iterative regularization method \(g_\alpha = g_n\), this means that the corresponding stopping index decreases by setting \(s < 0\). This motivates calling the operator \(L^{-2s}\) in (1.14) a preconditioner. In order to stay with the notation of the previous section, we continue to consider general regularization methods \(g_\alpha\) and mention some implications for iterative regularization methods in separate remarks.

Preconditioning of iterative regularization methods in Hilbert scales has been investigated in [6, 7] under the standard assumption (1.10) and under weaker conditions. Here, we are using scales of spaces over \(\mathcal{Y}\) and the assumption (4.1). The derivation of the convergence results of the following sections therefore requires different reasoning and a more sophisticated a posteriori stopping rule.

Let \(L\) and \(\mathcal{Y}_s\) be defined as above. Throughout this section we require the following assumption.

**Assumption 4.1.** \(T\) is injective and there exists a positive real constant \(a\) such that
\[
\|Tx\|_a \leq \|x\|
\]
holds uniformly for all \(x \in X\).

Unlike (3.1), where the injectivity of the operator \(T\) was implied by the left estimate, it has to be assumed here additionally. Similarly as in the previous section we have the following equivalent characterization of Assumption 4.1.

**Proposition 4.2.** Assumption 4.1 is equivalent to \(T\) being injective and
\[
\|T^*y\| \leq \|y\|_{-a} \quad \text{for all } y \in \mathcal{Y}_{-a}.
\]

**Proof.** Let \(x, y, \hat{y} \neq 0\). Assumption 4.1 can be restated as
\[
1 \geq \sup_{x \in X} \frac{\|L^aTx\|}{\|x\|} = \sup_{x \in X} \sup_{y \in \mathcal{Y}} \frac{\langle Tx, L^a y \rangle}{\|x\| \|y\|} = \sup_{\hat{y} \in \mathcal{Y}_{-a}} \sup_{x \in X} \frac{\langle x, T^* \hat{y} \rangle}{\|x\| \|\hat{y}\|_{-a}} = \sup_{\hat{y} \in \mathcal{Y}_{-a}} \|T^* \hat{y}\|_{-a}.
\]
The other implication follows in the same way. \(\square\)
Due to the relaxed assumption (4.1), Proposition 3.4 is no longer valid in its general form. Instead, only the following weaker implications hold.

**Proposition 4.3.** Let Assumption 4.1 hold, and let $B = L^{-s}T$ for some $s \geq -a$ with $\|B\| \leq 1$. Then the estimates

\begin{align}
(4.3) \quad &\| (BB^*)^{\frac{s}{2}} y \| \leq \| y \|-\nu(a+s) \quad \text{for all } y \in \mathcal{Y}_-\nu(a+s) \subset \mathcal{R}( (BB^*)^{-\frac{s}{2}}), \\
(4.4) \quad &\| (BB^*)^{-\frac{s}{2}} y \| \geq \| y \|_\nu(a+s) \quad \text{for all } y \in \mathcal{R}( (BB^*)^{\frac{s}{2}}) \subset \mathcal{Y}_\nu(a+s)
\end{align}

hold for $0 \leq \nu \leq 1$.

**Proof.** The results follow from the inequality of Heinz and the standard results on Hilbert scales; see [8, section 8.5] for details. □

Assumption 4.1 and Proposition 4.3 are sufficient to prove the following a priori error estimates corresponding to those of Theorem 3.6.

**Theorem 4.4.** Let $-a/2 \leq s \leq 0$ and let Assumption 4.1 hold. Additionally, let $g_\alpha$, $x^1$, $y^\delta$ satisfy the assumptions of Theorem 3.6. Then (3.6), (3.7), and (3.8) hold.

**Proof.** The estimate for the approximation error $x^\delta - x^1$ follows as in the proof of Theorem 3.6. Now observe that $L^{-s}(y^\delta - y) \in \mathcal{R}(L^{-s}) = \mathcal{Y}_s \subset \mathcal{R}( (BB^*)^{\frac{s}{2(s+a)}})$ is implied by (4.3) since $s \leq 0$. Hence, also the estimate for the propagated data error follows as in Theorem 3.6. □

**Remark 4.5.** In case $s \geq 0$, condition (4.1) has to be replaced by

\begin{align}
(4.5) \quad &\| x \| \leq \| T x \|_\alpha.
\end{align}

Proposition 4.3 then holds with opposite inclusions and the analogue of Theorem 4.4 follows with obvious modifications. Again, similarly as in the proof of Theorem 3.6, there exists a $v_\alpha \in \mathcal{Y}$ such that $L^{-s}(y^\delta - y) = (BB^*)^{\frac{s}{2(s+a)}} v_\alpha$, since for $s \geq 0$ and under condition (4.5),

\[ \mathcal{R}(L^{-s}) = \mathcal{Y}_s \subset \mathcal{R}( (BB^*)^{\frac{s}{2(s+a)}}) \]

follows from the analogue of Proposition 4.3 for this case.

### 4.1. A posteriori parameter choice.

The result of Theorem 4.4 is not of great use per se, since it would require exact knowledge of the smoothness of the solution $x^1$. We will therefore consider a posteriori parameter choice rules in the following.

A **discrepancy principle.** In contrast to the results of the previous section, the discrepancy principle (3.15) does not provide parameters that guarantee optimal convergence rates if only Assumption 4.1 is used. To see this, let us estimate the residual

\begin{align}
(4.6) \quad &\| Tx_\alpha^\delta - y^\delta \| = \| L^s(Bg_\alpha(B^*B)B^* - I)L^{-s}y^\delta \| \\
(4.7) \quad &\leq \| L^s r_\alpha(BB^*)B(B^*B)^{\frac{s}{2(s+a)}} w \| \\
&\quad + \| L^s g_\alpha(BB^*)BB^*L^{-s}(y^\delta - y) \| + \| y^\delta - y \| \\
(4.8) \quad &\leq c_1 \alpha^{\frac{s}{2(s+a)}} + c_2 \alpha^{\frac{s}{2(s+a)}} \delta
\end{align}

for some positive constants $c_1$ and $c_2$. A careful inspection of the proof of Theorem 3.8 reveals that the second term may disturb the convergence of the residual $Tx_\alpha^\delta - y^\delta$; i.e., one cannot guarantee that $\inf(\|Tx_\alpha^\delta - y^\delta\| : \alpha > 0) \leq \tau \delta$ holds for any $\tau$ (independent of $\delta$ and $y^\delta$) and for all possible data $y^\delta$. Hence, the discrepancy principle (3.15) might not yield a parameter $\alpha_\text{max}$ at all. A partial result can, however, be found for the following modified criterion: Let $\alpha_\text{max}$ be defined by

\begin{align}
(4.9) \quad &\alpha_\text{max} := \sup\{ \alpha : \|Tx_\alpha^\delta - y^\delta\| \leq \tau \delta \alpha^{\frac{s}{2(s+a)}} \}.
\end{align}
for some appropriate $\tau > 1$. We can estimate $\alpha_{\text{max}}$ from below by

$$\tau \delta \alpha_{\text{max}} \leq \|Tx_{\alpha}^\delta - y^\delta\| \leq c_1 \alpha^\frac{s+a+\gamma}{s+a} + c_2 \alpha^\frac{s-\gamma}{s+a} \delta$$

for all $\alpha > \alpha_{\text{max}}$. Hence, assuming that $\tau > c_2$, we obtain with an appropriate constant $C > 0$ that

$$\alpha_{\text{max}} \geq C \delta \frac{2s(a+s)}{s+a+\gamma}.$$

In view of Theorems 3.6 and 3.8, a parameter choice $\alpha_* = \alpha_{\text{max}}$ cannot provide optimal convergence rates in general, but at least the following suboptimal result holds.

**Proposition 4.6.** Let the iterative method (1.14) be stopped at $\alpha_* = \alpha_{\text{max}}$. Then the following (suboptimal) rate holds:

$$\|x_{\alpha}^\delta - x^\dagger\| = O\left(\delta \frac{s(a+s)}{a+u}\right).$$

**Proof.** In the same manner as (4.6) one obtains that

$$\|Tx_{\alpha} - y\| \leq \alpha^\frac{s+\gamma}{s+a+\gamma} \delta.$$

The result then follows similarly as in the proof of Theorem 3.8.

**Remark 4.7.** For $s = 0$ the rate (4.11) coincides with the optimal rate $O(\delta^{2s+1})$. If $s < 0$, the convergence rate observed for this parameter choice is not optimal; e.g., for $s = -a/2$, (4.11) yields

$$\|x_{\alpha}^\delta - x^\dagger\| = O(\delta^{\frac{n}{2s+1}}),$$

which is only half of the optimal rate (3.8). We only mention that even if the standard discrepancy principle (3.15) provides a parameter, i.e., if there exists an $\alpha$ such that $\|Tx_{\alpha}^\delta - y^\delta\| \leq \tau \delta$, such a choice will in general only yield suboptimal rates similar to (4.11).

We will show next how optimal rates can be restored.

**A Lepskij principle.** In view of Theorem 3.6, $\alpha_{\text{max}}$ is a reasonable upper bound for appropriate parameters yielding optimal convergence rates. On the other hand, according to Theorem 3.6 optimal rates are always obtained for the choice

$$\alpha_* = \alpha_{\text{opt}} := \overline{C} \delta^{\frac{2(a+s)}{s+a+\gamma}} \geq \overline{C} \delta^{\frac{2(a+s)}{s+\gamma}} =: \alpha_{\text{min}}$$

with $\overline{C}$ as in (4.10). Hence, $\alpha_{\text{min}}$ may serve as a reasonable lower bound for interesting regularization parameters. A specific parameter yielding optimal convergence rates can be found in the range $[\alpha_{\text{min}}, \alpha_{\text{max}}]$ by means of the Lepskij principle (cf. [19, 21]); we use here a formulation similar to the one presented in [1].

Let $\alpha_{\text{min}}, \alpha_{\text{max}}$ be determined according to (4.9), (4.12), and let $\alpha_n$ be defined by $\alpha_n := q^n \cdot \alpha_{\text{max}}$ for some $0 < q < 1$. We define the set $\mathcal{M}$ (of admissible parameters) by

$$\mathcal{M} := \left\{ \alpha_n \in [\alpha_{\text{min}}, \alpha_{\text{max}}]: \|x_{\alpha_n}^\delta - x_{\alpha_m}^\delta\| \leq 4C\alpha_n^{-\frac{s+\gamma}{s+a+\gamma}} \delta \right\},$$

for all $m = n+1, \ldots, N$. 

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where \( N \) is the largest integer such that \( \alpha_N > \alpha_{\min} \), and \( C \) is a generic constant such that
\[
\max\{\|x_\alpha - x\|, \|x_\alpha - x^\dagger\|\} \leq C \alpha^{-\frac{a}{s+\alpha}} \delta \quad \text{for all} \quad \alpha \leq \alpha_{\opt}.
\]

Note that by the error estimates (3.6), (3.7), such a constant can be chosen depending on (a bound on) \( \|w\| \), but without knowledge of \( u \) or \( \alpha_{\opt} \). As appropriate parameter, we select
\[
\alpha_* := \begin{cases} 
\max \mathcal{M} & \text{if } \mathcal{M} \neq \emptyset, \\
\alpha_N & \text{else}.
\end{cases}
\]

**Theorem 4.8.** Let the assumptions of Theorem 4.4 be satisfied and let \( \alpha_* \) be defined by (4.15). Then the rate
\[
\|x_{\alpha_*} - x^\dagger\| = O(\delta^{\frac{2}{2+a}})
\]
holds.

**Proof.** By the definition of \( \alpha_{\opt} \) we have \( \alpha_{\opt} \in [\alpha_{\min}, \alpha_{\max}] \). Additionally, it follows from the error estimates (3.6), (3.7), and (4.14) that \( \alpha_* \geq \alpha_{\opt} \). Hence, we obtain together with (4.13) that
\[
\|x_{\alpha_*} - x^\dagger\| \leq \|x_{\alpha_{\opt}} - x^\dagger\| + \|x_{\alpha_*} - x_{\alpha_{\opt}}\| = O(\delta^{\frac{2}{2+a}}). \quad \square
\]

**Remark 4.9.** A disadvantage of the Lepskij principle is that its realization requires a higher computational effort than the standard discrepancy principle (3.15). Note that the construction of the set \( \mathcal{M} \) requires us to generate solutions for a wide range of regularization parameters. Additionally, at least some solutions \( x_\alpha \) have to be stored in order to check the conditions for the construction of \( \mathcal{M} \). By using the upper bound \( \alpha_{\max} \) on admissible parameters, the number of solutions \( x_\alpha \) which have to be stored can be reduced significantly; note that usually one chooses \( \alpha_{\max} \sim 1 \). Additionally, the \( \gamma \)-scale approach allows us to choose the smallest regularization parameter \( \alpha_{\min} \) relatively large, e.g., \( \alpha_{\min} \sim \delta \) for \( s = -a/2 \), whereas one would need \( \alpha_{\min} \sim \delta^2 \) without preconditioning; cf. [1, 21].

**Remark 4.10.** The results derived for general regularization methods (1.3) hold with obvious modifications also for iterative regularization methods. If the strong condition (3.1) holds and \( s = -a/2 \), Theorem 3.8 yields optimal convergence rates when the method is stopped at iteration \( k_* \) defined by the discrepancy principle
\[
\|Tx_k^\delta - y^\delta\| \leq \tau \delta < \|Tx_k^\delta - y^\delta\| \quad \text{for all} \quad k < k_*.
\]
As can be seen from (3.8), respectively, (2.3), the number of iterations needed to get the optimal rates (2.3) can be reduced to about the square root by proper preconditioning \((s = -a/2)\).

Under the weaker assumption (4.1), and with \( s = -a/2 \), one has to perform \( k_{\max} := 1/\alpha_{\min} \sim \delta^{-1} \) iterations in order to apply the Lepskij principle (4.15). Note that if \( x^\dagger \) is not very smooth, i.e., if \( \mu \) in (1.9) is small, which is the case we are interested in, then one would expect \( k_* = O(\delta^{\frac{2}{2+a}}) \sim O(\delta^{-2}) \) iterations for the standard version of Landweber iteration. Hence also in this case, the number of iterations can be reduced to about the square root by preconditioning.

**Remark 4.11.** If instead of Assumption 4.1 condition (4.5) holds and \( s \geq 0 \), then the results concerning the discrepancy principle are slightly different from
those presented here; e.g., the standard discrepancy principle (3.15) can be used to determine \( \alpha_{\text{max}} \). However, as in the case \( s \leq 0 \), a choice \( \alpha_s = \alpha_{\text{max}} \) does not yield optimal convergence rates in general. The result for the Lepskij principle follows as above with obvious modification. For \( s \geq 0 \) and if iterative regularization methods are applied, the number of iterations will be increased by application of \( L^{-2s} \), which in case \( s \geq 0 \) is a smoothing operator. Such choice seems to be disadvantageous from a numerical point of view, at least for iterative methods.

We only mention that if a condition (4.5) holds for some \( \tilde{a} \geq a \) additionally to (4.1) (cf. Example 5.2), then some of the previous results can be strengthened. In particular, for \( \tilde{a} = a \), the results of section 3 apply.

5. Examples and numerical tests. In this section we verify Assumption 3.1, respectively, Assumption 4.1, for several examples and present the results of the numerical tests illustrating the theoretical results.

Example 5.1 (Radon transform). Let \( f \in C_0^\infty(\Omega^n) \) for some compact domain \( \Omega^n \subset \mathbb{R}^n \), \( n \geq 2 \). The Radon transform defined by

\[
Rf(\omega, s) = \int_{x \cdot \omega = s} f(x)dx
\]

is important in many applications, e.g., in computerized tomography, and has been studied extensively in the literature. The following very general stability estimate holds; cf. [24, 29]:

\[
(5.1) \quad c(\gamma, n) \| f \|_{H^\gamma_0(\Omega^n)} \leq \| Rf \|_{H^{\gamma+(n-1)/2}(Z)} \leq C(\gamma, n) \| f \|_{H^\gamma_0(\Omega^n)},
\]

where \( Z \) denotes the unit cylinder in \( \mathbb{R}^n \), and

\[
\| g \|_{H^{\gamma+(n-1)/2}(Z)} := \int_{S^{n-1}} \int_{\mathbb{R}} (1 + \sigma^2)^{\gamma/2} |\hat{g}(\theta, \sigma)|^2 d\sigma d\theta.
\]

Here, the Fourier transform is only taken with respect to the second variable. Due to (5.1) with \( \alpha = 0 \), the results of section 3 are applicable with \( Ly(\theta, s) = -\partial_{ss} y(\theta, s) + y(\theta, s) \); fractional powers of the operator \( L \) can be realized efficiently by Fourier transform.

Example 5.2 (an integral equation allowing a lower bound). Consider the Fredholm integral equation \( Tx = y \) with operator \( T : L_2(0, 1) \rightarrow L_2(0, 1) \) defined by

\[
Tx(s) = \int_0^1 k(s, t)dt \quad \text{with} \quad k(s, t) = \sqrt{t}\begin{cases} s(1-t), & 0 \leq s < t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}
\]

Since \( k(s, t) \in L_2([0, 1]^2) \), it follows that \( T \) is compact and \( Tx = y \) is ill-posed. With

\[
(Tx)(s) = (1-s) \int_0^s t^{3/2}x(t)dt + s \int_s^1 t^{1/2}(1-t)x(t)dt
\]

we get \( (Tx)(0) = (Tx)(1) = 0 \). By twice differentiation we obtain that

\[
(Tx)''(s) = -s^{1/2}x(s),
\]

which yields

\[
\mathcal{R}(T) = \{ y \in H^2(0, 1) \cap H^1_0(0, 1) : s^{-1/2}y''(s) \in L_2(0, 1) \}.
\]
We utilize the Hilbert scale induced by the operator \( L \) defined by
\[
L^{x} := \sum_{n=1}^{\infty} (n\pi)^{x} \langle x, x_{n} \rangle y_{n} \quad \text{with} \quad y_{n} := \sqrt{2} \sin(n\pi),
\]
i.e., \( L^{2} x = -x'' \) and \( \mathcal{Y}_{2} = H^{2}(0, 1) \cap H_{0}^{1}(0, 1) \). This choice yields \( \mathcal{R}(T) \subseteq \mathcal{Y}_{2} \) and \( \mathcal{R}(T) \supset \mathcal{Y}_{2.5} = \{ y \in H^{2.5}(0, 1) \cap H_{0}^{1}(0, 1) : \rho^{-1/2} y'' \in L_{2}(0, 1) \} \) with \( \rho(t) = t(1-t) \). By Theorem 11.7 in [20], we get
\[
\| x \| \leq \| Tx \|_{2.5} \quad \text{and} \quad \| Tx \|_{2} \leq m_{\infty} \| x \|
\]
for some positive constants \( m, m_{\infty} \), and the results of section 4 can be applied.

Next we compare Landweber iteration and the \( \nu \)-methods by Brakhage [4, 12] for the numerical solution of \( T x = y \). As parameter choice strategy we use the Lepskij principle (4.15). We shortly describe how this stopping rule is implemented for Landweber iteration: As soon as the modified discrepancy principle (4.9) is satisfied, we set
\[
\text{Landweber iteration: As soon as the modified discrepancy principle (4.9) is satisfied, we set}
\]
\[
\text{Landweber iteration: As soon as the modified discrepancy principle (4.9) is satisfied, we set}
\]
\[
\text{Lepskij principle (4.15). We shortly describe how this stopping rule is implemented for}
\]
\[
\text{Lepskij principle (4.15). We shortly describe how this stopping rule is implemented for}
\]
\[
\text{the iterates} \quad x^{k} \quad \text{rather than the iteration indices} \quad k.
\]
At each iteration \( k_{n} := 2^{n} \cdot k_{\min} \), \( n \in \mathbb{N} \), we add the element \( x^{k_{n}} \) to the set \( \mathcal{M}_{\xi} \) and check if the condition
\[
\| x^{k_{n}} - x^{k_{m}} \| \leq 4Ck_{n}^{\frac{m+n}{2}} \delta
\]
holds for all \( x^{k_{n}} \), \( x^{k_{m}} \) in the set \( \mathcal{M}_{\xi} \) with \( m > n \). If (5.3) does not hold for some \( n \), we eliminate all elements \( x^{k_{m}} \) with \( m < n \). This procedure is continued until the final iteration \( k_{N} \) is reached, where \( N \) is the largest integer such that \( k_{N} < k_{\max} = C\delta^{-1} \). As stopping index \( k_{\ast} \) we then choose the smallest \( k_{n} \) with \( x^{k_{n}} \in \mathcal{M}_{\xi} \).

For a concrete numerical test, we set \( x^{\dagger}(s) = \text{sign}(1-2s) \). With \( a = 2, s = -1 \), and the above choice of a Hilbert scale, we have
\[
x^{\dagger} \in \mathcal{R}(B^{\ast} B) \cap \mathcal{Y}_{2} = \left\{ x \in \mathcal{Y}_{2} : \frac{1}{\sqrt{\delta}} \left( \frac{x}{\sqrt{\delta}} \right)^{''} \in L_{2}(0, 1) \right\}
\]
for \( 0 \leq u < 1 \). The correct data \( y = Tx^{\dagger} \) are perturbed by uniformly distributed random noise such that \( \| y - x^{\dagger} \| = \delta \) for several values of \( \delta \). In a first test, we compare the performance of the discrepancy principle and the Lepskij rule suggested in the previous section: we set \( \delta = 0.002 \), \( \tau = 1.1 \) and plot the two error components (approximation error and propagated data error) for Landweber iteration with preconditioning \( (s = -a/2) \) in Figure 5.1. The optimal stopping index, i.e., the one where the total error \( \| x^{k} - x^{\dagger} \| \) attains its minimum, is 230, whereas the standard discrepancy principle stops only after 469 iterations; cf. (4.6) and Remark 4.9. The Lepskij principle (4.15) with \( C = 0.25 \) yields \( k_{\ast} = 96 \). The corresponding errors \( e_{k} = \| x^{k} - x^{\dagger} \| \) are \( e_{230} = 0.26 \), \( e_{469} = 0.30 \), and \( e_{96} = 0.27 \).

In a second test, we illustrate the effect of preconditioning and list the iteration numbers and errors for several values of \( \delta \) obtained for Landweber iteration and the \( \nu \)-methods by Brakhage [4, 12] with and without preconditioning in Table 5.1. Note that the numerical effort for an iteration step is about the same for all methods and consists of application of \( T \) and \( T^{\ast} \). The preconditioned iteration requires further application of a differential operator, which is cheap in comparison with application of the integral operator \( T \).
Y-scale regularization

Fig. 5.1. Approximation error, propagated data error, and total error, Example 5.2, with \( \delta = 0.002, \tau = 1.1 \), and iterates obtained by the preconditioned Landweber iteration. The vertical line denotes the actual stopping index \( n^* \) determined by the Lepskij rule.

Table 5.1

<table>
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<th>( \delta )</th>
<th>( (lw))</th>
<th>( (lw) )</th>
<th>( (hu lw) )</th>
<th>( (hu lw) )</th>
<th>( (nu) )</th>
<th>( (nu) )</th>
<th>( (hs nu) )</th>
<th>( (hs nu) )</th>
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</tr>
</tbody>
</table>

The corresponding convergence rates are \( e_{k^n} = \| x_n - \| = O(\delta^{0.18}) \) for all iterations. The iteration numbers behave like \( k^{(lw)} \sim \delta^{-1.7} \), \( k^{(nu)} \sim \delta^{-0.87} \) for the standard Landweber iteration and the \( \nu \)-method. In both cases the stopping indices can be reduced to the square root by preconditioning, i.e., \( k^{(hu lw)} \sim \delta^{-0.90} \) and \( k^{(hs nu)} \sim \delta^{-0.48} \).

The following two examples shall serve as models for inverse problems governed by partial differential equations. Note that in two dimensions or three dimensions, an assembly of the full operator is usually impossible for reasonable discretization levels, and even the application of the operator \( T \) which involves the solution of the governing equation may be rather expensive. Thus such problems will usually be solved by some iterative method, and the number of iterations should be kept as small as possible. As we will illustrate, this can effectively be achieved by preconditioning in \( Y \)-scales.

Example 5.3 (source reconstruction). Let \( \Omega \subset \mathbb{R}^n, n = 2, 3, \) be a bounded convex domain and let \( T : L^2(\Omega) \rightarrow L^2(\Omega) \) be defined by \( T f = u \) where \( u \) satisfies

\[-q \Delta u + cu = f, \quad u|_{\partial \Omega} = 0,\]

and \( c, q \in L^2(\Omega) \) are known functions bounded from below by some constants \( c \geq 0, \)
Table 5.2

Table 5.2: Iteration numbers and relative errors for Landweber iteration (lw), the \( \nu \)-method with \( \nu = 2 \), and the preconditioned versions, Example 5.3.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( k^*_{(lw)} )</th>
<th>( \varepsilon_{k^*_{(lw)}} )</th>
<th>( k^*_{(hslw)} )</th>
<th>( \varepsilon_{k^*_{(hslw)}} )</th>
<th>( k^*_{(nu)} )</th>
<th>( \varepsilon_{k^*_{(nu)}} )</th>
<th>( k^*_{(hsnu)} )</th>
<th>( \varepsilon_{k^*_{(hsnu)}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>424</td>
<td>0.451</td>
<td>34</td>
<td>0.587</td>
<td>55</td>
<td>0.456</td>
<td>14</td>
<td>0.538</td>
</tr>
<tr>
<td>0.01</td>
<td>1639</td>
<td>0.374</td>
<td>52</td>
<td>0.445</td>
<td>116</td>
<td>0.375</td>
<td>18</td>
<td>0.425</td>
</tr>
<tr>
<td>0.005</td>
<td>4723</td>
<td>0.307</td>
<td>91</td>
<td>0.359</td>
<td>192</td>
<td>0.310</td>
<td>26</td>
<td>0.355</td>
</tr>
<tr>
<td>0.0025</td>
<td>12794</td>
<td>0.252</td>
<td>161</td>
<td>0.282</td>
<td>321</td>
<td>0.253</td>
<td>33</td>
<td>0.275</td>
</tr>
</tbody>
</table>

\( q > 0 \). We consider the \( \mathcal{Y} \)-scale induced by

\[
L^2 : \mathcal{D}(L^2) \subset L_2(\Omega) \to L_2(\Omega) \quad \text{with} \quad L^2u = -\Delta u,
\]

and domain \( \mathcal{D}(L^2) = H^2(\Omega) \cap H_0^1(\Omega) =: \mathcal{Y}_2 \). It follows from standard regularity results for elliptic equations that the solution \( u \) satisfies \( u \in \mathcal{Y}_2 \), i.e., \( \mathcal{R}(T) \subset \mathcal{Y}_2 \). Moreover, if \( q \) is additionally bounded from above, then also the opposite inclusion holds; i.e., Assumption 3.1 is valid with \( a = 2 \).

For a numerical test, we consider \( \Omega = (0, 1)^2 \), \( q = 2 + \text{sign}(1 - 2x) \), \( c = 1 \). Let the true solution be defined by \( f^\dagger = \text{sign}(1 - 2y) \) and choose \( s = -a/2 = -1 \). In view of Theorem 3.8, we can apply the standard discrepancy principle for this example. The results of the numerical tests are listed in Table 5.2.

Example 5.4 (exponentially ill-posed problems). As a final example we consider the backward heat equation as a model problem for exponentially ill-posed problems:

Let \( T : L_2(0, 1) \to L_2(0, 1) \) defined by \( (Tg)(x) = y(x) := u(x, \bar{t}) \) with some \( \bar{t} > 0 \) and

\[
-u_t + qu_{xx} = 0, \quad u(0, t) = u(1, t) = 0, \quad u(x, 0) = g.
\]

The solution of this equation has the Fourier expansion

\[
u(x, \bar{t}; g) = \sum_{n=1}^{\infty} \exp(-q\bar{t}\pi^2 n^2) \langle g, y_n \rangle y_n
\]

with the basis functions \( y_n \) as in Example 5.2. Let \( L^s \) be defined by (5.2). Then we have

\[
\|Tg\|_r \leq c(r)\|g\| \quad \text{for all} \quad r \in \mathbb{R};
\]

thus Assumption 4.1 can be satisfied for arbitrary \( a \) (up to scaling by a constant).

For a numerical test, we let \( q = 0.01 \), set \( s = -2 \) for preconditioning, and try to reconstruct the initial condition

\[
g^\dagger = 2x - \text{sign}(2x - 1) - 1
\]

from measurements of \( u \) at time \( \bar{t} = 1 \).

Since the problem under consideration is exponentially ill-posed, only logarithmic convergence rates can be expected; cf. [14]. The solution of all considered methods look almost identical. Note that for this problem, our theory does not actually imply smaller stopping indices of the discrepancy principle for the preconditioned methods. However, as can be seen from Table 5.3, the number of iterations is reduced significantly for finite noise levels \( \delta > 0 \); cf. also [7] for a discussion of this phenomenon.
Table 5.3
Iteration numbers and relative errors for Landweber iteration (lw), the $\nu$-method with $\nu = 2$, and the preconditioned versions, Example 5.4.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$k^\ast$</th>
<th>$e_k^\ast$</th>
<th>$k^\ast_{lw}$</th>
<th>$e_k^\ast_{lw}$</th>
<th>$k^\ast_{nu}$</th>
<th>$e_k^\ast_{nu}$</th>
<th>$k^\ast_{hslw}$</th>
<th>$e_k^\ast_{hslw}$</th>
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</thead>
<tbody>
<tr>
<td>0.02</td>
<td>13</td>
<td>0.501</td>
<td>9</td>
<td>0.498</td>
<td>6</td>
<td>0.492</td>
<td>5</td>
<td>0.486</td>
</tr>
<tr>
<td>0.01</td>
<td>21</td>
<td>0.488</td>
<td>12</td>
<td>0.486</td>
<td>9</td>
<td>0.482</td>
<td>7</td>
<td>0.476</td>
</tr>
<tr>
<td>0.005</td>
<td>37</td>
<td>0.482</td>
<td>15</td>
<td>0.482</td>
<td>15</td>
<td>0.473</td>
<td>8</td>
<td>0.474</td>
</tr>
<tr>
<td>0.0025</td>
<td>402</td>
<td>0.435</td>
<td>58</td>
<td>0.433</td>
<td>76</td>
<td>0.434</td>
<td>21</td>
<td>0.432</td>
</tr>
</tbody>
</table>

REFERENCES


